



## An Application of The Method of Weighted Residual to the Boundary Layer Problems

### Ağırlıklı Artık Yöntemlerin Sınır Tabaka Problemlerine Uygulanması

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#### ABSTRACT

Falkner-Skan equation is a third order non-linear boundary value problem which describes the laminar boundary layer flow developing on a plate. The strong non-linear characteristics of the problem, sensitivity of the equation to the initial conditions and the semi-infinite domain of the problem have attracted many researchers. In this paper, the method of weighted residuals is used to solve Falkner-Skan equations. The residuals are minimized by the least squares approach. The procedure is very simple and suitable for solving boundary layer problems. The main aim of this paper is to demonstrate the success of the proposed method. We observe that even the simplest approach with only one unknown provide quite accurate results for the velocity profile in the boundary layer. Additionally, better results with any desired accuracy can be obtained by increasing the number of unknown coefficient. Moreover, this method provides analytical solutions which are valid for whole domain.

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#### MAKALE BİLGİSİ

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#### ÖZET

Falkner-Skan denklemi, akışkan içerisindeki bir levha üzerinde gelişen sınır tabaka akışını ifade eden üçüncü dereceden non-lineer bir sınır değer problemidir. Denklem, baskın non-lineer bir yapıya sahip olması, başlangıç koşullarına yüksek derecede hassas olması ve yarı sonsuz bir tanım kümesine sahip olması dolayısıyla birçok araştırmacının ilgisini çekmiştir. Bu çalışmada, ağırlıklı artık bir yöntem kullanılarak Falkner-Skan denklemi yaklaşık olarak çözülmüştür. Artıklar en küçük kareler tekniği kullanılarak minimize edilmiştir. Sunulan prosedür sınır tabaka problemlerinin çözümü için oldukça basit ve kullanışlıdır. Çalışmanın ana amacı uygulanan yöntemin başarısını ortaya koymaktır. Sadece bir bilinmeyen ile en basit yaklaşımın bile sınır tabakasındaki hız profili için oldukça doğru sonuçlar verdiğini gözlemlenmiş ve ek olarak, bilinmeyen katsayı sayısı artırılarak istenen herhangi bir doğrulukla daha iyi sonuçlar elde edilebilmiştir. Ayrıca, bu yöntem tüm alan için geçerli olan analitik çözümler sunmaktadır.

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## 1. INTRODUCTION

Ever since introduced by [1] boundary layer equations have been investigated by many researchers not only because they are able to give proper information for flows around immersed bodies [2] but also they can illustrate the main physical boundary layer phenomena [3].

Blasius introduced a similarity analysis and reduced the zero pressure gradient laminar boundary layer flow equations to an ordinary differential equation (ODE) which has a much simpler form [4]. Hiemenz presented a formulation for the stagnation point flow which yields to another ODE with a similar form [5]. Falkner and Skan found a more general similarity solution in which pressure gradient is also taken into account and showed that two dimensional incompressible boundary layer equations can be reduced to

$$f''' + ff'' + \beta[1 - (f')^2] = 0 \tag{1}$$

and the relative boundary conditions are

$$f(0) = f'(0) = 0 \tag{2a}$$

$$\lim_{\eta \rightarrow \infty} f'(\eta) = 1 \tag{2b}$$

where  $\beta$  is a constant and  $\eta$  is called similarity variable [6]. When  $\beta = 0$ , equation corresponds to the zero pressure gradient flow over flat plate given by [4] and when  $\beta = 1$ , it corresponds to the stagnation point flow of [5]. Equation 1 is physically meaningful where  $-0.19984 < \beta < 2$  [2]. Values of  $\beta > 0$  represents the accelerating flow with favourable pressure gradient while  $\beta < 0$  represents the decelerating flows with adverse pressure gradient. When  $\beta = -0.19984$ ,  $f''(0)$  becomes zero which means shear stress at the wall becomes zero [2].

No exact solution has found yet for equations 1 and 2. Therefore, approximate numerical or analytical techniques are generally applied to solve it. Cebeci and Keller presented three numerical procedure based on shooting and parallel shooting techniques and Runge-Kutta method [3]. They showed that, basic shooting technique is sensitive to the initial guess of  $f''(0)$  for accelerating flows and this sensitivity increases with increasing  $\beta$ . For another words, the initial estimate should be precise enough to get convergence. They introduced parallel shooting method in order to overcome this issue which requires seven initial guesses rather than one. Laine and Reinhart presented further solutions using shooting technique and Newton method coupled with a continuation method [7]. Fazio introduced a free boundary formulation and used iterative transformation method [8]. He successfully calculated the initial slope ( $f''(0)$ ) accurate up to 9 decimal places when  $\beta = 0.5$ . Asaithambi used a finite difference method to solve the Falkner-Skan equation [9]. In the study, a coordinate transformation was used to map the domain into unit interval  $[0, 1]$  while domain is truncated after a finite value of  $\eta_{\infty}$ . It has also been noted that the computational effort of the finite difference method is significantly less compared to the shooting methods. Motsa and Sibanda applied a spectral homotopy analysis method [10]. Once again they used domain truncation and transfer the domain into  $[-1, 1]$ . Fazio extended the Topfer's algorithm to solve Falkner-Skan equation [11]. Liu introduced an iterative numerical method which is based on eigenfunctions and adjoint eigenfunctions [12]. In this method first, equation (1) is transformed into an integral equation and solved taking the advantage of the bi-orthogonality of eigenfunctions and adjoint eigenfunctions. Baramia et. al. applied the homotopy analysis method with Pade approximation to obtain an approximate analytical solution [13] while Yun introduced an iterative analytical technique [14]. Khidir applied two semi-analytical methods namely the successive linearization method and spectral homotopy perturbation method [15]. Kinaci and Usta used the method of moments which is a subset of the methods of weighted residuals in order to solve Blasius equation [16].

In this study, an approximate analytical solution for Falkner-Skan equation is presented. Equation is solved by method of weighted residuals while least squares method is applied to minimize the residuals. The main purpose of the study is to present an accurate and easy to implement analytical method to solve these type of strongly nonlinear equations which are defined in infinite or semi-infinite domain.

## 2. METHODOLOGY

### 2.1. Method of Weighted Residuals

The method of weighted residuals (MWR) is an effective way to seek approximate solutions of differential equations. In general, a trial function with following form is used.

$$F = F_0 + \sum_{i=1}^N c_i F_i \tag{3}$$

where  $c_i$  are unknown constants to be determined. It should be noted that in MWR the trial function is required to satisfy all boundary conditions. Hence  $F_0$  is a function which satisfy the all boundary condition while  $F_i$  is required to satisfy homogenous version of the relative conditions. If the trial function has the same form of the exact solution of the problem, then it is possible to obtain exact analytical solution of a given problem. But if the trial function does not have the same form, there will always be a residual. The main objective of all types of MWRs is to find  $c_i$  which minimize the residual by using suitable weighting functions and orthogonality feature of weighting and trial functions [17].

There are many ways to select the weighting function and each way yields to another method such as least squares, subdomain, moment etc. In present paper, the well-known least squares method is used for weighting functions. This method is one of the best known method and widely used in many problems [17]. Consider the equation (1) with the boundary conditions (2). By choosing a trial function which has the form (3), residual function becomes

$$R(c_i, \eta) = F_0''' + \sum_{i=1}^N c_i F_i''' + \left( F_0 + \sum_{i=1}^N c_i F_i \right) \left( F_0'' + \sum_{i=1}^N c_i F_i'' \right) + \beta \left[ 1 - \left( F_0' + \sum_{i=1}^N c_i F_i' \right)^2 \right] \quad (4)$$

where  $R(c_i, \eta)$  is the residual function and  $F'$  denotes the derivative of  $F$  with respect to  $\eta$ . The idea of the least squares approach is to search for  $c_i$  which minimizes the integral of the square of the residual in the domain.

$$I = \int_0^{\infty} [R(c_i, \eta)]^2 d\eta \quad (5)$$

In order to minimize  $I$ , we use conjugate gradients method. In this method, we start with an initial guess for  $c_i$  values and calculate them iteratively by implementing the conjugate gradient algorithm. The applied algorithm is explained in detail by [18].

### 2.2. Numerical Integration

It is sometimes hard to express the residual in a closed form for large  $N$  values and take the definite integral given at equation (5). Thus, we calculate the residual for equally spaced discrete points in space and take the integral at equation (5) numerically using Boole's method.

Let the values of a function  $y(x)$  be calculated for equally spaced points  $x_i$  and  $\Delta x = x_{i+1} - x_i$ . Then Boole's method can be expressed as

$$\int_{x_1}^{x_5} y dx = \frac{2\Delta x}{45} [7y(x_1) + 32y(x_2) + 12y(x_3) + 32y(x_4) + 7y(x_5)] \quad (6)$$

Detailed information can be found in [19]. Additionally, it is important to note that we calculate the integral from 0 to a finite value of  $\eta_{\infty}$  instead of calculating it for infinite domain. The reason is not only because it would be more complex to calculate it for infinite domain, but also it is sufficient enough to minimize the residual in the boundary layer for a finite value of  $\eta_{\infty}$ .

### 3. APPLICATION TO THE FALKNER-SKAN EQUATION

We introduce following trial function to approximate the exact solution of the problem.

$$F = \eta + e^{-\eta} - 1 + \sum_{i=1}^N c_i [e^{-(i+1)\eta} - (i+1)e^{-\eta} + i] \quad (7)$$

The exponential functions are convenient to be used in problems with semi-infinite domains. By taking the derivatives and putting to the equation (1), the residuals can be obtained as

$$R = -e^{-\eta} + \sum_{i=1}^N c_i [-(i+1)^3 e^{-(i+1)\eta} + (i+1)e^{-\eta}] + \left[ e^{-\eta} + \sum_{i=1}^N c_i [(i+1)^2 e^{-(i+1)\eta} - (i+1)e^{-\eta}] \right] \left[ \eta + e^{-\eta} - 1 + \sum_{i=1}^N c_i [e^{-(i+1)\eta} - (i+1)e^{-\eta} + i] \right] \quad (8)$$

$$+\beta \left[ 1 - \left[ 1 - e^{-\eta} + \sum_{i=1}^N c_i [-(i+1)e^{-(i+1)\eta} + (i+1)e^{-\eta}] \right]^2 \right]$$

The easiest case occurs when  $N = 1$ . The trial function  $F$  and residual  $R$  in this case can be calculated as

$$F = \eta + e^{-\eta} - 1 + c_1(e^{-2\eta} - 2e^{-\eta} + 1) \tag{9}$$

$$R = -e^{-\eta} + c_1(-8e^{-2\eta} + 2e^{-\eta}) + [e^{-\eta} + c_1(4e^{-2\eta} - 2e^{-\eta})][\eta + e^{-\eta} - 1 + c_1(e^{-2\eta} - 2e^{-\eta} + 1)] + \beta \left[ 1 - [1 - e^{-\eta} + c_1(-2e^{-(i+1)\eta} + 2e^{-\eta})]^2 \right] \tag{10}$$

By putting equation (10) into the equation (5) and integrate with a sufficient  $\eta_\infty$  value, approximate solutions can be obtained. In practical applications, boundary layer thickness is usually defined as the thickness where local velocity reaches the 99% of the free stream velocity which corresponds to the  $f'(\eta_\infty)=0.99$ . When  $\beta = 0$  for example  $\eta_\infty$  value is around 5 [2]. In this study, we chose  $\eta_\infty = 20$  for the calculations which is sufficient enough for the calculations. As an initial guess  $c_1 = 0$  value has implemented and  $R^2$  values are calculated for each  $\eta$  from 0 to  $\eta_\infty$  with  $\Delta\eta = 0.01$  increments. Approximate solution of the equations (1) and (2) is calculated as

$$F = \eta + e^{-\eta} - 1 - 0.21995(e^{-2\eta} - 2e^{-\eta} + 1) \tag{11}$$

$$F' = \frac{u}{U_e} = 1 - e^{-\eta} - 0.21995(-2e^{-2\eta} + 2e^{-\eta}) \tag{12}$$

where,  $u$  is the horizontal velocity component with respect to the plate and  $U_e$  is the free stream velocity. Comparison of the equation 12 with the numerical solution given by [19] presented in Figure 1. It can be seen from the figure that even the easiest case ( $N=1$ ) gives quite accurate results. The shooting angle ( $f''(0)$ ) can be calculated as 0.5601 with around 19% relative error.

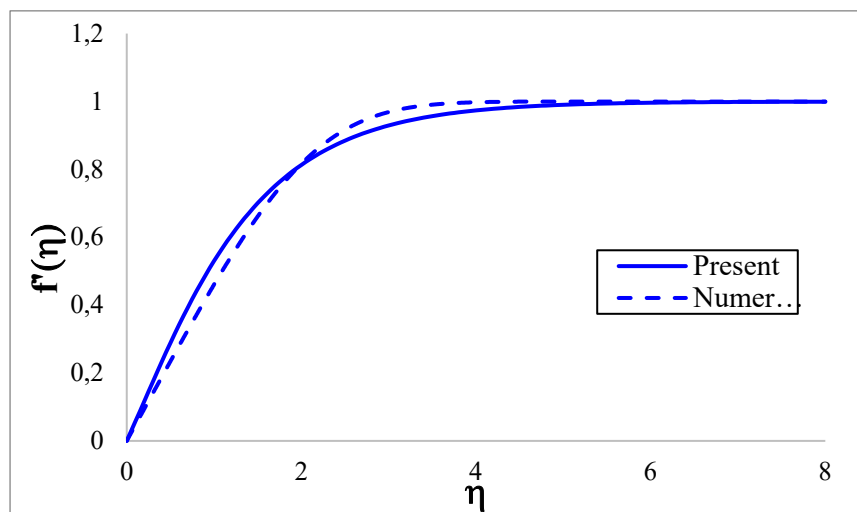


Figure 1. Comparison of the  $f'(\eta)$  with the numerical solution ( $\beta = 0, N = 1$ )

The accuracy of the solution can be increased by increasing  $N$ . Table 1 shows the  $c_i$ ,  $f''(0)$  and relative error (RE%) values for different numbers of  $N$ . It can be seen that current method provides sufficient results for high numbers of  $N$ .

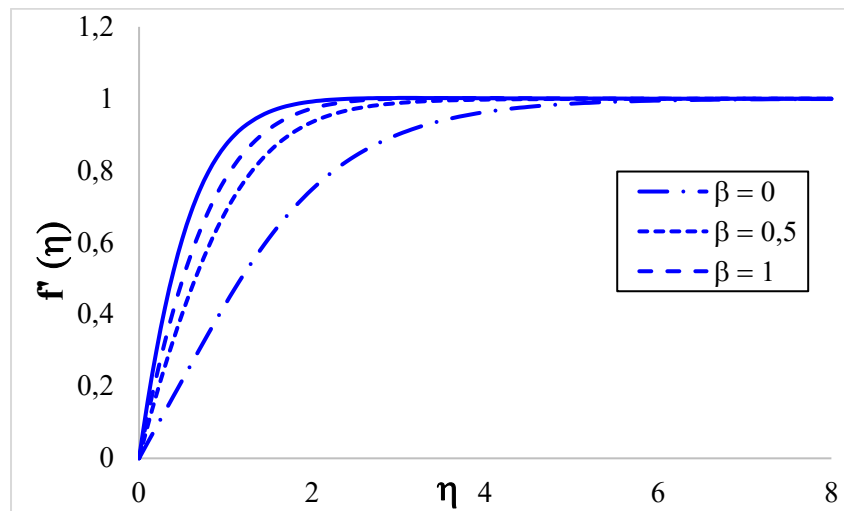
**Table 1.**  $c_i$  coefficients and shooting angle ( $f''(0)$ ) for different numbers of N ( $\beta = 0$ ).

N	$c_1$	$c_2$	$c_3$	$c_4$	$f''(0)$	RE%
1	-0.21995	-	-	-	0.5601	19.3
2	-0.65299	0.127151	-	-	0.45693	2.7
3	-1.10085	0.423943	-0.07647	-	0.42431	9.6
4	-0.72273	0.003824	0.165297	-0.05624	0.43625	7.1

Solution for different  $\beta$  values can be obtained by the same procedure. Table 2 shows calculated shooting angle values when N=4. Results are highly accurate when  $\beta > 0$  when comparing with the numerical values of [21] while accuracy reduces when  $\beta < 0$ . The possible reason for this is that the selected trial function is not capable of representing the exact solution for decelerating flows.

**Table 2.** Calculated shooting angle ( $f''(0)$ ) for different values of  $\beta$  (N=4).

$\beta$	Present	Numerical [21]
2.0000	1.68727	1.68722
1.0000	1.23252	1.23259
0.5000	0.93054	0.92768
0.0000	0.43625	0.46960
-0.1000	0.26501	0.31927
-0.1500	0.1934	0.21636
-0.1988	0.1391	0.005220



**Figure 2.** Velocity Profiles calculated by the presented method for different  $\beta \in [0,2]$ .

Calculated velocity ( $f'(\eta)$ ) profiles are shown at Fig. 2 when  $\beta > 0$ . The figure shows that the thickness of the boundary layer decreases and non-dimensional velocity values in the boundary layer increases with increasing  $\beta$ . This result is physically meaningful since pressure gradient in the flow becomes more favourable as  $\beta$  increases [2]. It can be seen from the Table 2 that the success of the applied method increases with increasing  $\beta$ . Calculated velocity profile ( $f'(\eta)$ ) and its comparison with the numerical results given by [21] for Hiemenz problem (when  $\beta = 1$ ) [5] is tabulated below. The maximum relative error is 1.785% occurs at  $\eta = 0.2$  which shows that the method is highly accurate.

**Table 3.** Calculated  $f'(\eta)$  values for Hiemenz problem ( $\beta = 1, N = 4$ ).

$\eta$	Present	Numerical [19]	RE%
0.0	0.0000	0.0000	0.000
0.1	0.1183	0.1183	0.025
0.2	0.2266	0.2226	1.785
0.5	0.4946	0.4946	0.003
1.0	0.7778	0.7778	0.000
1.2	0.8462	0.8467	0.055
1.6	0.9314	0.9324	0.107
2.0	0.9728	0.9732	0.037
2.2	0.9841	0.9841	0.003
2.6	0.9961	0.9946	0.147
3.0	1.0006	0.9985	0.210

#### 4. CONCLUSIONS

In this study, the method of weighted residuals is applied to the Falkner-Skan boundary layer flow problem. A simple method based on least squares approach is developed to minimize the residual. We observe that even the simplest approach with only one unknown provide quite accurate results for the velocity profile in the boundary layer. Additionally, better results with any desired accuracy can be obtained by increasing the number of unknown coefficient. Moreover, this method provides analytical solutions which are valid for whole domain.

Despite the simplicity of proposed method, when  $\beta > 0$ , our results are in good accordance with those reported in the literature. Additionally, method can be used as a tool to find the shooting angle ( $f''(0)$ ) of the problem. Once this value is calculated, the Falkner-Skan problem becomes an initial value problem which is much simpler to solve numerically.

It should be noted however that the accuracy of the solution might be affected significantly by selection of trial function. We observed that the presented trial function is not capable of presenting highly accurate results when  $\beta < 0$ . We are intended to cover this point and in search of better approximations for negative values of  $\beta$ .

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