Konuralp Journal of Mathematics
Volume 5 No. 2 pp. 1–11 (2017) ©KJM

SOME RESULTS ON THE INTEGER TRANSLATION OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

SAMTEN TAMANG AND NITYAGOPAL BISWAS

Abstract. In the paper, authors have studied the comparative growth properties of composite entire and meromorphic functions on the basis of integer translation applied upon them and established some newly developed results.

1. Introduction, Definitions, and Notations

Let $f(z)$ be a meromorphic function defined in the Complex Plane $\mathbb{C}$. For $n \in \mathbb{N}$, the translation of $f(z)$ be denoted by $f(z+n)$. We now describe or investigate the changes to Nevanlinna's Characteristic function of the translated meromorphic functions. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [7] and [3].

For each $n \in \mathbb{N}$, we may obtain a function with some properties. Let us denote this family by $f_n(z) = \{f(z+n) : n \in \mathbb{N}\}$. The Nevanlinna's Characteristic function of a meromorphic function f denoted by $T(r, f)$ is defined as

$$T(r,f) = N(r,f) + m(r,f)$$

where

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r$$

and

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta.$$

It is clear that the number of zeros of f may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$ i.e.,

$$N(r, f(z+n)) = N(r,f) + e_n, \quad \text{where } e_n \to 0 \text{ as } r \to \infty.$$

Date: September 3, 2016 and, in revised form, April 5, 2017.
2010 *Mathematics Subject Classification.* 30D35, 30D30.
Key words and phrases. Entire function, Meromorphic function, comparative growth property, integer translation.

Also

$$\begin{aligned} m\left(r, f\left(z+n\right)\right) &= \frac{1}{2\pi}\int_{0}^{2\pi} \log^{+}\left|f\left(re^{i\theta}+n\right)\right| d\theta \\ &= m\left(r,f\right)+e_n', \quad \text{where } e_n' \to 0 \text{ as } r\to\infty. \end{aligned}$$

Therefore on adding we get that

$$N\left(r, f\left(z+n\right)\right) + m\left(r, f\left(z+n\right)\right) = N\left(r,f\right) + m\left(r,f\right) + e_n + e_n'.$$

Now if n varies then the Nevanlinna's Characteristic function for the family f_n is

$$\begin{aligned} T\left(r, f_n\right) &= nT\left(r,f\right) + \sum_n \left(e_n + e_n'\right) \\ \text{i.e., } \log T\left(r, f_n\right) &= \log T\left(r,f\right) + \log n. \qquad (1.1) \end{aligned}$$

In order to express the rate of growth on the integer translation of composite entire and meromorphic functions more precisely we recall the following definitions:

Definition 1.1. The order ρ_f and lower order λ_f of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M\left(r,f\right)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M\left(r,f\right)}{\log r}$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, ..., n$ and $\log^{[0]} x = x$.

When f is meromorphic, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T\left(r,f\right)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T\left(r,f\right)}{\log r}.$$

Definition 1.2. The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of an entire function f is defined as

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M\left(r,f\right)}{\log r} \quad \text{and} \quad \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M\left(r,f\right)}{\log r}.$$

If f is meromorphic, then

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T\left(r,f\right)}{\log r} \quad \text{and} \quad \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T\left(r,f\right)}{\log r}.$$

Definition 1.3. The type σ_f of a meromorphic function f is defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T\left(r,f\right)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

If f is entire then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M\left(r,f\right)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Applying (1.1) on Definition 1.1, Definition 1.2, and Definition 1.3 we get that

$$\rho_{f_n} = \rho_f, \quad \lambda_{f_n} = \lambda_f, \ \overline{\rho}_{f_n} = \overline{\rho}_f, \quad \overline{\lambda}_{f_n} = \overline{\lambda}_f \ \text{ and } \ \sigma_{f_n} = n\sigma_f$$

and the relations can easily be verified on considering $f = \exp z$.

In this paper, we establish some new results in the connection with the comparative growth properties of composite entire and meromorphic functions by using integer translation upon them.

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. [4] *Let $g(z)$ be an integral function with $\lambda_g < \infty$, and assume that $a_i(z)$ $(i = 1, 2, ..., n;\ n \leq \infty)$ are entire functions satisfying $T(r, a_i(z)) = o\{T(r, g)\}$ and $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$, then*

$$\lim_{r\to\infty} \frac{T(r, g)}{\log M(r, g)} = \frac{1}{\pi}.$$

Lemma 2.2. [5] *Let f and g be two entire functions. If $M(r, g) > \frac{2+\epsilon}{\epsilon}|g(0)|$ for any ϵ $(\epsilon > 0)$ then*

$$T(r, f\circ g) < (1+\epsilon)\, T(M(r, g), f).$$

In particular, if $g(0) = 0$ then

$$T(r, f\circ g) < T(M(r, g), f).$$

for all $r > 0$.

Lemma 2.3. [1] *If f is meromorphic and g is entire then for all large values of r*

$$T(r, f\circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Lemma 2.4. [2] *Let f be meromorphic and g is entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity*

$$T(r, f\circ g) \geq T(\exp(r)^{\mu}, f).$$

Lemma 2.5. [6] *Let f and g be two entire functions. Then we have*

$$T(r, f\circ g) \geq \frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right) + O(1), f\right\}.$$

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let $f(z)$ and $g(z)$ be two non-constant integral functions such that ρ_f and λ_g are finite. Also suppose there exist entire functions $a_i(z)$ $(i = 1, 2, ..., n;\ n \leq \infty)$ such that (i) $T(r, a_i(z)) = o\{T(r, g)\}$ as $r \to \infty$ for $i = 1, 2, ..., n$ and (ii) $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$. If $f_n = \{f(z+n)\}$ and $g_n = \{g(z+n)\}$ for $n \in \mathbb{N}$, then*

$$\limsup_{r\to\infty} \frac{\log T(r, f_n\circ g_n)}{T(r, g_n)} \leq \frac{\pi}{n}\rho_f.$$

Proof. Since f and g are two non-constant and in view of $\rho_{f_n} = \rho_f$, we get for all large r and given $\epsilon(> 0)$ that

$$T(r, f_n\circ g_n) \leq \log M(M(r, g_n), f_n) \leq \{M(r, g_n)\}^{\rho_{f_n}+\epsilon}.$$

So, for all large r

$$\log T(r, f_n \circ g_n) \leq (\rho_{f_n} + \epsilon) \log M(r, g_n)$$

Hence we get for all large values of r

$$\limsup_{r\to\infty} \frac{\log T(r, f_n \circ g_n)}{T(r, g_n)} \leq (\rho_{f_n} + \epsilon) \limsup_{r\to\infty} \frac{\log M(r, g_n)}{T(r, g_n)}$$

$$i.e., \quad \limsup_{r\to\infty} \frac{\log T(r, f_n \circ g_n)}{T(r, g_n)} \leq (\rho_f + \epsilon) \limsup_{r\to\infty} \frac{\log M(r, g_n)}{T(r, g_n)}. \tag{3.1}$$

Now,

$$\limsup_{r\to\infty} \frac{\log M(r, g_n)}{T(r, g_n)} = \limsup_{r\to\infty} \frac{\log M(r, g)}{nT(r, g) + \sum_n (e_n + e_n')}$$

$$= \frac{1}{n} \limsup_{r\to\infty} \frac{\log M(r, g)}{T(r, g)} = \frac{\pi}{n} \text{ [from Lemma1]},$$

where $e_n \to 0$ & $e_n' \to 0$ as $r \to \infty$.

Since $\epsilon\,(>0)$ is arbitrary, it follows from (3.1)

$$\limsup_{r\to\infty} \frac{\log T(r, f_n \circ g_n)}{T(r, g_n)} \leq \frac{\pi}{n} \rho_f.$$

This proves the theorem. □

Example 3.1. Let us consider two functions $f(z) = e^z$ and $g(z) = e^{e^z}$. Then $f_n(z) = e^{z+n}$, $g_n(z) = e^{e^{z+n}}$ and $(f_n \circ g_n)(z) = e^{e^{e^{z+n}}}$. Here $\rho_{f_n} = 1$, $\lambda_{g_n} = \infty$.

$$T(r, g_n) = \frac{e^r e^n}{(2\pi^3 r)^{\frac{1}{2}}}$$

$$\log T(r, f_n \circ g_n) = e^r e^n + 0(1)$$

$$\limsup_{r\to\infty} \frac{\log T(r, f_n \circ g_n)}{T(r, g_n)} = \frac{e^r e^n + 0(1)}{\frac{e^r e^n}{(2\pi^3 r)^{\frac{1}{2}}}} = \infty.$$

Theorem 3.2. *Let $f(z)$ and $g(z)$ be two entire functions of finite order such that $g(0) = 0$ and $\rho_g < \lambda_f \leq \rho_f$. If $f_n(z) = \{f(z+n)\}$ and $g_n(z) = \{g(z+n)\}$ for $n \in \mathbb{N}$. Then*

$$\lim_{r\to\infty} \sup \frac{\log T(r, f_n \circ g_n)}{T(r, f_n)} \leq \rho_f$$

Proof. In view of $\rho_{f_n} = \rho_f$ and $\lambda_{f_n} = \lambda_f$, let us choose $\epsilon > 0$ such that $\rho_{g_n} + \epsilon < \lambda_{f_n} - \epsilon$.

By the definitions of $\lambda_{f_n} = \lambda_f$, we have for $\epsilon > 0$,

$$T(r, f_n) > r^{\lambda_{f_n} - \epsilon}$$

$$\text{i.e.,} \quad T(r, f_n) > r^{\lambda_f - \epsilon}. \tag{3.2}$$

Again by the Lemma 2.2 we have

$$T(r, f_n \circ g_n) \leq T(M(r, g_n), f_n)$$

$$< \{M(r, g_n)\}^{\rho_{f_n} + \epsilon}$$

$$\begin{aligned} \text{or,}\quad \log T(r, f_n \circ g_n) &< (\rho_{f_n} + \epsilon) \log M(r, g_n) \\ &< (\rho_{f_n} + \epsilon) r^{\rho_{g_n} + \epsilon} \\ &< (\rho_{f_n} + \epsilon) r^{\lambda_{f_n} - \epsilon} \end{aligned}$$

$$\text{i.e.,}\quad \log T(r, f_n \circ g_n) < (\rho_f + \epsilon) r^{\lambda_f - \epsilon} \tag{3.3}$$

From (3.2) and (3.3), we obtained for all large values of r

$$\frac{\log T(r, f_n \circ g_n)}{T(r, f_n)} < \frac{(\rho_f + \epsilon) r^{\lambda_f - \epsilon}}{r^{\lambda_f - \epsilon}} = (\rho_f + \epsilon)$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\lim_{r \to \infty} \sup \frac{\log T(r, f_n \circ g_n)}{T(r, f_n)} \leq \rho_f.$$

Thus the theorem is established. □

Example 3.2. Let us consider two functions $f(z) = e^z$ and $g(z) = e^z - 1$.

Then $f_n(z) = e^{z+n}$, $g_n(z) = e^{z+n} - 1$ and $(f_n \circ g_n)(z) = e^{e^{z+n}-1}$. Here $\rho_{g_n} = 1$, $\rho_{f_n} = 1$.

$$T(r, f_n) = \frac{r}{\pi} + n$$

$$\log T(r, f_n \circ g_n) = r - \frac{1}{2} \log\left(2\pi^3 r\right) + n$$

$$\frac{\log T(r, f_n \circ g_n)}{T(r, f_n)} = \frac{r - \frac{1}{2}\log\left(2\pi^3 r\right) + n}{\frac{r}{\pi} + n} = \pi$$

Thus

$$\lim_{r \to \infty} \sup \frac{\log T(r, f_n \circ g_n)}{T(r, f_n)} > \rho_f = 1$$

Theorem 3.3. *Let $f(z)$ and $g(z)$ be two entire functions of finite order with $\rho_g > \lambda_f$. If $f_n(z) = \{f(z+n)\}$ and $g_n(z) = \{g(z+n)\}$ for $n \in \mathbb{N}$. Then*

$$\lim_{r \to \infty} \sup \frac{\log T(r, f_n \circ g_n)}{T(r, f_n)} = \infty$$

Proof. Since $\rho_{f_n} = \rho_f$ and $\rho_{g_n} = \rho_g$. So we can choose $\epsilon(> 0)$ such that $\rho_g - \epsilon > \rho_f + \epsilon$.

By the Lemma 2.5 we get for a sequence of values of r tending to infinity

$$\begin{aligned} T(r, f_n \circ g_n) &\geq \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g_n\right) + O(1), f_n\right) \\ &\geq \frac{1}{3} \left\{\frac{1}{8} M\left(\frac{r}{4}, g_n\right) + O(1)\right\}^{\lambda_{f_n} - \epsilon} \\ &\geq \frac{1}{3} \left\{\frac{1}{9} M\left(\frac{r}{4}, g_n\right)\right\}^{\lambda_{f_n} - \epsilon} \\ &\geq \frac{1}{3} \left(\frac{1}{9}\right)^{\lambda_{f_n} - \epsilon} \left\{\exp\left(\frac{r}{4}\right)^{\rho_{g_n} - \epsilon}\right\}^{\lambda_{f_n} - \epsilon} \end{aligned}$$

$$\text{i.e.,}\quad \log T(r, f_n \circ g_n) \geq \log \frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_{f_n}-\epsilon} + (\lambda_{f_n} - \epsilon)\left(\frac{r}{4}\right)^{\rho_{g_n}-\epsilon}$$

$$\text{i.e.,}\quad \log T(r, f_n \circ g_n) \geq \log \frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_{f}-\epsilon} + (\lambda_{f} - \epsilon)\left(\frac{r}{4}\right)^{\rho_{g}-\epsilon}$$

$$\text{i.e.,}\quad \log T(r, f_n \circ g_n) \geq \log A + (\lambda_{f} - \epsilon)\left(\frac{r}{4}\right)^{\rho_{g}-\epsilon} \tag{3.4}$$

where $A = \frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_f - \epsilon}$.

Again for any $\epsilon > 0$

$$T(r, f_n) < r^{\rho_{f_n}+\epsilon}$$

$$\text{i.e.,}\quad T(r, f_n) < r^{\rho_{f}+\epsilon}. \tag{3.5}$$

From (3.4) and (3.5) it follows for a sequence of values of r tending to infinity that

$$\frac{\log T(r, f_n \circ g_n)}{T(r, f_n)} > \frac{\log A}{r^{\rho_f+\epsilon}} + \frac{(\lambda_f - \epsilon)\left(\frac{r}{4}\right)^{\rho_g-\epsilon}}{r^{\rho_f+\epsilon}}.$$

Since $\rho_g - \epsilon > \rho_f + \epsilon$,

$$\lim_{r\to\infty} \frac{\left(\frac{r}{4}\right)^{\rho_g-\epsilon}}{r^{\rho_f+\epsilon}} = \infty.$$

Hence

$$\lim_{r\to\infty} \sup \frac{\log T(r, f_n \circ g_n)}{T(r, f_n)} = \infty.$$

This proves the theorem. $\square$

Theorem 3.4. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order with $\rho_g > 0$. If $f_n(z) = \{f(z+n)\}$ and $g_n(z) = \{g(z+n)\}$ for $n \in \mathbb{N}$. Then*

$$\lim_{r\to\infty} \sup \frac{\log T(r, f_n \circ g_n)}{\log T(r, g_n)} = \infty.$$

Proof. In view of Lemma 2.5 we have for a sequence of values of r tending to infinity

$$\begin{aligned} T(r, f_n \circ g_n) &\geq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, g_n\right) + O(1), f_n\right) \\ &\geq \frac{1}{3}\left\{\frac{1}{8}M\left(\frac{r}{4}, g_n\right) + O(1)\right\}^{\lambda_{f_n}-\epsilon} \\ &\geq \frac{1}{3}\left\{\frac{1}{9}M\left(\frac{r}{4}, g_n\right)\right\}^{\lambda_{f_n}-\epsilon} \\ &\geq \frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_{f_n}-\epsilon}\left\{\exp\left(\frac{r}{4}\right)^{\rho_{g_n}-\epsilon}\right\}^{\lambda_{f_n}-\epsilon} \end{aligned}$$

$$\text{i.e.,}\quad \log T(r, f_n \circ g_n) \geq \log \frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_{f_n}-\epsilon} + (\lambda_{f_n} - \epsilon)\left(\frac{r}{4}\right)^{\rho_{g_n}-\epsilon}$$

$$\text{i.e.,}\quad \log T(r, f_n \circ g_n) \geq \log \frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_{f}-\epsilon} + (\lambda_{f} - \epsilon)\left(\frac{r}{4}\right)^{\rho_{g}-\epsilon}$$

$$\text{i.e.,}\quad \log T(r, f_n \circ g_n) \geq \log A + (\lambda_f - \epsilon)\left(\frac{r}{4}\right)^{\rho_g-\epsilon} \tag{3.6}$$

where $A = \frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_f - \epsilon}$.

Also for any $\epsilon > 0$

$$\begin{aligned} T(r, g_n) &< r^{\rho_{g_n} + \epsilon} \\ \log T(r, g_n) &< (\rho_{g_n} - \epsilon) \log r \\ \text{i.e.,} \quad \log T(r, g_n) &< (\rho_g - \epsilon) \log r. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) we obtained for a sequence of values of r tending to infinity

$$\frac{\log T(r, f_n \circ g_n)}{T(r, g_n)} \geq \frac{\log A}{(\rho_g - \epsilon) \log r} + \frac{(\lambda_f - \epsilon)}{4^{\rho_g - \epsilon}} \cdot \frac{r^{\rho_g - \epsilon}}{(\rho_g - \epsilon) \log r}.$$

Therefore

$$\lim_{r \to \infty} \sup \frac{\log T(r, f_n \circ g_n)}{\log T(r, g_n)} = \infty, \quad \text{since } \rho_g > 0 .$$

Thus the theorem is established. □

Theorem 3.5. *Let f be a meromorphic function and g be an entire function such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \rho_g < \infty$. If $f_n = \{f(z+n)\}$ and $g_n = \{g(z+n)\}$ for $n \in \mathbb{N}$, then*

$$\frac{\rho_g}{\rho_f} \leq \limsup_{r \to \infty} \frac{\log^{[2]} T(r, f_n \circ g_n)}{\log T(r, f_n)} \leq \frac{\rho_g}{\lambda_f}.$$

Proof. In view of $\rho_{f_n} = \rho_f$ and $\lambda_{f_n} = \lambda_f$, let us choose that $0 < \varepsilon < \min\{\rho_{f_n}, \lambda_{f_n}\}$ $= \min\{\rho_f, \lambda_f\}$.

Since $T(r, g_n) \leq \log^+ M(r, g_n)$, by Lemma 2.2 we obtain for all sufficiently large values of r,

$$\begin{aligned} T(r, f_n \circ g_n) &\leq \{1 + o(1)\} T(M(r, g_n), f_n) \\ \text{i.e.,} \quad \log T(r, f_n \circ g_n) &\leq \log T(M(r, g_n), f_n) + O(1) \\ \text{i.e.,} \quad \log T(r, f_n \circ g_n) &\leq (\rho_{f_n} + \varepsilon) \log M(r, g_n) + O(1) \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \quad \log^{[2]} T(r, f_n \circ g_n) &\leq \log^{[2]} M(r, g_n) + O(1) \\ \text{i.e.,} \quad \log^{[2]} T(r, f_n \circ g_n) &\leq (\rho_{g_n} + \varepsilon) \log r + O(1) \\ \text{i.e.,} \quad \log^{[2]} T(r, f_n \circ g_n) &\leq (\rho_g + \varepsilon) \log r + O(1). \end{aligned} \tag{3.8}$$

Again in the view of Lemma 2.4, we get for a sequence of values of r tending to infinity on taking $\mu = \rho_{g_n} - \varepsilon < \rho_{g_n}$ that

$$\begin{aligned} \log T(r, f_n \circ g_n) &\geq \log T\left(\exp\left(r^{\rho_{g_n} - \varepsilon}\right), f_n\right) \\ \text{i.e.,} \quad \log T(r, f_n \circ g_n) &\geq (\lambda_{f_n} - \varepsilon) \log\left(\exp\left(r^{\rho_{g_n} - \varepsilon}\right)\right) \\ \text{i.e.,} \quad \log T(r, f_n \circ g_n) &\geq (\lambda_f - \varepsilon) r^{\rho_g - \varepsilon}. \end{aligned} \tag{3.9}$$

$$\begin{aligned} \text{i.e.,} \quad \log^{[2]} T(r, f_n \circ g_n) &\geq (\rho_{g_n} - \varepsilon) \log r + O(1) \\ \text{i.e.,} \quad \log^{[2]} T(r, f_n \circ g_n) &\geq (\rho_g - \varepsilon) \log r + O(1). \end{aligned} \tag{3.10}$$

Also from the definition of $\rho_{f_n} = \rho_f$ and $\lambda_{f_n} = \lambda_f$, we have for all sufficiently large values of r,

$$\begin{aligned} \log T(r, f_n) &\leq (\rho_{f_n} + \varepsilon) \log r \\ \text{i.e.,} \quad \log T(r, f_n) &\leq (\rho_f + \varepsilon) \log r \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \log T\left(r, f_n\right) &\geq \left(\lambda_{f_n} - \varepsilon\right) \log r \\ \text{i.e.,} \quad \log T\left(r, f_n\right) &\geq \left(\lambda_f - \varepsilon\right) \log r. \end{aligned} \tag{3.12}$$

From (3.10) and (3.11) , it follows for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[2]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, f_n\right)} &\geq \frac{\left(\rho_{g_n} - \varepsilon\right) \log r + O(1)}{\left(\rho_{f_n} + \varepsilon\right) \log r} \\ \text{i.e.,} \quad \frac{\log^{[2]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, f_n\right)} &\geq \frac{\left(\rho_g - \varepsilon\right) \log r + O(1)}{\left(\rho_f + \varepsilon\right) \log r}. \end{aligned}$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, f_n\right)} \geq \frac{\rho_g}{\rho_f}. \tag{3.13}$$

From (3.8) and (3.12) it follows for all sufficiently large values of r

$$\frac{\log^{[2]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, f_n\right)} \leq \frac{\left(\rho_g + \varepsilon\right) \log r + O(1)}{\left(\lambda_f - \varepsilon\right) \log r}.$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, f_n\right)} \leq \frac{\rho_g}{\lambda_f}. \tag{3.14}$$

Thus the theorem follows from (3.13) and (3.14) . □

Remark 3.1. In addition to the conditions of Theorem 3.5, if f_n is of regular growth i.e., $\lambda_{f_n} = \rho_{f_n}$, equivalently $\lambda_f = \rho_f$ then

$$\limsup_{r\to\infty} \frac{\log^{[2]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, f_n\right)} = \frac{\rho_g}{\rho_f}.$$

Remark 3.2. Under the same conditions of Theorem 3.5,

$$\limsup_{r\to\infty} \frac{\log^{[2]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, g_n\right)} \geq 1.$$

Theorem 3.6. *Let f be a meromorphic function and g be a non constant entire function such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \rho_g < \infty$. If $f_n = \{f(z+n)\}$ and $g_n = \{g(z+n)\}$ for $n \in \mathbb{N}$, then*

$$\frac{\overline{\lambda}_g}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[3]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, g_n^{(k)}\right)} \leq \limsup_{r\to\infty} \frac{\log^{[3]} T\left(r, f_n \circ g_n\right)}{\log\left(r, g_n^{(k)}\right)} \leq \frac{\overline{\rho}_g}{\lambda_g}$$

where $k = 0, 1, 2, ...$

Proof. In view of $\rho_{f_n} = \rho_f$ and $\lambda_{f_n} = \lambda_f$, let us choose that $0 < \varepsilon < \min\{\rho_{f_n}, \lambda_{f_n}\}$ $= \min\{\rho_f, \lambda_f\}$.

For large values of r

$$\begin{aligned} \log M\left(r, f_n\right) &\geq r^{\lambda_{f_n} - \varepsilon} \\ \textit{i.e.,} \quad \log M\left(r, f_n\right) &\geq r^{\lambda_f - \varepsilon}. \end{aligned} \tag{3.15}$$

We know for all values of r that

$$T\left(r, f_n \circ g_n\right) \geq \frac{1}{3} \log M\left\{\frac{1}{8} M\left(\frac{r}{4}, g_n\right)+O(1), f_n\right\}.$$

So from (3.15), we get for all large values of r

$$\begin{aligned} T\left(r, f_n \circ g_n\right) & \geq \frac{1}{3}\left\{\frac{1}{8} M\left(\frac{r}{4}, g_n\right)+O(1)\right\}^{\lambda_{f_n}-\varepsilon} \\ & \geq \frac{1}{3}\left\{\frac{1}{9} M\left(\frac{r}{4}, g_n\right)\right\}^{\lambda_f-\varepsilon}. \end{aligned}$$

We obtain for all sufficiently large values of r,

$$\frac{\log^{[3]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, g_n^{(k)}\right)} \geq \frac{\log^{[3]} M\left(\frac{r}{4}, g_n\right)}{\log \frac{r}{4}} \frac{\log \frac{r}{4}}{\log T\left(r, g_n^{(k)}\right)}+O(1). \tag{3.16}$$

Also from the definition of $\rho_{f_n}=\rho_f$ and $\lambda_{f_n}=\lambda_f$, we have for all sufficiently large values of r,

$$\begin{aligned} \log T\left(r, g_n^{(k)}\right) & \leq \left(\rho_{g_n}+\varepsilon\right) \log r \\ \text{i.e., } \log T\left(r, g_n^{(k)}\right) & \leq \left(\rho_g+\varepsilon\right) \log r \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \log T\left(r, g_n^{(k)}\right) & \geq \left(\lambda_{g_n}-\varepsilon\right) \log r \\ \text{i.e., } \log T\left(r, g_n^{(k)}\right) & \geq \left(\lambda_g-\varepsilon\right) \log r. \end{aligned} \tag{3.18}$$

Since $\varepsilon(>0)$ is arbitrary, we get from (3.16) and (3.17)

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[3]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, g_n^{(k)}\right)} & \geq \frac{\overline{\lambda}_{g_n}}{\rho_{g_n}} \\ \text{i.e., } \liminf_{r \rightarrow \infty} \frac{\log^{[3]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, g_n^{(k)}\right)} & \geq \frac{\overline{\lambda}_g}{\rho_g}. \end{aligned} \tag{3.19}$$

Again for given $\varepsilon\left(0<\varepsilon<\lambda_g\right)$, and for all large values of r

$$T\left(r, f_n \circ g_n\right) \leq \log M\left\{M\left(r, g_n\right), f_n\right\} \leq\left\{M\left(r, g_n\right)\right\}^{\rho_{f_n}+\varepsilon}$$

or

$$\begin{aligned} \frac{\log^{[3]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, g_n^{(k)}\right)} & \leq \frac{\log^{[3]} M\left(r, g_n\right)}{\log T\left(r, g_n^{(k)}\right)}+O(1) \\ \frac{\log^{[3]} T\left(r, f_n \circ g_n\right)}{\log T\left(r, g_n^{(k)}\right)} & \leq \frac{\log^{[3]} M\left(r, g_n\right)}{\log r} \frac{\log r}{\log T\left(r, g_n^{(k)}\right)}+O(1). \end{aligned} \tag{3.20}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from (3.18) and (3.20)

$$\limsup_{r\to\infty}\frac{\log^{[3]}T\left(r,f_n\circ g_n\right)}{\log T\left(r,g_n^{(k)}\right)} \leq \frac{\overline{\rho}_{g_n}}{\lambda_{g_n}}$$

$$\text{i.e., }\limsup_{r\to\infty}\frac{\log^{[3]}T\left(r,f_n\circ g_n\right)}{\log T\left(r,g_n^{(k)}\right)} \leq \frac{\overline{\rho}_{g}}{\lambda_{g}}. \tag{3.21}$$

Thus the theorem follows from (3.19) and (3.21). □

Theorem 3.7. *Let $f(z)$ be meromorphic and $g(z)$ be an entire function such that $0<\rho_g<\infty$ and $\lambda_f>0$. If $f_n(z)=\{f(z+n)\}$ and $g_n(z)=\{g(z+n)\}$ for $n\in\mathbb{N}$. Then*

$$\lim_{r\to\infty}\sup\frac{\log T\left(r,f_n\circ g_n\right)}{\log T\left(\exp\left(r\right)^{\mu},g_n\right)}=\infty$$

where $0<\mu<\rho_g$.

Proof. Let $0<\mu<\mu'<\rho_g$. By the Lemma 2.4 we get for a sequence of values of r tending to infinity

$$\begin{aligned} T\left(r,f_n\circ g_n\right) &\geq T\left(\exp\left(r\right)^{\mu'},f_n\right)\\ &> \left(\exp\left(r\right)^{\mu'}\right)^{\lambda_{f_n}-\epsilon}. \end{aligned}$$

$$\text{i.e., }\log T\left(r,f_n\circ g_n\right) > \left(\lambda_{f_n}-\epsilon\right)\log\left(\exp\left(r\right)^{\mu'}\right)=\left(\lambda_{f_n}-\epsilon\right)r^{\mu'}$$

$$\text{i.e., }\log T\left(r,f_n\circ g_n\right) > \delta r^{\mu'}. \tag{3.22}$$

where $0<\delta=\lambda_{f_n}-\epsilon<\lambda_{f_n}$ i.e., $0<\delta=\lambda_f-\epsilon<\lambda_f$.

Again for all sufficiently large values of r, we have

$$\log T\left(\exp\left(r\right)^{\mu},g_n\right) < \log\left(\exp\left(r\right)^{\mu}\right)^{\rho_{g_n}+\epsilon}=\left(\rho_{g_n}+\epsilon\right)r^{\mu}$$

$$\text{i.e., }\log T\left(\exp\left(r\right)^{\mu},g_n\right) < \left(\rho_g+\epsilon\right)r^{\mu}. \tag{3.23}$$

From (3.22) and (3.23) it follows for a sequence of values of r tending to infinity

$$\frac{\log T\left(r,f_n\circ g_n\right)}{\log T\left(\exp\left(r\right)^{\mu},g_n\right)}>\frac{\delta r^{\mu'}}{\left(\rho_g+\epsilon\right)r^{\mu}}.$$

As $\epsilon\,(>0)$ is arbitrary, it follows from the above that

$$\lim_{r\to\infty}\sup\frac{\log T\left(r,f_n\circ g_n\right)}{\log T\left(\exp\left(r\right)^{\mu},g_n\right)}=\infty.$$

This proves the theorem. □

Corollary 3.1. *Under the assumptions of Theorem 3.7*

$$\lim_{r\to\infty}\sup\frac{T\left(r,f_n\circ g_n\right)}{T\left(\exp\left(r\right)^{\mu},g_n\right)}=\infty,\quad \textit{where } 0<\mu<\rho_g.$$

Proof. From Theorem 3.7 we see that for $K\,(>1)$ there exist a sequence of values of r tending to infinity such that

$$\begin{aligned} \log T\left(r,f_n\circ g_n\right) &> K\log T\left(\exp\left(r\right)^{\mu},g_n\right)\\ \text{i.e., }\quad T\left(r,f_n\circ g_n\right) &> T\left(\exp\left(r\right)^{\mu},g_n\right)^{K}. \end{aligned}$$

It follows that

$$\lim_{r\to\infty} \sup \frac{T(r, f_n \circ g_n)}{T(\exp(r)^{\mu}, g_n)} = \infty.$$

□

References

[1] Bergweiler, W. : On the Nevanlinna characteristic of a composite function, Complex Var. Elliptic Equ., Vol. 10(1988), pp.225-236.

[2] Bergweiler, W. : On the growth rate of composite meromorphic functions, Complex Var. Elliptic Equ., Vol. 14 (1990), pp.187-196.

[3] Hayman, W.K.: Meromorphic Functions, The Clarendon Press, Oxford (1964) .

[4] Qun Lin and Chong Ji Dai: On a conjecture of Shah concerning small functions, *Kexue Tongbao (English Edn.),Vol. 31 (1986), No. 4, pp. 220-224.*

[5] Niino, K. and Suita, N.:Growth of a composite function of entire functions, Kodai Math. J. 3(1980), 374-79.

[6] Niino, K. and Yang, C.C.: Factorization theory of meromorphic functions and related topics, Marcel Dekker Inc. New York (1982) , 95-99.

[7] Valiron, G.: Lectures on the General Theory of Integral Functions, Chelsea Publishing Company (1949) .

Department of Mathematics, The University of Burdwan, Burdwan, Pin-713104, West Bengal, India.

E-mail address: samtentamang@yahoo.in, stamang@math.buruniv.ac.in

Department of Mathematics, University of Kalyani, Kalyani, Dist. Nadia, PIN - 741235, West Bengal, India

E-mail address: nityamaths@gmail.com