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Summability of Fourier Series and its Derived Series by Matrix Means

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Abstract – This Paper introduces the concept of matrix operators and establishes two new theorems on matrix summability of Fourier series and its derived series. the results obtained in the paper further extend several known results on linear operators. Various types of criteria, under varying conditions, for the matrix summability of the Fourier series, In this paper quite a different and general type of criterion for summability of the Fourier Series has been obtained, in the theorem function f is integrable in the sense of Lebesgue to the interval $[-\pi,\pi]$ and period with period 2π.

Keywords – *Summability, matrix summability, Fourier series, derived Fourier series.*

1. Introduction

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of partial sum $\{S_n\}$. Let $A=(a_{n,k})$ be an infinite triangular matrix of real constants, The sequence-to-sequence transformation [4].

$$t_n^A=\sum_{k=0}^{n} a_{n,k}\,S_k=\sum_{k=0}^{n} a_{n,n-k}\;S_{n-k} \qquad (1)$$

Defines the sequence t_n^A of matrix means of the sequence $\{S_n\}$, generated by the sequence of coefficients $(a_{n,k})$. The series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the sum S by We can write $t_n^A \to S \quad (A), \text{as } n\to\infty$.

The necessary and sufficient conditions for A-transform to be regular

$$(\text{i.e.} \lim_{n\to\infty} S_n = S \Longrightarrow \lim_{n\to\infty} t_n^A = S)$$

are the well-known Silverman-Toeplitz conditions? [1][4] where the triangular matrix $A=(a_{n,k}),\ n,k=0,1,2,3\,...$ and $a_{n,k}=0$ for $k>n$ is regular if

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$$\lim_{n\to\infty} a_{n,k} = 0 \quad ; k = 1,2, \dots$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{n,k} = 1$$

$$\sum_{k=0}^{n} |a_{n,k}| \le M \;\; ; \;\; n = 1,2, \dots (M \text{ independent of } n)$$

and

$$t_n^{AB} = \sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,r} S_r.$$

Examples:

(1). Matrix Hankel [2] . Let $\{h_n\}_{n=0}^{\infty}$ be a positive sequence of real constants let

$$H = (a_{n,k}) = (h_{n+k-1})$$

2). Matrix Toeplitz [4] $T = (a_{n,k}) \quad ; (a_{n,k} = 0 \;\; ; \;\; k > n)$(Thus from (1), We get [2]

(1). $\quad \sum_{n=0}^{\infty} u_n = A \quad (T) \Rightarrow \sum_{n=0}^{\infty} a u_n = aA \quad (T)$

(2). $\quad \sum_{n=0}^{\infty} u_n = A_1 (T) \;\&\; \sum_{n=0}^{\infty} v_n = A_2 (T) \quad \Rightarrow \sum_{n=0}^{\infty} (u_n + v_n) = A_1 + A_2 \; (T)$

(3). $\quad \sum_{n=0}^{\infty} u_n = A_1 \;\; (S) \wedge \sum_{n=0}^{\infty} u_n = A_2 \;\; (T) \;\; \Rightarrow A_1 = A_2$

Proof (3): Now by (1) we have

$$(R) = (r_{m,n}) = (a_{m,0}.b_{m,n} + a_{m,1}.b_{m,n-1} + \cdots + a_{m,n-1}.b_{m,1} + a_{m,n}.b_{m,0})$$
$$t_m = \sum_{k=0}^{m} r_{m,k} . S_k$$

$$\Longrightarrow t_m = r_{m,0} . S_0 + r_{m,1} . S_1 + r_{m,2} . S_2 + \cdots + r_{m,m-1} . S_{m-1} + r_{m,m} . S_m$$

$$\Longrightarrow t_m = a_{m,0}.b_{m,0}(S_0) + (a_{m,0}.b_{m,1} + a_{m,1}.b_{m,0}).(S_1) + (a_{m,0}.b_{m,2} + a_{m,1}.b_{m,1} + a_{m,2}.b_{m,0})(S_2) + (a_{m,0}.b_{m,3} + a_{m,1}.b_{m,2} + a_{m,2}.b_{m,1} + a_{m,3}.b_{m,0}).(S_3) + (a_{m,0}.b_{m,4} + a_{m,1}.b_{m,3} + a_{m,2}.b_{m,2} + a_{m,3}.b_{m,1} + a_{m,4}.b_{m,0}).(S_4) + (a_{m,0}.b_{m,m} + a_{m,1}.b_{m,m-1} + a_{m,2}.b_{m,m-2} + \cdots + a_{m,m-2}.b_{m,1} + a_{m,m-1}.b_{m,0}).(S_{m-1}) + (a_{m,0}.b_{m,m-1} + a_{m,1}.b_{m,m-2} + a_{m,2}.b_{m,m-3} + \cdots + a_{m,m-1}.b_{m,1} + a_{m,m}.b_{m,0}).(S_m)$$

$$\Longrightarrow t_m = a_{m,0}.(b_{m,0}.S_0 + b_{m,1}.S_1 + b_{m,2}.S_2 + b_{m,3}.S_3 + b_{m,4}.S_4 + \cdots + b_{m,m-1}.S_{m-1} + b_{m,m}.S_m) + a_{m,1}.(b_{m,0}.S_1 + b_{m,1}.S_2 + b_{m,2}.S_3 + b_{m,3}.S_4 + \cdots + b_{m,m-2}.S_{m-1} + b_{m,m-1}.S_m) + a_{m,2}.(b_{m,0}.S_2 + b_{m,1}.S_3 + b_{m,2}.S_4 + b_{m,3}.S_5 + \cdots + b_{m,m-3}.S_{m-1} + b_{m,m-2}.S_m) + \cdots + a_{m,m-1}.(b_{m,0}.S_{m-1} + b_{m,1}.S_m) + a_{m,m}.(b_{m,0}.S_m)$$

$$\Longrightarrow t_m = a_{m,0}.\sum_{k=0}^{m} b_{m,k} . S_{k+0} + a_{m,1}.\sum_{k=0}^{m-1} b_{m,k} . S_{k+1} + a_{m,2}.\sum_{k=0}^{m-2} b_{m,k} . S_{k+2} + \cdots + a_{m,m-1}.\sum_{k=0}^{m-(m-1)} b_{m,k} . S_{k+m-1} + a_{m,m}.\sum_{k=0}^{m-m} b_{m,k} . S_{k+m}$$

$$\Longrightarrow t_m = \sum_{n=0}^{m} a_{m,n}.\sum_{k=0}^{m-n} b_{m,k} . S_{k+n}$$

$$\Longrightarrow \lim_{m\to\infty} t_m = \sum_{n=0}^{\infty} a_{m,n}.A_2 = A_2$$

$$\sum_{n=0}^{\infty} u_n = A_2(R) \quad ; (R) = (r_{m,n})$$

Similarly

$$\sum_{n=0}^{\infty} u_n = A_1(R) \Longrightarrow A_2 = A_2$$

2 Preliminaries

Theorem 2.1. Let $A = (a_{n,k}), B = (b_{n,k})$ be an infinite triangular matrix with $a_{n,k} \geq 0, b_{n,k} \geq 0$then$t_n^{AB} \in \mathcal{B}(\mathcal{A}_r)$ Where $\mathcal{B}(\mathcal{A}_r)$ Space call to bounded linear operator on $\mathcal{A}_r$, $T: \mathcal{A}_r \to \mathcal{A}_r$ and $\mathcal{A}_r = \{(s_n)_{n=0}^{\infty}; \sum_{n=0}^{\infty}(n+1)^{r-1}. |u_n|^r < \infty; u_n = \Delta s_n = s_n - s_{n-1}\}$

Proof of Theorem 2.1. Let τ_n^{AB} mn-denote the mn-term of the AB-transform, in terms of $(n+1)u_n$, that is $\tau_n^{AB} = \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)u_j = (n+1)(t_n^{AB} - t_{n-1}^{AB})$ to prove the theorem, it will be sufficient to show that $\sum_{n=0}^{\infty} \frac{1}{n+1} |\tau_n^{AB}|^r < \infty$ Using Hölder's inequality, we have

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |\tau_n^{AB}|^r = \sum_{n=0}^{\infty} \frac{1}{n+1} \left|\sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)u_j\right|^r \leq$$
$$\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)^r |u_j|^r \times \left\{\sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j}\right\}^{r-1}$$

Since $\sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} = 1$ we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |\tau_n^{AB}|^r = \sum_{n=0}^{\infty} \frac{1}{n+1} \left|\sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)u_j\right|^r \leq$$
$$\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} b_{k,j} (j+1)^r |u_j|^r \leq \sum_{j=0}^{\infty} (j+1)^r |u_j|^r \sum_{n=j}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{n,k} . b_{k,j}$$

For $m, n \geq 1$,

$$\sum_{n=j}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_{n,k} . b_{k,j} = \frac{1}{j+1}$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |\tau_n^{AB}|^r = O(1). \sum_{j=0}^{\infty} (j+1)^r |u_j|^r \frac{1}{j+1} = O(1). \sum_{j=0}^{\infty} (j+1)^r |u_j|^r = O(1) < \infty$$

This completes the proof of the theorem (1).

3 Particular Cases

Several authors such as ([4]-[6]), (see also [7]) studied the matrix summability method and obtained many interesting results.

The important particular cases of the triangular matrix means are:

(i) Cesàro mean of order 1 or $(C, 1)$ mean if, $a_{n,k} = \frac{1}{n+1}$.

(ii) Harmonic means $(H, 1)$when, $a_{n,k} = \frac{1}{(n-k+1) \log n}$.

(iii) (C,δ) where $0 \le \delta \le 1$ means when, $a_{n,k} = \frac{\binom{n-k+\delta+1}{\delta-1}}{\binom{n+\delta}{\delta}}$.

(v) Nörlund means (N, p_n) when, $a_{n,k} = \frac{p_{n-k}}{P_n}$ where $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$.

(vi) Riesz means $(\overline{N}, p_n)$ when, $a_{n,k} = \frac{p_k}{P_n}$ where $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$.

4 Results and Discussion

Let $f(t)$ be a periodic function with period 2π, integrable in the sense of Lebesgue over $[-\pi, \pi]$. The Fourier series and derived Fourier series of $f(t)$ are given by [3][4][5]

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n\,(x)$$

With partial sums s_n and

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) \qquad (2)$$

We shall use following notations

$$\tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right].$$

We use the following notations

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$g(t) = f(x+t) - f(x-t) - 2t\hat{f}(x)$$

$$K_{AB}(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}.\sum_{r=0}^{k} b_{k,k-r}\frac{\sin\left(k-r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$$

$$M_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}.\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \qquad (3)$$

$$\tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right].$$

Theorem 4.1. Let $\{p_n\}_{n=0}^{\infty}$ be a real non-negative and non-increasing sequence of real constants such that $P_n = \sum_{k=0}^{n} p_k \to \infty; (n \to \infty)$ and $A = (a_{n,k}), B = (b_{k,r})$ be an infinite triangular matrix with $a_{n,k} \ge 0$, If

$$\Phi(t) = \int_0^t |\phi(u)|du = o\left(\frac{t}{\alpha\left(\frac{1}{t}\right)P_\tau}\right), \text{ as } t \to +0 \ , \tau = \left[\frac{1}{t}\right]$$

where $\alpha(t)$ is a positive, monotonic and non-increasing function of $t \to +0$

$$\log n = O(\alpha(n).P_n); (n \to \infty).$$

and

$$\lim_{n\to\infty}\int_1^n \frac{B_{n,t}}{t.\alpha(t)P_t}dt = O(1); B_{n,\tau} = \sum_{k=0}^{\tau} b_{n,n-k} = \sum_{k=n-\tau}^{n} b_{n,k}$$

Then the Fourier series (1) is summable AB to f (x).

Theorem 4.2. Let $\{a_{n,k}\}_{k=0}^{\infty}$ be a real non-negative and non-decreasing sequence with respect to k such that $T = (a_{n,k})$ be an infinite triangular matrix with $a_{n,k} \geq 0$. If

$$\int_0^t |dg(u)| = o\left(\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}}\right), \text{ as } t \to +0$$

Then the derived Fourier Series (2) is sumable (T) to the sum $\acute{f}(x)$, where $\acute{f}(x)$ is the derivative of $f(x)$,provided $\alpha(t)$ is a positive monotonic decreasing function of $t \to +0$ such $\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}}$ increases monotonically as $t \to +0$.

For the proof of our theorems, following lemmas are required.

Lemma 4.1. [5] If $\{a_{n,k}\}$ is non-negative and non-decreasing with k then for $0 \leq t \leq \pi, 0 \leq a \leq b \leq \infty$ and for any n, we have $\left|\sum_{k=a}^{b} a_{n,n-k}.e^{i(n-k)t}\right| \leq O(A_{n,\tau})$ Where $A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}$, $A_{n,n} = 1 \quad (\forall n \geq 0)$.

Lemma 4.2. [5] If $\{b_{n,k}\}$ is non-negative and non-decreasing with k then for $0 \leq t \leq \pi, 0 \leq a \leq b \leq \infty$ and for any n, we have $\left|\sum_{k=a}^{b} b_{n,n-k}.e^{i(n-k)t}\right| \leq O(B_{n,\tau})$ Where $B_{n,\tau} = \sum_{k=0}^{\tau} b_{n,n-k}$, $B_{n,n} = 1 \quad (\forall n \geq 0)$.

Lemma 4.3. For $0 < t \leq \frac{1}{n}$, $K_{AB}(n,t) = O(n)$.

Proof. For $0 < t \leq \frac{1}{n}$, $\sin(n+1)t \leq (n+1)t$, $\left[\sin\left(\frac{t}{2}\right)\right]^{-1} \leq \frac{\pi}{t}$

$$K_{AB}(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\sum_{r=0}^{k} b_{k,k-r}\frac{\sin\left(k-r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \leq \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\sum_{r=0}^{k} b_{k,k-r}\frac{\left|\sin\left(k-r+\frac{1}{2}\right)t\right|}{\left|\sin\frac{1}{2}t\right|} \leq \frac{2n+1}{2\pi}\sum_{k=0}^{n} a_{n,k}\sum_{r=0}^{k} b_{k,k-r}$$

Since $\sum_{k=0}^{n} a_{n,k}\sum_{r=0}^{k} b_{k,k-r} = 1$

Thus $K_{AB}(n,t) = O(n)$

Lemma 4.4. For $\frac{1}{n} \leq t \leq \delta < \pi$, $K_{AB}(n,t) = O\left(\frac{B_{n,\tau}}{t}\right)$; $\tau \leq n$

Proof. $K_{AB}(n,t) \leq \frac{1}{2\pi}\left|\sum_{k=0}^{n} a_{n,k}\sum_{r=0}^{k} b_{k,k-r}\frac{\sin\left(k-r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right|$

$$\leq \frac{1}{2t}\left|\sum_{k=0}^{n} a_{n,k}\sum_{r=0}^{k} b_{k,k-r}\sin\left(k-r+\frac{1}{2}\right)t\right| \text{; (by Jordan's lemma)}$$

$$\leq \frac{1}{2t}\left|\sum_{k=0}^{n} a_{n,k}\,\mathrm{Im}\sum_{r=0}^{k} b_{k,k-r}e^{i\left(k-r+\frac{1}{2}\right)t}\right|$$

$$\leq \frac{1}{2t}\sum_{k=0}^{n} a_{n,k}\left|\mathrm{Im}\sum_{r=0}^{k} b_{n,n-k}.e^{i(k-r)t}.e^{i\frac{t}{2}}\right|$$
$$\leq \frac{1}{2t}\sum_{k=0}^{n} a_{n,k}\left|\sum_{r=0}^{k} b_{n,n-k}.e^{i(k-r)t}\right| \;;\; \left|e^{i\frac{t}{2}}\right| = 1$$
$$\leq \frac{1}{2t}\sum_{k=0}^{n} a_{n,k}O(B_{k,\tau});\;\; \text{by lemma (1)}$$
$$\leq O\left(\frac{B_{n,\tau}}{t}\right)\sum_{k=0}^{n} a_{n,k}$$
$$= O\left(\frac{B_{n,\tau}}{t}\right)$$

Lemma 4.5. For $\frac{1}{n} \leq t \leq \delta < \pi$, $M_n(t) = O\left(\frac{A_{n,\tau}}{t}\right)$; $\tau \leq n$.

Proof. Now by (3)

$$M_n(t) \leq \left|\sum_{k=0}^{n} a_{n,n-k}.\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right| \leq \frac{1}{\sin\frac{1}{2}t}\left|\mathrm{Im}.\sum_{k=0}^{n} a_{n,n-k}.e^{i\left(n-k+\frac{1}{2}\right)t}\right| \leq$$
$$\frac{\pi}{t}\left|\mathrm{Im}\sum_{k=0}^{n} a_{n,n-k}.e^{i(n-k)t}.e^{i\frac{t}{2}}\right| \text{ (by Jordan's lemma) } \leq \frac{\pi}{t}\left|\sum_{k=0}^{n} a_{n,n-k}.e^{i(n-k)t}.\right| \;;\; \left|e^{i\frac{t}{2}}\right| \leq 1$$
$$= \frac{\pi}{t}.O(A_{n,\tau}) \quad \text{by lemma (1)}$$
$$= O\left(\frac{A_{n,\tau}}{t}\right)$$

Lemma 4.6. For $0 \leq t \leq \frac{1}{n}$, $M_n(t) = O(n)$.

Proof of Theorem 4.1. Let $s_n(x)$ denote the n^{th} partial sum of the (1). Then we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$
$$t_n^B - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n} b_{n,k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$$
$$t_n^{AB} - f(x) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\int_0^{\pi}\phi(t)\left\{\sum_{r=0}^{k} b_{k,r}\frac{\sin\left(r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt$$
$$= \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\int_0^{\pi}\phi(t)\left\{\sum_{r=0}^{k} b_{k,k-r}\frac{\sin\left(k-r+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt = \int_0^{\pi}\phi(t)K_{AB}(n,t)dt$$
$$= \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right)\phi(t)K_{AB}(n,t)dt$$
$$= I_{1.1} + I_{1.2} + I_{1.3}$$
$$I_{1.1} \leq |I_{1.1}| \leq \int_0^{\frac{1}{n}}|\phi(t)|.|K_{AB}(n,t)|dt = O(n)\left\{\int_0^{\frac{1}{n}}|\phi(t)|dt\right\}$$

By Lemma 4.3

$$= O(n).\left\{o\left(\frac{\frac{1}{n}}{\alpha(n).P_n}\right)\right\} = o\left(\frac{1}{\alpha(n).P_n}\right)$$
$$= o\left(\frac{1}{\log n}\right) = o(1);(n \to \infty)$$
$$I_{1.2} \leq |I_{1.2}| \leq O\left(\int_{\frac{1}{n}}^{\delta}|\phi(t)||K_{AB}(n,t)|dt\right)$$

$$= O\left(\int_{\frac{1}{n}}^{\delta} |\phi(t)| \frac{B_{n,t}}{t} dt\right)$$

By Lemma 4.4

Integrating by parts

$$I_{1.2} \le O\left\{\left(\frac{B_{n,t}}{t}.\Phi(t)\Big|_{\frac{1}{n}}^{\delta}\right) - \int_{\frac{1}{n}}^{\delta} \frac{d}{dt}\left(\frac{B_{n,t}}{t}\right)\Phi(t)dt\right\}$$
$$= O\left\{o\left(\frac{B_{n,t}}{t}\frac{t}{\alpha(\frac{1}{t})P_\tau}\Big|_{\frac{1}{n}}^{\delta}\right) + \int_{\frac{1}{n}}^{\delta} \frac{B_{n,t}}{t^2}\frac{t}{\alpha(\frac{1}{t})P_\tau} dt \int_{\frac{1}{n}}^{\delta} \frac{1}{t}\frac{t}{\alpha(\frac{1}{t})P_\tau} d(B_{n,\tau})\right\}$$
$$\left(u = \frac{1}{t}\right)$$
$$I_{1.2} = O\left\{o\left(\frac{B_{n,[\frac{1}{\delta}]}}{\alpha\frac{1}{\delta}P_{[\frac{1}{\delta}]}}\right) + o\left(\frac{B_{n,n}}{\alpha(n)P_n}\right) + o\left(\int_{\frac{1}{\delta}}^{n} \frac{B_{n,u}}{u.\alpha(u)P_u} du\right) + o\left(\int_{\frac{1}{\delta}}^{n} \frac{1}{\alpha(u)P_u} d(B_{n,u})\right)\right\}$$
$$\le O\left\{o(1) + o(1) + o\left(\int_1^n \frac{B_{n,u}}{u.\alpha(u)P_u} du\right) + o\left(\frac{1}{\alpha(n)P_n}\right)\left(\int_1^n d(B_{n,u})\right)\right\} = O\{o(1) + o(1) + o(O(1)) + o(1)B_{n,n}\} = O\{o(1) + o(1) + o(1) + o(1)\} = o(1); \quad (n \to \infty)$$

Lastly, by the Riemann-Lebesgue theorem and the regularity condition of matrix summability, we obtain

$$I_{1.3} \le |I_{1.3}| \le \int_{\delta}^{\pi} |\phi(t)||K_{AB}(n,t)|dt = o(1) \quad ; \quad (n \to \infty)$$

Next

$$t_n^{AB} - f(x) = o(1) \qquad (n \to \infty) \Longrightarrow \lim_{n\to\infty}\{t_n^{AB} - f(x)\} = 0$$

This completes the proof of the theorem.

Proof of Theorem 4.2. Let $\acute{s}_n(x)$ denote the n^{th} partial sum of the (2). Then

$$\acute{s}_n(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} dg(t) + \acute{f}(x)$$

We have

$$\acute{s}_n(x) = \sum_{k=1}^{n} k(b_k \cos kx - a_k \sin kx) = \sum_{k=1}^{n} \frac{1}{\pi}\int_0^{2\pi} k(\sin ku . \cos kx - \cos ku . \sin kx). f(u)du = \sum_{k=1}^{n} \frac{k}{\pi}\int_0^{2\pi} \sin k(u-x) . f(u)du =$$
$$\frac{1}{2\pi}\int_0^{2\pi} f(u). \sum_{k=1}^{n} 2k \sin k(u-x)\, du$$
$$= -\frac{1}{2\pi}\int_0^{2\pi} f(u). \sum_{k=1}^{n} 2k \sin k(x-u)\, du$$

where

$$\sum_{k=1}^{n} k \sin ky = -\frac{d}{dy}\left(\frac{\sin\left(n+\frac{1}{2}\right)y}{2\sin\frac{1}{2}y}\right)$$

Next

$$\sum_{k=1}^{n} 2k \sin k(x-u) = -2\frac{d}{dx}\left(\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{2\sin\frac{1}{2}(x-u)}\right) = 2\frac{d}{du}\left(\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{2\sin\frac{1}{2}(x-u)}\right)$$

Thus

$$\acute{s}_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d}{dx}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}.f(u)du = -\frac{1}{2\pi}\int_0^{2\pi} f(u).\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du$$

$$= -\frac{1}{2\pi}\left(\int_{-\pi}^{0} f(u)\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du + \int_0^{\pi} f(u)\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du\right) = G_1 + G_2$$

Now

$$\acute{s}_n(x) = -\frac{1}{2\pi}\int_0^{\pi}\{f(x+t) - f(x-t)\}\left\{\frac{d}{dt}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt$$

Integrating by parts

$$\acute{s}_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}d\{f(x+t) - f(x-t)\}$$

where

$$\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} = \frac{1}{2} + \cos t + \cos 2t + \cos 3t + \cdots + \cos nt = D_n(t) \Rightarrow \frac{1}{\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt = \frac{1}{2}$$

and

$$g(t) = f(x+t) - f(x-t) - 2t\acute{f}(x) \Longrightarrow f(x+t) - f(x-t) = g(t) + 2t\acute{f}(x)$$
$$\Longrightarrow d\{f(x+t) - f(x-t)\} = dg(t) + 2\acute{f}(x)dt + 0$$

Next

$$\acute{s}_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t) + 2\acute{f}(x).\frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$

$$\Longrightarrow \acute{s}_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t) + \acute{f}(x)$$

$$\Longrightarrow \acute{s}_n(x) - \acute{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t)$$

$$\Longrightarrow \acute{s}_{n-k}(x) - \acute{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t)$$

$$\Longrightarrow \sum_{k=0}^{n} a_{n,n-k}\left\{\acute{s}_{n-k}(x) - \acute{f}(x)\right\}$$

$$= \int_0^{\pi} dg(t).\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}.\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$$

$$= \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right)dg(t).M_n(t) = I_1 + I_2 + I_3$$

$$I_1 = \int_0^{\frac{1}{n}} dg(t).M_n(t) = O\left(\int_0^{\frac{1}{n}}|dg(t)|.|M_n(t)|\right) = O\left(n.\int_0^{\frac{1}{n}}|dg(t)|\right) = O\left(n.o\left(\frac{\frac{1}{n}.\alpha\left(\frac{1}{\frac{1}{n}}\right)}{\log\frac{1}{\frac{1}{n}}}\right)\right)$$

$$= O\left(n.o\left(\frac{\alpha(n)}{n.\log n}\right)\right) = o\left(\frac{\alpha(n)}{\log n}\right) = o(1) \;\; ; \;\; (n \to \infty)$$

Lastly, by the Riemann-Lebesgue theorem and the regularity condition of matrix sumability, we obtain

$$I_3 \le |I_3| = \int_{\delta}^{\pi} |M_n(t)|.|dg(t)| = o(1), \text{ as } (n \to \infty)$$
$$I_2 \le \left| \int_{\frac{1}{n}}^{\delta} dg(t).M_n(t) \right| = O\left(\int_{\frac{1}{n}}^{\delta} |dg(t)|.|M_n(t)| \right) = O\left(\int_{\frac{1}{n}}^{\delta} \frac{A_{n,\tau}}{t}.|dg(t)| \right)$$

Integrating by parts, where $u = \frac{A_{n,\tau}}{t}$, $dv = dg(t)$. Therefore

$$I_2 = O\left(\frac{A_{n,\tau}}{t}.o\left(\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}} \right) \Bigg|_{\frac{1}{n}}^{\delta} \right) + O\left(\int_{\frac{1}{n}}^{\delta} \frac{A_{n,\tau}}{t^2}.o\left(\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}} \right) dt \right) + O\left(\int_{\frac{1}{n}}^{\delta} \frac{1}{t}.o\left(\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}} \right) d(A_{n,\tau}) \right)$$

(Using condition)

$$\Longrightarrow I_2 \le o\left(A_{n,\tau} \frac{\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}} \Bigg|_{\frac{1}{n}}^{\delta} \right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{n}}^{\delta} \frac{A_{n,\tau}}{t^2} dt \right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{n}}^{\delta} \frac{d(A_{n,\tau})}{t} \right)$$

where

$$\frac{t.\alpha\left(\frac{1}{t}\right)}{\log\frac{1}{t}} = \frac{\alpha\left(\frac{1}{t}\right)}{\frac{1}{t}.\log\frac{1}{t}} \text{ increases monotonically as } t \to +0.$$

$$\Longrightarrow I_2 \le o(1) + o\left(A_{n,n}.\frac{\alpha(n)}{\log n} \right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{\delta}}^{n} A_{n,u} du \right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{\delta}}^{n} u.d(A_{n,u}) \right)$$

$$\Longrightarrow I_2 = o(1) + o\left(O\left(\frac{\alpha(n)}{\log n} \right) \right) + o\left(\frac{\alpha(n)}{n.\log n}.(u.A_{n,u})\Big|_{\frac{1}{\delta}}^{n} \right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{n}}^{\delta} u.d(A_{n,u}) \right) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{\delta}}^{n} u.d(A_{n,u}) \right)$$

Integrating by parts, where $u_1 = A_{n,u}$, $dv_1 = du$

$$\Longrightarrow I_2 \le o(1) + o\left(\frac{\alpha(n)}{\log n} \right) + o\left(\frac{\alpha(n)}{n.\log n}.n.A_{n,n} \right) + o(1) + o\left(\frac{\alpha(n)}{n.\log n}.\int_{\frac{1}{\delta}}^{n} u.d(A_{n,u}) \right) = o(1) + o\left(\frac{\alpha(n)}{\log n} \right) + o\left(\frac{\alpha(n)}{\log n} \right) + o\left(\frac{\alpha(n)}{n.\log n}.n.\int_{\frac{1}{\delta}}^{n} d(A_{n,u}) \right)$$
$$\Longrightarrow I_2 \le o(1) + o\left(\frac{\alpha(n)}{\log n} \right) + o\left(\frac{\alpha(n)}{\log n}.A_{n,n} \right) \quad ; \quad A_{n,n} = 1$$
$$= o(1) + o(1) + o(1) = o(1) \; ; \; (n \to \infty)$$

Then

$$\sum_{k=0}^{n} a_{n,n-k} \{\acute{s}_{n-k}(x) - \acute{f}(x)\} = o(1) \; ; \; (n \to \infty) \Longrightarrow \acute{t}_n(x) - \acute{f}(x) = o(1) \; ; \; (n \to \infty)$$

where

$$\acute{t}_n(x) = \sum_{k=0}^{n} a_{n,n-k}.\acute{s}_{n-k}(x)$$

Next

$$\lim_{n \to \infty} \acute{t}_n(x) = \acute{f}(x)$$

This completes the proof of the theorem.

5. Conclusions

One of the most important outcomes of this study is that the product of any two matrix methods of the methods of summability is a matrix method and that this method is a bounded linear operator which transforms each sequence of a given space to a sequence of the space itself. And

$$t_n^{AB} \neq t_n^{A.B}$$

where

$$t_n^{AB} = \sum_{k=0}^{n} a_{n,k} \sum_{r=0}^{k} b_{k,r} s_r = \sum_{k=0}^{n} \sum_{r=0}^{k} a_{n,k} b_{k,r} s_r \,,\, t_n^{A.B} = \sum_{r=0}^{k} \sum_{k=0}^{n} a_{n,k} b_{k,r} s_r$$

The third characteristic of the matrix method showed that, no matter how different the method used to collect the studied series, we would obtain the same sum for that series. We have demonstrated two theorems. The first speaks of the sum of Fourier series using product matrix methods, and the second speaks of the sum of a Fourier series derivative using a matrix method only.

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