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Oscillation theorems for fractional neutral differential equations

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Abstract

In this paper we study the oscillation of the fractional neutral differential equation

$$D_t^\alpha[a(t)D_t^\alpha(x(t)+p(t)x(\tau(t)))]+q(t)x(\sigma(t))=0,$$

where D_t^α is a modified Riemann-Liouville derivative. The obtained results are based on the new comparison theorems, which enable us to reduce the oscillatory problem of 2α-order fractional differential equation to the oscillation of the first order equation. The results are easily verified.

Keywords: Oscillation; Fractional differential equation; Modified Riemann-Liouville derivative; Comparison theorem

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1. Introduction

In this paper, we shall study the oscillation behavior of a class of fractional neutral differential equations with the form

$$D_t^\alpha[a(t)D_t^\alpha(x(t)+p(t)x(\tau(t)))]+q(t)x(\sigma(t))=0, \quad t\geq t_0>0,\ 0<\alpha<1, \tag{1.1}$$

where D_t^α denotes the modified Riemann-Liouville derivative [1] with respect to the variable t, $q(t)\in C([t_0,+\infty))$, $D_t^\alpha a(t)\in C([t_0,+\infty))$, $D_t^{2\alpha}p(t)\in C([t_0,+\infty))$, and we define $z(t)=x(t)+p(t)x(\tau(t))$. The equation also satisfies that:

(H_1) $a(t)>0,\ q(t)>0,\ 0\leq p(t)\leq p_0<\infty$;
(H_2) $\lim\limits_{t\to+\infty}\tau(t)=+\infty,\ \lim\limits_{t\to+\infty}\sigma(t)=+\infty$;
(H_3) $\tau'(t)\geq\tau_0>0,\ \tau\circ\sigma=\sigma\circ\tau$;
(H_4) $\frac{t}{\tau(t)}\geq l>0$.

In recent years, there has been much research activity concerning the fractional differential equation and many useful achievement have been obtained. Due to the fractional differential equation is more realistic in describing some practical models, it has been used widely in establishing mathematical models in electrochemistry, control, electromagnetic field theories and other natural phenomena and physical problems. Furthermore, it can also provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a “memory" term in the model. Its initial and boundary value problems, stability of solutions, explicit and numerical solutions and many other properties have obtained significant development [2–6]. Particularly, the oscillation of fractional differential equations as a new research field has been received attention, and some interesting results have already been obtained. The relative works we refer to [7–17].

In 2012, Grace *et al.* [7] studied the oscillation theory for fractional differential equations by considering equations of the form

$$D_a^q x+f_1(t,x)=v(t)+f_2(t,x),\ \lim_{t\to a^+}J_a^{1-q}x(t)=b_1,$$

under the conditions

$$xf_i(t,x)>0 \quad \text{for} \quad i=1,2,x\neq0, \quad \text{and} \quad t\geq a,$$

and

$$|f_1(t,x)|>p_1(t)|x|^\beta \quad \text{and} \quad |f_2(t,x)|>p_2(t)|x|^\gamma \quad \text{for} \quad x\neq0, \quad \text{and} \quad t\geq a,$$

where D_a^q denotes the Riemann-Liouville differential operator of order q with $0<q\leq1$, and the operator J_a^p is the Rieman-Liouville fractional integral operator. The authors obtained some new oscillation criteria by reducing the fractional differential equation to the equivalent Volterra fractional integral equation and by applying inequality technique.

In 2012, Chen *et al.* [8] studied the oscillatory behavior of the following fractional differential equation

$$[r(t)(D_-^\alpha y)^\eta(t)]'-q(t)f(\int_t^\infty(v-t)^{-\alpha}y(v)dv)=0 \quad \text{for} \quad t>0,$$

where $D_-^\alpha y$ denotes the Liouville right-sided fractional derivative of order α with the form

$$(D_-^\alpha y)(t):=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty(v-t)^{-\alpha}y(v)dv \quad \text{for} \quad t\in\mathbb{R}_+:=(0,\infty).$$

By the Riccati transformation technique the authors obtained some sufficient conditions, which guarantee that every solution of the equation is oscillatory.

Using the same method, in 2013, Chen [9] studied oscillatory behavior of the fractional differential equation in the form

$$(D_-^{1+\alpha}y)(t) - p(t)(D_-^{\alpha}y)(t) + q(t)f(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv) = 0 \quad \text{for} \quad t>0,$$

where $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of order $\alpha \in (0,1)$ of y.

Zheng [10] considered the oscillation of the nonlinear fractional differential equation with damping term

$$[a(t)(D_-^{\alpha}x(t))^{\gamma}]' + p(t)(D_-^{\alpha}x(t))^{\gamma} - q(t)f(\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi) = 0,\ t \in [t_0,\infty),$$

where $D_-^{\alpha}x(t)$ denotes the Liouville right-sided fractional derivative of order α of x. Using a generalized Riccati function and inequality technique, he established some new oscillation criteria.

Han *et al.* [11] considered the oscillation for a class of fractional differential equation

$$\left[r(t)g\left((D_-^{\alpha}y)(t)\right)\right]' - p(t)f\left(\int_t^{\infty}(s-t)^{-\alpha}y(s)ds\right) = 0, \quad \text{for} \quad t>0,$$

where $0<\alpha<1$ is a real number, $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of order α of y. By generalized Riccati transformation technique, oscillation criteria for the nonlinear fractional differential equation are obtained.

In this paper we focus on the fractional neutral differential equations involving a modified Riemann-Liouville derivative, which is given by Jumarie in [1] (see also in [18–22]). The modified Riemann-Liouville derivative is defined as

$$D_t^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \le \alpha < n+1,\ n \ge 1. \end{cases}$$

And it has some properties that

$$D_t^{\alpha}t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{1.2}$$

$$D_t^{\alpha}(f(t)g(t)) = g(t)D_t^{\alpha}f(t) + f(t)D_t^{\alpha}g(t), \tag{1.3}$$

$$D_t^{\alpha}f[g(t)] = f_g'[g(t)]D_t^{\alpha}g(t) = D_g^{\alpha}f[g(t)](g'(t))^{\alpha}. \tag{1.4}$$

Due to having these especial properties, it can be more appropriately used in studying the oscillatory behavior of the fractional differential equations.

In [12], Feng *et al.* considered the fractional differential equation involving the derivative of this type in the form

$$D_t^{\alpha}[r(t)\psi(x(t))D_t^{\alpha}x(t)] + q(t)f(x(t)) = e(t),\ t \ge t_0 > 0,\ 0<\alpha<1,$$

where $D_t^{\alpha}(\cdot)$ denotes the modified Riemann-Liouville derivative. Based on a transformation of variables and properties of the modified Riemann-Liouville derivative, they transformed the fractional differential equation into a second-order ordinary differential equation. Then by a generalized Riccati transformation, inequalities, and an integration average technique, they established some oscillation criteria for the fractional differential equation.

In [13], Liu *et al.* concerned with oscillation of a class of fractional differential equations under the modified Riemann-Liouville derivative

$$D_t^{\alpha}[a(t)(D_t^{\alpha}(r(t)D_t^{\alpha}x(t)))^{\gamma}] + q(t)f(x(t)) = 0,\ t \ge t_0 > 0,\ 0<\alpha<1,$$

where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative and they put some sufficient conditions about the oscillation of the equation.

Although the oscillation of fractional differential equation has been initiated to study by some authors, to the best of our knowledge very little is known in the literature regarding the oscillatory behavior of fractional neutral differential equations up to now.

Regarding the integer case of our equation (1.1), that is, $\alpha = 1$, B. Baculíková *et al.* in their article [23] have studied the second-order neutral differential equation

$$(1.5) \qquad (r(t)[x(t)+p(t)x(\tau(t))]')' + q(t)x(\sigma(t)) = 0.$$

By comparison theorem, they established some oscillation criteria for the equation (1.5). They proved that: when $\sigma(t) \le t$, if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)R(\sigma(s))ds > \frac{\tau_0 + p_0}{\tau_0 e},$$

where $Q(t) = \min\{q(t), q(\tau(t))\}$, $R(t) = \int_{t_0}^{t} \frac{1}{r(s)} ds$, then (1.5) is oscillatory.

Moreover, in article [24], B. Baculíková *et al.* investigated the oscillation for the nonlinear case. They studied the equation in the form

$$(1.6) \qquad (a(t)[z'(t)]^\gamma)' + q(t)x^\beta(\sigma(t)) = 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$. Also by comparison theorem, they established some sufficiently conditions for the oscillation of equation (1.6).

In this paper we will consider the oscillation of fractional neutral differential equation (1.1). Comparing to the method used by Feng and Liu [12,13], we will reduce a fractional differential equations to an integer one by appropriate variable transforms and establish some new comparison theorems and then use them to reduce the problem of the fractional order differential equation to the problem of second-order differential equations. In order to treat the delay or advance term in our equations, in this paper, we establish some new variable transformations so that the variable transformation method in [12, 13] can be applied for more classes of fractional differential equations, such as fractional neutral differential equations and fractional differential equations with delays. We also extend B. Baculíková and J. Džurina's results to the fractional order differential equations.

We organize this article as follows. In the next section, we give a transformation of variables to the fractional differential equation similar to that in the references [12, 13], and provide a new transformation on account of the delay term. So we can translate our fractional neutral differential equation to a second-order neutral differential equation. In Section 3, we first establish some new comparison theorems and then use them to get some sufficient conditions for oscillation of all solutions of (1.1). At the last we provide some examples to show applications of our criteria.

A solution of the equation is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation is said to be oscillatory if all its solutions are oscillatory.

2. Some preliminary lemmas

First we will use a variable substitution. Denote $\xi = y(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $\xi_i = y(t_i) = \frac{t_i^\alpha}{\Gamma(1+\alpha)}$, $i = 0, 1$, $x(t) = \tilde{x}(\xi)$, $a(t) = \tilde{a}(\xi)$, $p(t) = \tilde{p}(\xi)$, $q(t) = \tilde{q}(\xi)$.

Towards to $\tau(t)$, $\sigma(t)$, we have the next transformations.

2.1. Lemma. *Suppose* (H_3), (H_4) *hold, we define the functions* $\tilde{\tau}(\xi)$, $\tilde{\sigma}(\xi)$ *as the following forms*

$$\tilde{\tau}(\xi) = y(\tau(y^{-1}(\xi))),$$

$$\tilde{\sigma}(\xi) = y(\sigma(y^{-1}(\xi))),$$

then it satisfies

$$x(\tau(t)) = \tilde{x}(\tilde{\tau}(\xi)),\ x(\sigma(t)) = \tilde{x}(\tilde{\sigma}(\xi));$$

and a new condition

$$(H_3'): \quad \tilde{\tau}'(\xi) \geq \tau_0 l^{1-\alpha} = \tilde{\tau}_0 > 0,\ \tilde{\tau} \circ \tilde{\sigma} = \tilde{\sigma} \circ \tilde{\tau}.$$

Proof. From the defines of $\tilde{\tau}$, $\tilde{\sigma}$ we get

$$\tilde{x}(\tilde{\tau}(\xi)) = \tilde{x}(y(\tau(y^{-1}(\xi)))) = \tilde{x}(y(\tau(t))).$$

Due to

$$x(t) = \tilde{x}(\xi) = \tilde{x}(y(t)),$$

substituting t with $\tau(t)$ we get

$$\tilde{x}(y(\tau(t))) = x(\tau(t)).$$

Thus

$$\tilde{x}(\tilde{\tau}(\xi)) = x(\tau(t)).$$

The same is

$$\tilde{x}(\tilde{\sigma}(\xi)) = x(\sigma(t)).$$

On the other hand, from H_3, H_4 and the defines of $\tilde{\tau}$ we get

$$\tilde{\tau}\circ\tilde{\sigma} = y(\tau(y^{-1}(\tilde{\sigma}(\xi)))) = y(\tau(y^{-1}(y(\sigma(y^{-1}(\xi)))))) = y(\tau(\sigma(y^{-1}(\xi)))) = y(\sigma(\tau(y^{-1}(\xi)))) = \tilde{\sigma}\circ\tilde{\tau}.$$

Also we have that,

$$\begin{aligned}
\tilde{\tau}'(\xi) = \frac{\partial}{\partial \xi} y(\tau(y^{-1}(\xi))) \quad &= \tfrac{\partial y(\tau(y^{-1}(\xi)))}{\partial \tau(y^{-1}(\xi))} \times \tfrac{\partial \tau(y^{-1}(\xi))}{\partial y^{-1}(\xi)} \times \tfrac{\partial y^{-1}(\xi)}{\partial \xi} \\
&= \tfrac{\partial y(\tau(t))}{\partial \tau(t)} \times \tfrac{\partial \tau(t)}{\partial t} \times \tfrac{\partial y^{-1}(\xi)}{\partial \xi} \\
&\geq \tfrac{\alpha(\tau(t))^{\alpha-1}}{\Gamma(1+\alpha)} \times \tau_0 \times \tfrac{1}{\alpha}(\Gamma(1+\alpha))^{\frac{1}{\alpha}} \xi^{\frac{1}{\alpha}-1} \\
&= \tfrac{\alpha(\tau(t))^{\alpha-1}}{\Gamma(1+\alpha)} \times \tau_0 \times \tfrac{1}{\alpha}(\Gamma(1+\alpha))^{\frac{1}{\alpha}} \left(\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right)^{\frac{1}{\alpha}-1} \\
&= \tau_0 \left(\tfrac{t}{\tau(t)}\right)^{1-\alpha} \\
&\geq \tau_0 l^{1-\alpha} = \tilde{\tau}_0.
\end{aligned}$$

The proof is complete.

2.2. Lemma. *If x(t) is a eventually positive solution of* (1.1)*, and a sufficient large* t_1 *such that*

$$R(t) = \int_{t_1}^{t} \frac{1}{a(s)} ds \to +\infty \qquad \text{as } t \to +\infty, \tag{2.1}$$

then the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ *satisfies*

$$z(t) > 0, \qquad a(t) D_t^\alpha(z(t)) > 0, \qquad D_t^\alpha[a(t) D_t^\alpha(z(t))] < 0,$$

eventually.

Proof. Let $x(t) = \tilde{x}(\xi)$, where $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$. Then from (1.2) we get $D_t^\alpha \xi(t) = 1$, and furthermore by use of (1.4) and Lemma 2.1 we have

$$D_t^\alpha x(t) = D_t^\alpha \tilde{x}(\xi) = \tilde{x}'(\xi) D_t^\alpha \xi(t) = \tilde{x}'(\xi),$$
$$D_t^\alpha x(\tau(t)) = D_t^\alpha \tilde{x}(\tilde{\tau}(\xi)) = (\tilde{x}(\tilde{\tau}(\xi)))' D_t^\alpha \xi(t) = (\tilde{x}(\tilde{\tau}(\xi)))'.$$

Similarly we have $D_t^\alpha a(t) = \tilde{a}'(\xi)$, $D_t^\alpha p(t) = \tilde{p}'(\xi)$, $D_t^\alpha q(t) = \tilde{q}'(\xi)$ and $D_t^\alpha x(\sigma(t)) = (\tilde{x}(\tilde{\sigma}(\xi)))'$. Then we get $D_t^\alpha z(t) = (\tilde{x}(\xi) + \tilde{p}(\xi)\tilde{x}(\tilde{\tau}(\xi)))'$. We define $\tilde{z}(\xi) = \tilde{x}(\xi) + \tilde{p}(\xi)\tilde{x}(\tilde{\tau}(\xi))$, then $D_t^\alpha z(t) = \tilde{z}'(\xi)$. So the equation (1.1) can be transformed into the following form:

$$(\tilde{a}(\xi)\tilde{z}'(\xi))' + \tilde{q}(\xi)\tilde{x}(\tilde{\sigma}(\xi)) = 0, \qquad \xi \geq \xi_0 > 0. \tag{2.2}$$

Since $x(t)$ is an eventually positive solution of (1.1), $\tilde{x}(\xi)$ is an eventually positive solution of (2.2). Hence there exists $\xi_1 > \xi_0$ such that $\tilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. Also we know $\tilde{z}(\xi) > 0$ on $[\xi_1, \infty)$. It follows from (2.2) that

$$(\tilde{a}(\xi)\tilde{z}'(\xi))' = -\tilde{q}(\xi)\tilde{x}(\tilde{\sigma}(\xi)) < 0,$$

holds eventually. Consequently, $\tilde{a}(\xi)\tilde{z}'(\xi)$ is decreasing and thus either $\tilde{z}'(\xi) > 0$ or $\tilde{z}'(\xi) < 0$ eventually. We claim $\tilde{z}'(\xi) > 0$. Otherwise if $\tilde{z}'(\xi) < 0$, then also $\tilde{a}(\xi)\tilde{z}'(\xi) < -c < 0$ and integrating this from ξ_1 to ξ, we have

$$\tilde{z}(\xi) \leq \tilde{z}(\xi_1) - c\int_{\xi_1}^{\xi} \frac{1}{\tilde{a}(s)}ds = \tilde{z}(\xi_1) - c\int_{t_1}^{t} \frac{1}{a(s)}ds \to -\infty \quad \text{as} \quad t \to +\infty.$$

This contradicts the positivity of $\tilde{z}(\xi)$ and the proof is complete.

3. Main results

To simplify our notation, let us denote

$$Q(\xi) = \min\{\tilde{q}(\xi), \tilde{q}(\tilde{\tau}(\xi))\}, \quad Q^*(\xi) = Q(\xi)\int_{\xi_1}^{\xi} \frac{1}{\tilde{a}(s)}ds, \tag{3.1}$$

where ξ_1 is defined in Lemma 2.2.

3.1. Theorem. *If the first order neutral differential inequality*

$$(u(t) + \frac{p_0}{\tilde{\tau}_0}u(\tilde{\tau}(t)))' + Q^*(t)u(\tilde{\sigma}(t)) \leq 0, \qquad t \geq \xi_1 = \frac{t_1^\alpha}{\Gamma(1+\alpha)}, \tag{3.2}$$

where $\tilde{\tau}(t)$ is defined in Lemma 2.1, $Q^*(t)$ is defined in (3.1)*, has no positive solution, then* (1.1) *is oscillatory.*

Proof. Assume to the contrary that there exists a non-oscillatory solution x of equation (1.1). Without loss of generality, we only consider the case when $x(t)$ is eventually positive, since the case when $x(t)$ is eventually negative is similar. Then let $x(t) > 0$ on $[t_1, \infty)$. It is equivalent to $\tilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. Then from (H_1) and (H_3') the corresponding function $\tilde{z}(\xi)$ satisfies

$$\begin{aligned} \tilde{z}(\tilde{\sigma}(\xi)) &= \tilde{x}(\tilde{\sigma}(\xi)) + \tilde{p}(\tilde{\sigma}(\xi))\tilde{x}(\tilde{\tau}(\tilde{\sigma}(\xi))) \\ &\leq \tilde{x}(\tilde{\sigma}(\xi)) + p_0\tilde{x}(\tilde{\sigma}(\tilde{\tau}(\xi))). \end{aligned} \tag{3.3}$$

On the other hand from (2.2) we have

$$0 = (\tilde{a}(\xi)\tilde{z}'(\xi))' + \tilde{q}(\xi)\tilde{x}(\tilde{\sigma}(\xi)), \tag{3.4}$$

which in view of (H_1) and (H_3') yields

$$\begin{aligned} 0 &= \tfrac{p_0}{\tilde{\tau}'(\xi)}(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi)))' + p_0\tilde{q}(\tilde{\tau}(\xi))\tilde{x}(\tilde{\sigma}(\tilde{\tau}(\xi))) \\ &\geq \tfrac{p_0}{\tilde{\tau}_0}(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi)))' + p_0\tilde{q}(\tilde{\tau}(\xi))\tilde{x}(\tilde{\sigma}(\tilde{\tau}(\xi))). \end{aligned} \tag{3.5}$$

Then combining (3.4) and (3.5) we get

$$(\tilde{a}(\xi)\tilde{z}'(\xi))' + \tilde{q}(\xi)\tilde{x}(\tilde{\sigma}(\xi)) + \frac{p_0}{\tilde{\tau}_0}(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi)))' + p_0\tilde{q}(\tilde{\tau}(\xi))\tilde{x}(\tilde{\sigma}(\tilde{\tau}(\xi))) \leq 0. \tag{3.6}$$

Furthermore using (3.1) and (3.3) we obtain

$$(3.7)\qquad (\tilde{a}(\xi)\tilde{z}'(\xi))' + \frac{p_0}{\tilde{\tau}_0}(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi)))' + Q(\xi)\tilde{z}(\tilde{\sigma}(\xi)) \leq 0,$$

where $Q(\xi)$ is defined in (3.1). Now we denote $u(\xi) = \tilde{a}(\xi)\tilde{z}'(\xi)$. From Lemma 2.2 we get $u(\xi) > 0$ eventually. Also we have

$$(3.8)\qquad \tilde{z}(\xi) \geq \int_{\xi_1}^{\xi} \frac{\tilde{a}(s)\tilde{z}'(s)}{\tilde{a}(s)} ds \geq \tilde{a}(\xi)\tilde{z}'(\xi) \int_{\xi_1}^{\xi} \frac{1}{\tilde{a}(s)} ds = u(\xi) \int_{\xi_1}^{\xi} \frac{1}{\tilde{a}(s)} ds.$$

Then taking (3.8) into (3.7) we get that $u(\xi)$ is a positive solution of

$$(u(\xi) + \frac{p_0}{\tilde{\tau}_0} u(\tilde{\tau}(\xi)))' + Q^*(\xi)u(\tilde{\sigma}(\xi)) \leq 0,$$

which is a contradiction and the proof is complete.

Next, by using the conclusion of Theorem 3.1, we will deduce oscillatory problem of our equation into the problem of first-order nonlinear delay differential equations, and establish some new oscillatory criteria for equation (1.1). We shall discuss both cases when τ is a delayed or advanced argument.

3.2. Theorem. *Assume that $\tau(t) \geq t$ and $\sigma(t) \leq t$ is increasing. Assumptions $(H_1) - (H_4)$ hold. Then if the first-order delay differential equation*

$$(3.9)\qquad w'(\xi) + \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0} Q^*(\xi)w(\tilde{\sigma}(\xi)) = 0$$

is oscillatory, the equation (1.1) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (1.1) eventually. Then it follows from Lemma 2.2 and the proof of Theorem 3.1 that $u(\xi) = \tilde{a}(\xi)\tilde{z}'(\xi) > 0$ is decreasing eventually and satisfies (3.2). We define

$$(3.10)\qquad w(\xi) = u(\xi) + \frac{p_0}{\tilde{\tau}_0} u(\tilde{\tau}(\xi)).$$

From the definition of $\tilde{\tau}(\xi)$ and $\tau(t) \geq t$, we can easily get that $\tilde{\tau}(\xi) \geq \xi$. Similarly we have $\tilde{\sigma}(\xi) \leq \xi$. Then

$$w(\xi) \leq u(\xi)(1 + \frac{p_0}{\tilde{\tau}_0}),$$

$$\frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0} w(\xi) \leq u(\xi).$$

Substituting this into (3.2), we get that $w(\xi)$ is the positive solution of the delay differential inequality

$$(3.11)\qquad w'(\xi) + \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0} Q^*(\xi)w(\tilde{\sigma}(\xi)) \leq 0.$$

Then from [25, Theorem 1] we know that the equation (3.9) also has a positive solution, which is a contradiction. The proof is complete.

3.3. Theorem. *Assume that $\tau(t) \leq t$ and $\sigma(t) \leq \tau(t) \leq t$. Conditions $(H_1) - (H_4)$ hold. Then if the first-order delay differential equation*

$$(3.12)\qquad w'(\xi) + \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0} Q^*(\xi)w(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))) = 0$$

is oscillatory, the equation (1.1) *is oscillatory.*

Proof. We assume that $x(t)$ is a positive solution of (1.1) eventually. Then it follows from Lemma 2.2 and the proof of Theorem 3.1 that $u(\xi) = \tilde{a}(\xi)\tilde{z}'(\xi) > 0$ is decreasing eventually and satisfies (3.2). Also from Lemma 2.1 we have

$$\tilde{\sigma}(\xi) \leq \tilde{\tau}(\xi) \leq \xi.$$

Then it follows from (3.10) that

$$w(\xi) \leq u(\tilde{\tau}(\xi))(1 + \frac{p_0}{\tilde{\tau}_0}),$$

which is equivalent to

$$u(\tilde{\sigma}(\xi)) \geq \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0} w(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))).$$

Substituting this into (3.2), we obtain that $w(\xi)$ is a positive solution of the delay differential inequality

$$w'(\xi) + \frac{\tilde{\tau}_0}{\tilde{\tau}_0 + p_0} Q^*(\xi) w(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))) \leq 0.$$

Then from [25, Theorem 1] we know that the equation (3.12) also has a positive solution, and a contradiction. The proof is complete.

Next we will give some sufficient conditions such that equations (3.9) and (3.12) have only oscillatory solutions.

3.4. Lemma. *Assume that $e(\xi)$ is a positive continuous function on $[\xi_0, \infty)$. If*

$$\lim_{\xi\to\infty} \inf \int_{\tilde{\sigma}(\xi)}^{\xi} e(s)ds > \frac{1}{e}, \tag{3.13}$$

then the first-order delay differential equation

$$w'(\xi) + e(\xi)w(\tilde{\sigma}(\xi)) = 0 \tag{3.14}$$

is oscillatory.

Proof. From (3.13) we can get that

$$\int_{\xi_0}^{\infty} e(s)ds = +\infty. \tag{3.15}$$

Then assume to the contrary that there exists a positive solution $w(\xi)$ of equation (3.14) on $[\xi_1, \infty)$. Since $w(\xi)$ is decreasing, there exists $\lim\limits_{\xi\to+\infty} w(\xi) = k \geq 0$. If $k > 0$, then integrating (3.14) from t_1 to t. We have

$$w(\xi_1) \geq \int_{\xi_1}^{\xi} e(s)w(\tilde{\sigma}(s))ds \geq k \int_{\xi_1}^{\xi} e(s)ds \to +\infty \qquad \text{as} \quad \xi \to +\infty.$$

This is a contradiction. So we get that $\lim\limits_{\xi\to+\infty} w(\xi) = 0$. But from the Theorem 2.1.1 in [26], the condition (3.13) yields that the equation (3.14) has no positive solution, which is a contradiction. The proof is complete.

3.5. Theorem. *Let $\tau(t) \geq t$ and $\sigma(t) \leq t$. Conditions $(H_1) - (H_4)$ hold. If*

$$\lim_{\xi\to\infty} \inf \int_{\tilde{\sigma}(\xi)}^{\xi} Q^*(s)ds > \frac{\tilde{\tau}_0 + p_0}{\tilde{\tau}_0 e}, \tag{3.16}$$

then (1.1) *is oscillatory.*

Proof. From the condition (3.16) and Lemma 3.4 we get that equation (3.9) is oscillatory. Then from Theorem 3.2 we have equation (1.1) is oscillatory, the proof is complete.

3.6. Theorem. *Let $\sigma(t) \le \tau(t) \le t$ and conditions $(H_1)-(H_4)$ hold. If*

$$\lim_{\xi\to\infty}\inf\int_{\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))}^{\xi} Q^*(s)ds > \frac{\tilde{\tau}_0+p_0}{\tilde{\tau}_0 e}, \tag{3.17}$$

then (1.1) *is oscillatory.*

Proof. The proof is similar to the proof of Theorem 3.5.

4. Examples

In this section, we will show the application of our main results.

Example 4.1 Consider the fractional differential equation

$$D_t^{\frac{1}{2}}[\sqrt{t}D_t^{\frac{1}{2}}(x(t)+\frac{1}{t}x(t+3))]+tx(t-5)=0, \qquad t\in[5,+\infty), \tag{4.1}$$

where $D_t^{\alpha}x(t)$ is the modified Riemann-Liouville differential operator. In (4.1), we set $a(t)=\sqrt{t}$, $p(t)=\frac{1}{t}$, $\tau(t)=t+3$, $q(t)=t$, $\sigma(t)=t-5$. Then using a variable substitution we have

$$\xi=y(t)=\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}, \quad y^{-1}(\xi)=\Gamma^2(\frac{3}{2})\xi^2, \quad \xi_1=\frac{\sqrt{5}}{\Gamma(\frac{3}{2})}.$$

And we also have

$$\tilde{a}(\xi)=a(y^{-1}(\xi))=\Gamma(\frac{3}{2})\xi,$$

$$\tilde{\sigma}(\xi)=y(\sigma(y^{-1}(\xi)))=\frac{(\Gamma^2(\frac{3}{2})\xi^2-5)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}=(\xi^2-\frac{5}{\Gamma^2(\frac{3}{2})})^{\frac{1}{2}},$$

$$\tilde{q}(\xi)=q(y^{-1}(\xi))=\Gamma^2(\frac{3}{2})\xi^2.$$

Easily we see the equation (4.1) satisfies $(H_1)-(H_4)$, furthermore we have

$$\begin{cases} 0\le p(t)=\frac{1}{t}\le\frac{1}{5}=p_0, \\ \tau_0=(t+3)'=1, \\ \lim\limits_{t\to\infty}\frac{t}{\tau(t)}=\frac{t}{t+3}=l=1, \\ \tilde{\tau}_0=\tau_0 l^{1-\frac{1}{2}}=1. \end{cases}$$

We know $\tilde{q}(\xi)$ is increasing and $\tau(t)>t,\ \tilde{\tau}(\xi)>\xi$, so

$$Q(\xi)=\tilde{q}(\xi)=\Gamma^2(\frac{3}{2})\xi^2,$$

$$\begin{aligned} Q^*(\xi) &= \Gamma^2(\tfrac{3}{2})\xi^2\int_{\xi_1}^{\xi}\tfrac{1}{\Gamma(\frac{3}{2})s}ds \\ &= \Gamma^2(\tfrac{3}{2})\xi^2(\tfrac{1}{\Gamma(\frac{3}{2})}\ln\xi-\tfrac{1}{\Gamma(\frac{3}{2})}\ln\xi_1). \end{aligned}$$

Then we have

$$\lim_{\xi\to\infty}\inf\int_{\tilde{\sigma}(\xi)}^{\xi} Q^*(s)ds$$

$$=\lim_{\xi\to\infty}\inf\int_{\tilde{\sigma}(\xi)}^{\xi}\Gamma(\tfrac{3}{2})(s^2\ln s-s^2\ln m)ds$$

$$=\lim_{\xi\to\infty}\inf[\tfrac{\Gamma(\frac{3}{2})}{3}(\xi^3\ln\tfrac{\xi}{m}-\tilde{\sigma}^3(\xi)\ln\tfrac{\tilde{\sigma}(\xi)}{m})-\tfrac{5}{9\Gamma(\frac{3}{2})}]$$

$$\geq\lim_{\xi\to\infty}\inf[\tfrac{\Gamma(\frac{3}{2})}{3}(\xi^3-\tilde{\sigma}^3(\xi))\ln\tfrac{\tilde{\sigma}(\xi)}{m})-\tfrac{5}{9\Gamma(\frac{3}{2})}]$$

$$\geq\lim_{\xi\to\infty}\inf[\tfrac{\Gamma(\frac{3}{2})}{3}(\xi^2-\tilde{\sigma}^2(\xi))\ln\tfrac{\tilde{\sigma}(\xi)}{m}-\tfrac{5}{9\Gamma(\frac{3}{2})}]$$

$$=\geq\lim_{\xi\to\infty}\inf[\tfrac{\Gamma(\frac{3}{2})}{3}\tfrac{5}{\Gamma^2(\frac{3}{2})}\ln\tfrac{\tilde{\sigma}(\xi)}{m}-\tfrac{5}{9\Gamma(\frac{3}{2})}]$$

$$=\infty>\tfrac{1+\frac{1}{5}}{e},$$

where $m=\xi_1=\frac{5^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}$. From Theorem 3.5 we get that (4.1) is oscillatory.

Example 4.2 Consider the fractional differential equation

$$D_t^{\frac{1}{3}}[tD_t^{\frac{1}{3}}(x(t)+2x(\frac{t}{2}))]+tx(\frac{t}{8})=0,\qquad t\in[1,+\infty), \tag{4.2}$$

where $D_t^{\alpha}x(t)$ is the modified Riemann-Liouville differential operator. In (4.2), we set $a(t)=t,\ p(t)=2,\ q(t)=t,\ \tau(t)=\frac{t}{2},\ \sigma(t)=\frac{t}{8}$. Then using a variable substitution we have

$$\xi=y(t)=\frac{t^{\frac{1}{3}}}{\Gamma(\frac{4}{3})},\quad y^{-1}(\xi)=\Gamma^3(\frac{4}{3})\xi^3,\quad \xi_1=\frac{1}{\Gamma(\frac{4}{3})}.$$

Then we get

$$\tilde{a}(\xi)=a(y^{-1}(\xi))=\Gamma^3(\frac{4}{3})\xi^3,$$

$$\tilde{\sigma}(\xi)=y(\sigma(y^{-1}(\xi)))=y(\frac{\Gamma^3(\frac{4}{3})\xi^3}{8})=\frac{\xi}{2},$$

$$\tilde{q}(\xi)=q(y^{-1}(\xi))=\Gamma^3(\frac{4}{3})\xi^3,$$

$$\tilde{\tau}(\xi)=y(\tau(y^{-1}(\xi)))=y(\frac{\Gamma^3(\frac{4}{3})\xi^3}{2})=\frac{\xi}{2^{\frac{1}{3}}},$$

$$\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))=2^{\frac{1}{3}}\tilde{\sigma}(\xi)=\frac{\xi}{2^{\frac{2}{3}}}.$$

Easily we see the equation (4.2) satisfies $(H_1)-(H_4)$, and

$$\begin{cases}0\leq p(t)=2=p_0,\\ \tau_0=(\frac{t}{2})'=\frac{1}{2},\\ \lim\limits_{t\to\infty}\frac{t}{\tau(t)}=\frac{t}{\frac{t}{2}}=2=l,\\ \tilde{\tau}_0=\tau_0 l^{1-\frac{1}{3}}=2^{-\frac{1}{3}}.\end{cases}$$

In this time $\tilde{q}(\xi)$ is increasing and $\tau(t)<t,\ \tilde{\tau}(\xi)<\xi$, so

$$Q(\xi)=\tilde{q}(\tilde{\sigma}(\xi))=\Gamma^3(\frac{4}{3})\frac{\xi^3}{2},$$

$$Q^*(\xi) \quad = \Gamma^3(\tfrac{4}{3})\tfrac{\xi^3}{2}\int_{\xi_1}^{\xi} \tfrac{1}{\Gamma^3(\frac{4}{3})s}ds$$

$$= \tfrac{\xi^3}{2}(\ln\xi - \ln\xi_1).$$

Following from (3.17) we have

$$\lim_{\xi\to\infty}\inf\int_{\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))}^{\xi} Q^*(s)ds$$

$$= \lim_{\xi\to\infty}\inf\int_{\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))}^{\xi} \tfrac{s^3}{2}(\ln s - \ln s_1)ds$$

$$= \lim_{\xi\to\infty}\inf[\tfrac{1}{8}\xi^4(\ln\xi - \tfrac{1}{4} - \ln\xi_1) - \tfrac{1}{8}\cdot\tfrac{1}{2^{\frac{8}{3}}}t^4(\ln\xi - \tfrac{2}{3}\ln 2 - \tfrac{1}{4} - \ln\xi_1)]$$

$$= \infty > \tfrac{2^{-\frac{1}{3}}+2}{2^{-\frac{1}{3}}e}.$$

According to Theorem 3.6 we get that (4.2) is oscillatory.

5. Conclusion

We have established some new oscillation criteria for a fractional neutral differential equation. First we can see, the variable transformation used in ξ is very important, transforms a fractional differential equation into an ordinary differential equation of integer order. Then toward to this differential equation with neutral term, we solve it by the comparison theorem, such that we can judge whether its solutions oscillatory by investigating some first-order delay differential equations. And some classical results can be used easily. Finally, we note that the oscillation for other fractional differential equations possessing the modified Riemann-Liouville derivative can also be used this method.

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