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Some statistical cluster point theorems

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Abstract

In this paper, we present results related to sets of statistical limit points and cluster points of sequences and their matrix transformations, single and double sequences and stretchings of sequences.

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1. Introduction

In real analysis there are many characterization theorems of the type- if $f : R \longrightarrow R$, then F, the set of discontinuities of f, is a closed set and conversely, if F is any closed subset of R then there exists a function $f : R \longrightarrow R$, whose set of discontinuities is precisely F.

R.C. Buck ([1] , [2] , [3]), in a series of articles considered similar questions concerning subsequential limit points of a given sequence. Pratulananda Das [5], at the suggestion of Brian Thompson, continued the inquiries of Buck [1].Our initial result showed that if (x_n) is a bounded sequence of reals, having L as its set of (subsequential) limit points, and if M, $M \neq \emptyset$, is any closed subset of L, then there is a subsequence (x_{n_k}) of (x_n), whose set of (subsequential) limit points is precisely M. Fortunately Cihan Orhan pointed out to us, that in fact, we had rediscovered a known result, pointing us to Theorem 1.62 II, on page 142 in Cooke [4].

Here we consider statistical cluster point analogues of the result mentioned above. Statistical limit points and statistical cluster points were first considered by Fridy in [6]. Our results are concerned with single sequences as well as with double sequences. Similarities between our results and core theorems (for example [10], [12]) will be apparent

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to the reader. Also we present a result about stretchings of sequences. Miller and Patterson have previously had stretching results.

2. Preliminaries

If K is a subset of the positive integers $\mathbb{N}$, then following Fridy [6], K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density of K (see [4] , Chapter 11) is given by $\delta(K) = \lim_{n\to\infty} n^{-1}|K_n|$. In the case that $\delta(K) = 0$ we say that K is thin, and otherwise we say that K is non-thin. We continue following Fridy [6] .

2.1. Definition. We say that a number λ is a statistical limit point of a sequence of reals (x_n) if $\lim_{k\to\infty} x_{n_k} = \lambda$ for some non-thin subsequence (x_{n_k}) of (x_n).

2.2. Definition. Given a sequence of reals (x_n), a stretching of that sequence is any sequence of the form $x_1, x_1, \ldots x_1, x_2, x_2 \ldots, x_2, \ldots, x_n, x_n, \ldots, x_n \ldots$.

2.3. Definition. We say that a number γ is a statistical cluster point of a sequence of reals (x_n) if for every $\epsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \gamma| < \epsilon\}$ is non-thin.

We also consider bounded double sequences $x = (x_{n,k})$ and 4-dimensional bounded regular matrix transformations.

2.4. Definition. A double sequence $x = (x_{n,k})$ of reals is said to be bounded if there exists an M such that $|x_{n,k}| < M$ for all n, k.

2.5. Definition. A double sequence $x = (x_{n,k})$ has Pringsheim limit L (or in what follows, just limit L) denoted by $\lim x_{n,k} = L$, if given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{n,k} - L| < \epsilon$ whenever $n, k > \mathbb{N}$. Briefly, we say that x is convergent and has limit L.

Let $A = (a_{s,t,n,k})$, denote a four dimensional summability method (see [8]) that maps the double real sequence x into the double real sequence Ax where the s,t-th term of Ax is defined as follows:

$$(Ax)_{s,t} = \sum_{n,k=1,1}^{\infty,\infty} a_{s,t,n,k} x_{n,k}$$

and is called an A-mean. For the above definition and for what follows see Móricz [11].

We say that a double sequence is A-summable to the limit L if the A-means exist for all $s, t = 1, 2, 3 \ldots$ and

$$\lim_{s,t} (Ax)_{s,t} = L.$$

2.6. Definition. The four dimensional real matrix A is said to be bounded regular if every bounded convergent double sequence with real entries x is A-summable to the same limit and the A-means are also bounded.

Finally, a classical theorem, ([8], [13]) characterizes bounded regular four dimensional matrices.

2.7. Theorem. Necessary and sufficient conditions for $A = (a_{s,t,n,k})$ to be bounded regular are

$\lim_{s,t} a_{s,t,n,k} = 0$ for each n and k
$\lim_{s,t} \sum_{n.k=1,1}^{\infty,\infty} a_{s,t,n,k} = 1$
$\lim_{s,t} \sum_{n=1}^{\infty} |a_{s,t,n,k}| = 0$
$\lim_{s,t} \sum_{k=1}^{\infty} |a_{s,t,n,k}| = 0$
$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{s,t,n,k}|$ is convergent; and
there exist positive integers A and B such that $\sum_{n,k>B} |a_{s,t,n,k}| < A$ for each s, t.

3. Results

Our first result is the statistical cluster point analogue of the result (see [4]) mentioned in our introduction.

3.1. Theorem. Suppose $x = (x_n)$ is a bounded sequence and L is the set of limit points of x. If $M \subseteq L$, M is closed and nonempty, there exists a subsequence $y = (y_n)$ of x such that M is the set of statistical cluster points of y.

Proof. Since M is closed and separable, there is a countable subset of M, $\{a_m : m \in \mathbb{N}\}$ such that its closure is M. Now for $m \in \mathbb{N}$, fix a subsequence of (x_n), $(x_{n_{k,m}})_{k=1}^{\infty})$, converging to a_m and contained in $(a_m - \frac{1}{m}, a_m + \frac{1}{m})$.

We construct $y = (y_n)$ as follows:

$y_1, y_3, y_5, \dots, y_{2j+1}, \dots$ will be chosen from $(x_{n_{k,1}})$,
$y_2, y_6, y_{10}, \dots, y_{2(2j+1)}, \dots$ will be chosen from $(x_{n_{k,2}})$,
$y_4, y_{12}, y_{20}, \dots, y_{4(2j+1)}, \dots$ will be chosen from $(x_{n_{k,3}})$,
$\dots$
$y_{2^{m-1}}, y_{2^{m-1}.3}, y_{2^{m-1}.5}, \dots, y_{2^{m-1}.(2j+1)}, \dots$
will be chosen from $(x_{n_{k,m}})$,
$\dots$

where

$y_1 = x_{n_{1,1}}$, and
$y_2 = x_{n_{k,2}}$ where k is the smallest number so that $n_{k,2} > n_{1,1}$
if $y_1, y_2 \dots y_{i-1}$ have been chosen, and $i = 2^{m-1}(2j+1)$ we choose $y_i = x_{n_{k,m}}$ so that k is the smallest number such that the index $n_{k,m}$ is bigger than the indices of $y_1, y_2 \dots y_{i-1}$ in terms of x.

Hence (y_n) is a subsequence of x. Also $(y_{2^{m-1}.(2j+1)})_{j=0}^{\infty}$ has density $\frac{1}{2^m}$ in (y_n) so a_m is a statistical limit point (and cluster point) of (y_n). Also it is clear that every $a \in M$ is a statistical cluster point of (y_n). Likewise for every $a \in R \setminus M$, there is a sufficiently small neighborhood around it that is disjoint from (y_n) (since M is closed), so the set M is precisely the set of statistical cluster points of (y_n). □

3.2. Corollary. If $x = (x_n)$ is bounded and $M \subset L$, M closed and nonempty, where L is the limit point set of (x_n), then there exists a regular summability method A such that M is the set of statistical cluster points of (Ax).

Proof. Suppose that (x_{n_k}) is the subsequence of (x_n) with M as the set of its statistical cluster points from Theorem 3.1. If $A = (a_{km})$ has entries $a_{kn_k} = 1$ for all k and $a_{km} = 0$ otherwise, then $(Ax) = (x_{n_k})$ and the corollary follows. □

Next, we show the analogous result for stretchings of sequences.

3.3. Theorem. Suppose $x = (x_n)$ is a bounded sequence and L is the set of limit points of x. If $M \subseteq L$, M is closed and nonempty, there exists a stretching of x, y such that M is the set of statistical cluster points of y.

Proof. Suppose that (x_{n_k}) is the subsequence of (x_n) with M as the set of its statistical cluster points constructed in the proof of Theorem 3.1. We construct the following stretching of the sequence (x_n):

$x_1, x_2, x_3, \dots, x_{n_1-1}$ remain as before,
$x_{n_1}, x_{n_1}, x_{n_1}, \dots, x_{n_1}, \dots$ will be repeated $2n_2$ times, followed by
$x_{n_1+1}, x_{n_1+2}, x_{n_1+3}, \dots, x_{n_2-1}$,
$x_{n_2}, x_{n_2}, x_{n_2}, \dots, x_{n_2}, \dots$ will be repeated $4n_3$ times, followed by

$x_{n_2+1}, x_{n_2+2}, x_{n_2+3}, \dots, x_{n_3-1}$
$\dots$
$x_{n_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}, \dots, x_{n_{k-1}}, \dots$ will be repeated $2^{k-1} \cdot n_k$ times, followed by
$x_{n_{k-1}+1}, x_{n_{k-1}+2}, x_{n_{k-1}+3}, \dots, x_{n_k-1}$,
$x_{n_k}, x_{n_k}, x_{n_k}, \dots, x_{n_k}, \dots$ will be repeated $2^k \cdot n_{k+1}$ times, ,
$\dots$

It is easy to check that the part of the new sequence that was not stretched (the odd numbered rows above) is thin (has density 0). Also since $2 \cdot n_2 < 4 \cdot n_3 < \dots < 2^{k-1} \cdot n_k < 2^k \cdot n_{k+1} < \dots$ we see that M is still the set of all statistical cluster points of the stretched part of the new sequence (even rows) and consequently of the whole new sequence (stretching).

□

Now, as mentioned in the introduction, we consider the two-dimensional analogue of our corollary. First we mention that we say l is a limit point of $(x_{n,k})$ if there exist $n_j \to \infty$ and $k_j \to \infty$ such that $\lim_{j\to\infty} x_{n_j,k_j} = l$ and we notice that L the set of limit points of $(x_{n,k})$ is always closed.

3.4. Theorem. If $x = (x_{n,k})$ is a bounded double sequence, and M is a closed nonempty subset of L, the set of limit points of x, then there exists a four dimensional bounded regular matrix transformation A of double sequences such that the set of limit points of Ax is exactly M.

Proof. Without loss of generality, assume that x is contained in the interval $[0,1]$. We define

$I^1 = [0, \frac{1}{2}]$, $I^2 = [\frac{1}{2}, 1]$;
$I^3 = [0, \frac{1}{4}]$, $I^4 = [\frac{1}{4}, \frac{1}{2}]$, $I^5 = [\frac{1}{2}, \frac{3}{4}]$, $I^6 = [\frac{3}{4}, 1]$;
$I^7 = [0, \frac{1}{8}]$, $I^8 = [\frac{1}{8}, \frac{1}{4}]$, etc.
and so on....

Let (s_i) be the sequence of integers satisfying $I^{s_i} \bigcap M \neq \emptyset$. For each i, pick a $y_i \in I^{s_i} \bigcap M$. Since $y_i \in M \subseteq L$, for each i, there exists $u_i \to \infty$, $v_i \to \infty$, such that

$$|y_i - x_{u_i,v_i}| < \frac{1}{i}.$$

Now we define the required matrix $A = (a_{m,n,u,v})$. For any m, n, if $m + n = i$ $(i = 2, 3, \dots)$, let $a_{m,n,u_i,v_i} = 1$ for $(i = 2, 3, \dots)$, and $a_{m,n,u,v} = 0$ otherwise. It is easy to see that A satisfies the conditions in Theorem 2.7 and that the set of limit points of Ax is exactly M. □

References

[1] R.C. Buck, *A note on subsequences*, Bull.Amer.Math.Soc., 49(1943), 898-899.
[2] R.C. Buck, *Limit points of subsequences*, Bull.Amer.Math.Soc., 50(1944), 395-397.
[3] R.C. Buck, H.Pollard, *Convergence and summability properties of subsequences*, Bull.Amer.Math.Soc., 49(1943), 924-931.
[4] R.G. Cooke, *Infinite matrices and sequence spaces*, Macmillan, 1950.
[5] P. Das, oral communication
[6] Fridy, J.A.,*On statistical convergence*, Analysis, 5 (4)(1985), 301-313.
[7] Fridy, J.A., Orhan, C. *Statistical core theorems*, J.Math.Anal. and Appl., Vol. 208, 2(1997), 520-527.
[8] Hamilton, H.J.,*Transformations of multiple sequences*, Duke Math.J., 2(1936), 29-60.
[9] Miller, H.I., Miller-Van Wieren, L. *Core theorems for double complex sequences*, Bull.Belg.Math.Soc.-Simon Stevin, 18(2011),no.2, 345-352.

[10] Miller, H.I., Patterson, R. *Core theorem for double subsequences and rearrangements*, Acta.Math.Hungar., 119(2008), no.1-2, 71-80.
[11] Móricz, F., *Statistical convergence of multiple sequences*, Arch.Math.(Basel), 81(2003), 82-89.
[12] Patterson, R., *Double sequence core theorems*, Inter. J.Math. and Math.Sci., Vol. 22, 4(1999), 785-793.
[13] Robison, G.M., *Divergent double sequences and series*, Trans. Amer. Math. Soc., 28(1926), 50-73.