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Projective Curvature Tensor on $N(\kappa)-$Contact Metric Manifold Admitting Semi-Symmetric Non-Metric Connection

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Abstract

The object of the present paper is to classify $N(\kappa)$-contact metric manifolds admitting the semi-symmetric non-metric connection with certain curvature conditions the projectively curvature tensor. We studied projective flat, $\xi-$projectively flat, $\phi-$projectively flat $N(\kappa)$-contact metric manifolds admitting the semi-symmetric non-metric connection. Also, we examine such manifolds under some local symmetry conditions related to projective curvature tensor.

1. Introduction

An almost contact metric manifold is a $(2n+1)-$dimensional differentiable manifold with a structure (ϕ,ξ,η,g) such as

$$\phi^2(W_1)=-W_1+\eta(W_1)\xi,\eta(\xi)=1,\quad \phi(\xi)=0,\quad \eta(\phi(W_1))=0, g(\phi(W_1),\phi(W_2))=g(W_1,W_2)-\eta((W_1))\eta((W_2)) \tag{1.1}$$

for any vector fields $W_1,W_2\in\chi(M)$, where g is Riemannian metric, ϕ is a $(1,1)-$tensor field, ξ is a vector field and η is a $1-$form on M [1]. Blair, et al. [2] introduced the (κ,μ)-nullity distribution of an almost contact metric manifold M that is defined by

$$N(\kappa,\mu): p\longrightarrow N_p(\kappa,\mu)$$
$$N_p(\kappa,\mu)=\left\{W_3\in\Gamma(T_pM): R(W_1,W_2)W_3=(\kappa I+\mu h)\left[g(W_2,W_3)W_1-g(W_1,W_3)W_2\right]\right\}$$

for all $W_1,W_2\in\Gamma(TM)$, where κ and μ are real constants and $p\in M$. If $\xi\in N(\kappa,\mu)$, then M is called $(\kappa,\mu)-$contact metric manifold. If $\mu=0$, the (κ,μ)-nullity distribution reduces to κ-nullity distribution.

The idea of κ-nullity distribution on a contact metric manifold was firstly presented by Tanno in 1988 [3]. κ-nullity distribution of an almost contact metric manifold (M,ϕ,ξ,η,g) is a distribution defined as

$$N(\kappa): p\longrightarrow N_p(\kappa)=\left\{W_3\in\Gamma(T_pM): R(W_1,W_2)W_3=\kappa\left[g(W_2,W_3)W_1-g(W_1,W_3)W_2\right]\right\}$$

for any $W_1,\ W_2\in\Gamma(T_pM)$ and $\kappa\in\mathbb{R}$, where R is the Riemannian curvature tensor of M. If ξ belongs to $\kappa-$nullity distribution then M is called $N(\kappa)-$contact metric manifold. Thus on a $N(\kappa)$ contact metric manifold, we have

$$R(W_1,W_2)\xi=\kappa\left[\eta(W_2)W_1-\eta(W_1)W_2\right].$$

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A $N(\kappa)$-contact metric manifold is Sasakian if and only if $k=1$. Also, if $k=0$, then the manifold is locally isometric to the product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$ [4]. The Riemannian geometry of $N(\kappa)-$contact metric manifolds is studied in [2], [5]-[9]. Levi-Civita connection ∇ is a torsion free, i.e has zero torsion, and a metric connection, i.e $\nabla g=0$. There are some kinds of linear connections except for Levi-Civita connection which is not need to be torsion free or metric. One of them is semi-symmetric non-metric connection [10]. Manifolds with semi-symmetric non-metric connection have been studied by many researchers [11]-[15]. In the Riemannian geometry of contact manifolds curvature tensors-such as conformal, concircular, projective curvature tensor etc.-have important applications. Some of geometric properties of structure on manifolds have been examined by the certain conditions on these curvature tensors. Many works on contact manifolds are stated in [16]-[23].

In this paper we study projective curvature tensor on $N(\kappa)$-contact metric manifolds with semi-symmetric non metric connection. In [24], Barman gave the curvature relations on such as manifolds. We use these properties and we examine flatness conditions of projective curvature tensor. Specifically, we given results for ξ-projectively flat, pseudo-quasi-projectively flat and $\phi-$projectively flat on $N(\kappa)$-contact metric manifolds with semi-symmetric non metric connection. After we investigate $\phi-$projectively semi-symmetric on $N(\kappa)$-contact manifolds admitting the semi-symmetric non-metric connection, we characterize this manifolds satisfying $\overset{\star}{Q}.\overset{\star}{P}=0$ and $\overset{\star}{S}.\overset{\star}{P}=0$, where $\overset{\star}{P},\overset{\star}{Q},\overset{\star}{Ric}$ are projective curvature tensor, Ricci tensor, Ricci curvature tensor, with a semi-symmetric non metric connection, respectively.

2. Preliminaries

Let (M,ϕ,ξ,η,g) be an almost contact metric manifold. The $h=\frac{1}{2}\mathscr{L}_\xi\phi$, $\mathscr{L}_\xi$ denotes the Lie derivative along vector field ξ. For any $W_1\in\Gamma(TM)$, we have

$$\nabla_{W_1}\xi=-\phi W_1-\phi hW_1$$

An almost contact metric manifold M is called $K-$contact if ξ is Killing vector field. M is called normal contact metric manifold if $N_\phi+2d\eta\otimes\xi=0$, where N_ϕ is the Nijenhuis tensor of ϕ. A normal contact metric manifold is called Sasakian. On the other hand a contact metric manifold is Sasakian if and only if

$$R(W_1,W_2)\xi=[\eta(W_2)W_1-\eta(W_1)W_2]$$

for all $W_1,W_2\in\Gamma(TM)$. On a $K-$contact and Sasakian manifold $h=0$.

A $N(\kappa)$-contact metric manifold is Sasakian if $\kappa=1$. $N(\kappa)$-contact metric manifolds are characterized the different values of κ. As we mentioned in the introduction when $\kappa=0$ then the manifold M is locally isometric to $E^{(n+1)}(0)\times S^{(n)}(4)$. On a $N(\kappa)$-contact metric manifold M^{2n+1}, we have following relations (for details see [1]):

$$(\nabla_{W_1}\phi)W_2=g(W_1+hW_1,W_2)\xi-\eta(W_2)(W_1+hW_1),$$
$$(\nabla_{W_1}\eta)W_2=g(W_1+hW_1,\phi W_2).$$

The Riemannian curvature R of a $N(\kappa)-$contact metric manifold has following properties:

$$R(W_1,W_2)\xi=\kappa[\eta(W_2)W_1-\eta(W_1)W_2], \tag{2.1}$$
$$R(\xi,W_1)W_2=\kappa[g(W_1,W_2)\xi-\eta(W_2)W_1] \tag{2.2}$$

for all $W_1,W_2\in\Gamma(TM)$. On the other hand the Ricci curvature of M is stated as [1];

$$Ric(W_1,W_2)=2(n-1)g(W_1,W_2)+2(n-1)g(hW_1,W_2)+2(n\kappa-(n-1))\eta(W_1)\eta(W_2) \tag{2.3}$$
$$Ric(\phi W_1,\phi W_2)=Ric(W_1,W_2)-2n\kappa\eta(W_1)\eta(W_2)-4(n-1)g(hW_1,W_2) \tag{2.4}$$
$$Ric(W_1,\xi)=2\kappa n\eta(W_1) \tag{2.5}$$

and the scalar curvature is given by

$$r=2n(2n+\kappa-2).$$

Example 2.1. *E. Boeckx [25] gave a classification for non-Sasakian $(\kappa,\mu)-$spaces. The number $I_M=\frac{1-\frac{\mu}{2}}{\sqrt{1-k}}$ is called by Boeckx invariant. D.E. Blair, et al. [26] gave an example of $N(\kappa)-$contact metric manifolds by using Boeckx invariant. They constructed $(2n+1)$-dimensional $N(1-\frac{1}{n})$-contact metric manifold, $n>1$. For details see [26].*

Let define a map $\overset{\star}{\nabla}$on a Riemann manifold M as

$$\overset{\star}{\nabla}_{W_1}W_2=\nabla_{W_1}W_2+\eta(W_2)W_1$$

where ∇ is Levi-Civita connection on M. This map is a linear connection. The torsion of $\overset{\star}{\nabla}$ is given by

$$\overset{\star}{T}(W_1,W_2)=\eta(W_2)W_1-\eta(W_1)W_2$$

for all $W_1,W_2\in\Gamma(TM)$. Also we have

$$(\overset{\star}{\nabla}_U g)(W_1,W_2)=-\eta(W_1)g(W_2,U)-\eta(W_2)g(W_1,U)\neq 0.$$

Thus $\overset{\star}{\nabla}$ is not symmetric and not metric connection. This type of connection is called by semi-symmetric non-metric connection [10].

$N(\kappa)-$ contact metric manifolds with a semi-symmetric non-metric connection were studied by Barman [24]. For the sake of brevity we denote $(M,\overset{\star}{\nabla})$ by a $N(\kappa)-$contact metric manifolds with a semi-symmetric non-metric connection. Barman gave the curvature of $(M,\overset{\star}{\nabla})$ as follow:

$$\begin{aligned}\overset{\star}{R}(W_1,W_2)W_3=&R(W_1,W_2)W_3+g(W_1,\phi W_3)W_2+g(hW_1,\phi W_3)W_2-\eta(W_1)\eta(W_3)W_2-g(W_2,\phi W_3)W_1\\&-g(hW_2,\phi W_3)W_1+\eta(W_3)\eta(W_2)W_1.\end{aligned}\tag{2.6}$$

Thus, we have following curvature properties [24]:

$$\overset{\star}{R}(\xi,W_2)W_3=\kappa g(W_2,W_3)\xi-(\kappa+1)\eta(W_3)W_2-g(W_2,\phi W_3)\xi-g(hW_2,\phi W_3)\xi+\eta(W_3)\eta(W_2)\xi\tag{2.7}$$

$$\overset{\star}{R}(\xi,W_2)\xi=(\kappa+1)(\eta(W_2)\xi-W_2)$$

$$\overset{\star}{R}(W_1,W_2)\xi=(\kappa+1)(\eta(W_2)W_1-\eta(W_1)W_2).$$

The Ricci curvature of a $(M,\overset{\star}{\nabla})$ is given by

$$\overset{\star}{Ric}(W_2,W_3)=Ric(W_2,W_3)-2ng(W_2,\phi W_3)-2ng(hW_2,\phi W_3)+2n\eta(W_3)\eta(W_2).\tag{2.8}$$

Thus, we have

$$\overset{\star}{Ric}(W_2,\xi)=2n(\kappa+1)\eta(W_2)\tag{2.9}$$

$$\overset{\star}{r}=2n+r$$

where $\overset{\star}{Ric}$, $\overset{\star}{R}$ and $\overset{\star}{r}$ are the Ricci tensor, the Riemann curvature tensor and scalar curvature admitting the semi-symmetric non-metric connection respectively [24].

The projective curvature tensor P admitting the semi-symmetric non-metric connection is defined by

$$\overset{\star}{P}(W_1,W_2)W_3=\overset{\star}{R}(W_1,W_2)W_3-\frac{1}{2n}\left(\overset{\star}{Ric}(W_2,W_3)W_1-\overset{\star}{Ric}(W_1,W_3)W_2\right),\tag{2.10}$$

for all $W_1,W_2,W_3\in TM$.

3. Flatness conditions of projective curvature tensor on $(M,\overset{\star}{\nabla})$

In this section, we examine that a $(M,\overset{\star}{\nabla})$ is ξ-projectively flat, pseudo-quasi-projectively flat and $\phi-$projectively flat.

Definition 3.1. *A $(M,\overset{\star}{\nabla})$ is called*

- *ξ-projectively flat if we have $\overset{\star}{P}(W_1,W_2)\xi=0$ for all $W_1,W_2\in\Gamma(TM)$,*
- *pseudo-quasi-projectively flat if we have $g(\overset{\star}{P}(\phi W_1,W_2)W_3,\phi W_4)=0$ for all $W_1,W_2,W_3\in\Gamma(TM)$,*
- *ϕ-projectively flat if we have $g(\overset{\star}{P}(\phi W_1,\phi W_2)\phi W_3,\phi W_4)=0$ for all $W_1,W_2,W_3\in\Gamma(TM)$.*

Theorem 3.2. *A $(M,\overset{\star}{\nabla})$ is always ξ-projectively flat.*

Proof. By putting $W_3 = \xi$ in (2.10), we obtain

$$\overset{\star}{P}(W_1,W_2)\xi = \overset{\star}{R}(W_1,W_2)\xi - \frac{1}{2n}\left(\overset{\star}{Ric}(W_2,\xi)W_1 - \overset{\star}{Ric}(W_1,\xi)W_2\right).$$

Also from (2.7) and (2.9), we get

$$\overset{\star}{P}(W_1,W_2)\xi = R(W_1,W_2)\xi - \eta(W_1)W_2 + \eta(W_2)W_1 - \frac{1}{2n}\left(2n(k+1)\eta(W_2)W_1 - 2n(k+1)\eta(W_1)W_2\right).$$

and take into account (2.1), we have

$$\overset{\star}{P}(W_1,W_2)\xi = 0 \tag{3.1}$$

for all $W_1, W_2 \in \Gamma(TM)$. □

Theorem 3.3. *If* $(M, \overset{\star}{\nabla})$ *is pseudo-quasi-projectively flat, then* M *is an Einstein manifold admitting Levi-Civita connection.*

Proof. Using (2.10), we have

$$g(\overset{\star}{P}(\phi W_1,W_2)W_3,\phi W_4) = \overset{\star}{R}(\phi W_1,W_2,W_3,\phi W_4) - \frac{1}{2n}[\overset{\star}{Ric}(W_2,W_3)g(\phi W_1,\phi W_4) - \overset{\star}{Ric}(\phi W_1,W_3)g(W_2,\phi W_4)]. \tag{3.2}$$

Let $(M, \overset{\star}{\nabla})$ be a pseudo-quasi-projectively flat. Then, by using (2.8) in (3.2) , it follows that

$$\begin{aligned}\overset{\star}{R}(\phi W_1,W_2,W_3,\phi W_4) = \frac{1}{2n}[&(Ric(W_2,W_3) - 2ng(W_2,\phi W_3) - 2ng(hW_2,\phi W_3) + 2n\eta(W_2)\eta(W_3))g(\phi W_1,\phi W_4)\\ &-(Ric(\phi W_1,W_3) - 2ng(\phi W_1,\phi W_3) - 2ng(h\phi W_1,\phi W_3))\,g(W_2,\phi W_4)]\end{aligned}$$

and from (2.6) we get

$$R(\phi W_1,W_2,W_3,\phi W_4) = \frac{1}{2n}\left(Ric(W_2,W_3)g(\phi W_1,\phi W_4) - Ric(\phi W_1,W_3)g(W_2,\phi W_4)\right). \tag{3.3}$$

Take a local orthonormal basis set of M as $\{e_1,e_2,...,e_{2n},\xi\}$, then $\{\phi e_1,\phi e_2,...,\phi e_{2n},\xi\}$ is also a local orthonormal basis. Putting $W_1 = W_4 = e_i$ in (3.3) and summing over $i = 1$ to $2n$, we get

$$\sum_{i=1}^{2n} R(\phi e_i,W_2,W_3,\phi e_i) = \frac{1}{2n}\left[\sum_{i=1}^{2n}\left(Ric(W_2,W_3)g(\phi e_i,\phi e_i) - Ric(\phi e_i,W_3)g(W_2,\phi e_i)\right)\right].$$

From (2.2) and (2.5), we obtain

$$Ric(W_2,W_3) = 2n\kappa g(W_2,W_3).$$

□

Theorem 3.4. *Let a* $(M, \overset{\star}{\nabla})$ *be* ϕ*-projectively flat. If* ξ *is Killing vector field, then the manifold is an Einstein manifold.*

Proof. Firstly, putting $W_2 = \phi W_2$ and $W_3 = \phi W_3$ in (3.2), we get

$$g(\overset{\star}{P}(\phi W_1,\phi W_2)\phi W_3,\phi W_4) = \overset{\star}{R}(\phi W_1,\phi W_2,\phi W_3,\phi W_4) - \frac{1}{2n}\left(\overset{\star}{Ric}(\phi W_2,\phi W_3)g(\phi W_1,\phi W_4) - \overset{\star}{Ric}(\phi W_1,\phi W_3)g(\phi W_2,\phi W_4)\right). \tag{3.4}$$

Now, by using (2.8) in (3.4) and from definition of ϕ-projectively flat, it follows that

$$\begin{aligned}\overset{\star}{R}(\phi W_1,W_2,W_3,\phi W_4) = \frac{1}{2n}[&\left(Ric(\phi W_2,\phi W_3) - 2ng(\phi W_2,\phi^2 W_3) - 2ng(h\phi W_2,\phi^2 W_3)\right)g(\phi W_1,\phi W_4)\\ &-\left(Ric(\phi W_1,\phi W_3) - 2ng(\phi W_1,\phi^2 W_3) - 2ng(h\phi W_1,\phi^2 W_3)\right)g(\phi W_2,\phi W_4)]\end{aligned}$$

and from (1.1), we get

$$R(\phi W_1,\phi W_2,\phi W_3,\phi W_4) = \frac{1}{2n}\left(Ric(\phi W_2,\phi W_3)g(\phi W_1,\phi W_4) - Ric(\phi W_1,\phi W_3)g(\phi W_2,\phi W_4)\right). \tag{3.5}$$

For local orthonormal basis $\{e_1,e_2,...,e_{2n},\phi e_1,\phi e_2,...,\phi e_{2n},\xi\}$ of M by putting $W_1 = W_4 = e_i$ in (3.5) and summing over $i = 1$ to $2n$, we get

$$\sum_{i=1}^{2n} R(\phi e_i,\phi W_2,\phi W_3,\phi e_i) = \frac{1}{2n}\left[\sum_{i=1}^{2n}\left(Ric(\phi W_2,\phi W_3)g(\phi e_i,\phi e_i) - Ric(\phi e_i,\phi W_3)g(\phi W_2,\phi e_i)\right)\right]$$

From (2.2) and (2.5), we obtain

$$Ric(\phi W_2,\phi W_3) = 2n\kappa g(\phi W_2,\phi W_3).$$

Also, from (2.4) we have

$$Ric(W_2,W_3) = 2n\kappa g(W_2,W_3) + 4(n-1)g(hW_2,W_3)$$

If ξ is Killing vector field then M is an Einstein manifold. □

4. Symmetry conditions admitting projective curvature tensor on $(M,\overset{\star}{\nabla})$

In this section, we study on a $(M,\overset{\star}{\nabla})$ under certain symmetry conditions. We firstly examine $\phi-$projectively semi-symmetric $(M,\overset{\star}{\nabla})$ and then we characterize this manifolds satisfying $\overset{\star}{Q}.\overset{\star}{P}=0$ and $\overset{\star}{Ric}.\overset{\star}{P}=0$, where $\overset{\star}{Q}$ is the Ricci operator defined by $\overset{\star}{Ric}(W_1,W_2)=g(\overset{\star}{Q}W_1,W_2)$.

Definition 4.1. *A $(M,\overset{\star}{\nabla})$ is said to be $\phi-$projectively semisymmetric if $\overset{\star}{P}(W_1,W_2)\phi=0$ for all $W_1,W_2\in\Gamma(M)$.*

Theorem 4.2. *A $\phi-$projectively $(M,\overset{\star}{\nabla})$ is isometric to Example 2.1.*

Proof. Suppose $(M,\overset{\star}{\nabla})$ be a $\phi-$projectively . Then, we have

$$\overset{\star}{P}(W_1,W_2)\phi W_3-\phi(\overset{\star}{P}(W_1,W_2)W_3)=0. \tag{4.1}$$

From (2.10), it follows that

$$\overset{\star}{P}(W_1,W_2)\phi W_3=\overset{\star}{R}(W_1,W_2)\phi W_3-\frac{1}{2n}\left(\overset{\star}{Ric}(W_2,\phi W_3)W_1-\overset{\star}{Ric}(W_1,\phi W_3)W_2\right). \tag{4.2}$$

Using (2.8) in (4.2), we obtain

$$\begin{aligned}\overset{\star}{P}(W_1,W_2)\phi W_3=&\overset{\star}{R}(W_1,W_2)\phi W_3-\frac{1}{2n}\left\{Ric(W_2,\phi W_3)W_1-2ng(W_2,\phi^2W_3)W_1-2ng(hW_2,\phi^2W_3)W_1\right\}\\&+\frac{1}{2n}\left\{Ric(W_1,\phi W_3)W_2-2ng(W_1,\phi^2W_3)W_2-2ng(hW_1,\phi^2W_3)W_2\right\}.\end{aligned}$$

From (2.1), (2.2) and (2.6), we have

$$\overset{\star}{P}(W_1,W_2)\phi W_3=\kappa g(W_2,\phi W_3)W_1-\kappa g(W_1,\phi W_3)W_2-\frac{1}{2n}Ric(W_2,\phi W_3)W_1+\frac{1}{2n}Ric(W_1,\phi W_3)W_2. \tag{4.3}$$

Also, by applying ϕ to $\overset{\star}{P}$, we get

$$\phi(\overset{\star}{P}(W_1,W_2)W_3)=\phi(\overset{\star}{R}(W_1,W_2)W_3)-\frac{1}{2n}\phi\left[\overset{\star}{Ric}(W_2,W_3)W_1-\overset{\star}{Ric}(W_1,W_3)W_2\right], \tag{4.4}$$

and using (2.8) in (4.4) yields

$$\begin{aligned}\phi(\overset{\star}{P}(W_1,W_2)W_3)=&\phi(\overset{\star}{R}(W_1,W_2)W_3)-\frac{1}{2n}\left\{Ric(W_2,W_3)-2ng(W_2,\phi W_3)-2ng(hW_2,\phi W_3)+2n\eta(W_2)\eta(W_3)\right\}\phi W_1\\&+\frac{1}{2n}\left\{Ric(W_1,W_3)-2ng(W_1,\phi W_3)-2ng(hW_1,\phi W_3)+2n\eta(W_1)\eta(W_3)\right\}\phi W_2\end{aligned}$$

Thus from (2.4) and (2.6), we have

$$\phi(\overset{\star}{P}(W_1,W_2)W_3)=\kappa g(W_2,W_3)\phi W_1-\kappa g(W_1,W_3)\phi W_2-\frac{1}{2n}Ric(W_2,W_3)\phi W_1+\frac{1}{2n}Ric(W_1,W_3)\phi W_2. \tag{4.5}$$

Putting (2.3), (4.3) and (4.5) in (4.1), we have

$$\begin{aligned}\overset{\star}{P}(W_1,W_2)\phi W_3-\phi(\overset{\star}{P}(W_1,W_2)W_3)=&\kappa g(W_2,\phi W_3)W_1-\kappa g(W_1,\phi W_3)W_2-\frac{2(n-1)}{2n}\left[g(W_2,\phi W_3)+g(hW_2,\phi W_3)\right]W_1\\&+\frac{2(n-1)}{2n}\left[g(W_1,\phi W_3)+g(hW_1,\phi W_3)\right]W_2-\kappa g(W_2,W_3)\phi W_1+\kappa g(W_1,W_3)\phi W_2\\&+\frac{1}{2n}[2(n-1)(g(W_2,W_3)+g(hW_2,W_3))+2(n\kappa-(n-1)\eta(W_2)\eta(W_3))]\phi W_1\\&-\frac{1}{2n}[2(n-1)(g(W_1,W_3)+g(hW_1,W_3))+2(n\kappa-(n-1)\eta(W_1)\eta(W_3))]\phi W_2\end{aligned} \tag{4.6}$$

Let take inner product with W_4 of (4.6) and then to contract W_2 and W_4, we obtain

$$\left\{2\kappa(1-n)+2(\frac{n^2-2n+1}{n})\right\}g(W_1,\phi W_3)+\{2(n-1)\}g(hW_1,\phi W_3)=0. \tag{4.7}$$

Now, putting $W_3 = \phi W_3$ in (4.7) and from (1.1), we get

$$\left\{2\kappa(1-n)+2(\frac{n^2-2n+1}{n})\right\}g(\phi W_1,\phi W_3)+\{2(n-1)\}g(hW_1,W_3)=0. \tag{4.8}$$

Taking trace in both sides of (4.8) and using $trh = 0$, we obtain

$$\kappa = \frac{n-1}{n}.$$

Thus M is isometric to Example 2.1. □

Theorem 4.3. *On a* $(M,\overset{\star}{\nabla})$, *we have* $\overset{\star}{Q}.\overset{\star}{P} = 0$.

Proof. For all $W_1,W_2,W_3 \in \Gamma(TM)$, we have

$$(\overset{\star}{Q}(W_1).\overset{\star}{P})(W_2,W_3) = \overset{\star}{Q}(\overset{\star}{P}(W_1,W_2)W_3) - \overset{\star}{P}(\overset{\star}{Q}W_1,W_2)W_3 - \overset{\star}{P}(W_1,\overset{\star}{Q}W_2)W_3 - \overset{\star}{P}(W_1,W_2)\overset{\star}{Q}W_3. \tag{4.9}$$

From (2.8) and (2.9), we have

$$\overset{\star}{Q}W_2 = 2(n-1)(W_2+hW_2)+2n(\phi W_2+\phi hW_2)+2(n\kappa+1)\eta(W_2)\xi \tag{4.10}$$

and, so

$$\overset{\star}{Q}\xi = 2n(\kappa+1)\xi. \tag{4.11}$$

Thus, for $W_3 = \xi$ in (4.9) we get

$$(\overset{\star}{Q}(W_1).\overset{\star}{P})(W_2,\xi) = \overset{\star}{Q}(\overset{\star}{P}(W_1,W_2)\xi) - \overset{\star}{P}(\overset{\star}{Q}W_1,W_2)\xi - \overset{\star}{P}(W_1,\overset{\star}{Q}W_2)\xi - \overset{\star}{P}(W_1,W_2)\overset{\star}{Q}\xi.$$

From (3.1), (4.10) and (4.11), it follows that

$$\overset{\star}{Q}.\overset{\star}{P} = 0.$$

□

Theorem 4.4. *A* $(M,\overset{\star}{\nabla})$ *satisfies* $\overset{\star}{P}.\overset{\star}{Ric} = 0$ *if and only if* M *is an Einstein manifold.*

Proof. Let $\overset{\star}{P}.\overset{\star}{Ric} = 0$ satisfies on $(M,\overset{\star}{\nabla})$, then we get

$$\overset{\star}{Ric}(\overset{\star}{P}(W_4,W_2)W_3,W_1)+\overset{\star}{Ric}(W_3,\overset{\star}{P}(W_4,W_2)W_1) = 0. \tag{4.12}$$

Putting $W_1 = W_4 = \xi$ in (4.12), we have

$$\overset{\star}{Ric}(\overset{\star}{P}(\xi,W_2)W_3,\xi)+\overset{\star}{Ric}(W_3,\overset{\star}{P}(\xi,W_2)\xi) = 0. \tag{4.13}$$

Also, from (2.10), we get

$$\overset{\star}{P}(\xi,W_2)W_3 = \overset{\star}{R}(\xi,W_2)W_3 - \frac{1}{2n}\left(\overset{\star}{Ric}(W_2,W_3)\xi - \overset{\star}{Ric}(\xi,W_3)W_2\right),$$

from (2.7), (2.8), (2.9), it follows that

$$\overset{\star}{P}(\xi,W_2)W_3 = \kappa g(W_2,W_3)\xi - \frac{1}{2n}Ric(W_2,W_3)\xi. \tag{4.14}$$

Again putting $W_3 = \xi$ in (4.14) and using (2.5), we obtain

$$\overset{\star}{P}(\xi,W_2)\xi = 0. \tag{4.15}$$

Using (2.9), (4.14) and (4.15) in (4.13), it follows that

$$Ric(W_2,W_3) = 2n\kappa g(W_2,W_3).$$

Conversely, let M be an Einstein manifold , i.e $Ric(W_2,W_3) = 2n\kappa g(W_2,W_3)$. Then, we get

$$\overset{\star}{P}(W_1,W_2)W_3 = \kappa(g(W_2,W_3)W_1 - g(W_1,W_3)W_2) - \frac{1}{2n}(2n\kappa g(W_2,W_3)W_1 - 2n\kappa g(W_1,W_3)W_2).$$

which implies $\overset{\star}{P}(W_1,W_2)W_3 = 0$. This also give us $\overset{\star}{P}.\overset{\star}{Ric} = 0$.

□

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