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Application of Extended Transformed Rational Function Method to Some (3+1) Dimensional Nonlinear Evolution Equations

Genişletilmiş Dönüştürülmüş Rasyonel Fonksiyon Metodunun Bazı (3+1) Boyutlu Lineer Olmayan Oluşum Denklemlerine Uygulanması

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Abstract

The transformed rational function method can be considered as unification of the tanh type methods, the homogeneous balance method, the mapping method, the exp-function method and the F-expansion type methods. In this paper, we present complexiton solutions of (3+1) dimensional Korteweg-de Vries (KdV) equation and a new (3+1) dimensional generalized Kadomtsev-Petviashvili equation by using extended transformed rational function method which provides very useful and effective way to obtain complexiton solutions of nonlinear evolution equations.

Keywords: Complexiton solutions, (3+1) dimensional KdV equation, Extended transformed rational function method, New (3+1) dimensional generalized Kadomtsev-Petviashvili equation

Öz

Dönüştürülmüş rasyonel fonksiyon metodu; tanh tipi metodlar, homojen denge metodu, resmetme metodu, üstel fonksiyon metodu ve F-açılım tipi metodların birleşimi olarak düşünülebilir. Biz bu çalışmada, lineer olmayan oluşum denklemlerinin kompleksiton çözümlerinin elde edilmesinde kullanışlı ve etkili bir yol olan genişletilmiş dönüştürülmüş rasyonel fonksiyon metodunu kullanarak (3+1) boyutlu KdV ve yeni (3+1) boyutlu genelleştirilmiş Kadomtsev-Petviashvili denklemlerinin kompleksiton çözümlerini elde edeceğiz.

Anahtar Kelimeler: Kompleksiton çözümler, (3+1) boyutlu KdV denklemi, Genişletilmiş dönüştürülmüş rasyonel fonksiyon metodu, Yeni (3+1) boyutlu genelleştirilmiş Kadomtsev-Petviashvili denklemi

1. Introduction

In last decades, searching exact solutions and integrability of nonlinear differential equations has become very popular in applied sciences such as mathematical physics, applied mathematics (1). So far, different methods have been used to search analytical solutions of nonlinear differential equations, such as the homogeneous balance method (2), the F-expansion method (3), the tanh function method (4), the sech-function method (5), the extended tanh function method (6-9), tanh-coth method (10) and some others (11-17).

In (18), the transformed rational function method which unifies the above exact solution methods is introduced. This method is based on idea of using rational function transformations. In (18), it's shown that the transformed rational function method is very effective tool to obtain exact travelling solutions of nonlinear differential equations. Afterwards, the transformed rational function method has been improved and the improved one was called the extended transformed rational function method (19). In (19), this method was applied to bilinear forms of (3+1) dimensional generalized KP equation, the Boiti-Leon-Manna-Pempinelli equation, the (3+1) dimensional BKP equation, the (3+1) dimensional Jimbo-Miwa equation to obtain complexiton solutions. In literature, Wen-Xiu Ma named and used complexiton solutions for the first time (20). In (20,21), a novel class of explicit exact solutions to the Korteweg-de Vries equation is given through its bilinear form.

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In this paper, we present complexiton solutions to two (3+1) dimensional nonlinear evolution equations. The rest of this paper is presented in following arrangement. In Section 2, we simply give the mathematical framework of the transformed rational function method and the extended transformed rational function method. In Section 3, in order to illustrate the method, complexiton solutions of nonlinear evolution equations are obtained through their bilinear form. Finally, some conclusions are provided.

2. Extended Transformed Rational Function Method

The transformed rational function method which underlies the method we use in this paper, is used to find travelling wave solutions of nonlinear equations and introduced in (18), as follows.

Let's start with the partial differential equation

$$P(u,u_x,u_t,u_{xx},\ldots)=0. \tag{2.1}$$

Step 1: We seek travelling wave solutions of Eq. (2.1) in the form of

$$u=u(\xi),\xi=k(x-ct), \tag{2.2}$$

where *k* and *c* are real constants to be determined. By using the transformation (2.2), Eq. (2.1) can be transformed into an ordinary differential equation

$$P(u,ku',-kcu',k^2u'',\ldots)=0, \tag{2.3}$$

where $u'=du/d\xi$.

Step 2: We search for travelling wave solutions determined by

$$u^{(r)}(\xi)=v(\eta)=\frac{p(\eta)}{q(\eta)}=\frac{p_m\eta^m+p_{m-1}\eta^{(m-1)}+\ldots+p_0}{q_n\eta^n+q_{n-1}\eta^{(n-1)}+\ldots+q_0} \tag{2.4}$$

where $p(\eta)$ and $q(\eta)$ are polynomials, $r>0$ represents the minimal differential number in (2.3). Introducing a new variable $\eta=\eta(\xi)$ by a solvable ordinary differential equation is very important part of the solution process. For instance, a first-order differential equation:

$$\eta'=T=T(\xi,\eta), \tag{2.5}$$

where T is a function of ξ and η. The prime is used to imply the derivative with respect to ξ. Thus we obtain,

$$\frac{du^{(r)}(\xi)}{d\xi}=T\frac{dv}{d\eta},\frac{du^{(r+1)}(\xi)}{d\xi}=T^2\frac{d^2v}{d\eta^2}+T'\frac{dv}{d\eta},\ldots \tag{2.6}$$

From now on, we just need to equate the numerator of the resulting rational function in the transformed equation to zero. This gives a system of algebraic equations.

Step 3: One can easily obtain travelling wave solutions to Eq. (2.1) by solving the above mentioned algebraic equations in Step 2.

In [18], it is showed that the transformed rational function method will be the exp-function method if we choose $\eta'=\eta$ and $\eta=e^{\xi}$ and that the transformed rational function method will be the extended tanh-function method if we choose $\eta'=\alpha+\eta^2$, where α is a constant. It is clear that the transformed rational function method unifies the existing methods using tanh-function type functions, tan-function type functions and the exponential functions.

However, it is not appropriate to construct complexiton solutions to nonlinear equations, since complexiton solutions have different travelling wave speeds of new type. In (19), so as to find complexiton solutions, the transformed rational function method is improved as follows.

For a partial differential equation (2.1);

Step 1: Suppose Eq. (2.1) has a Hirota bilinear form:

$$H(D_x,D_t,\ldots)f\cdot f=0, \tag{2.7}$$

where $D_x,D_t,\ldots,$ are Hirota's differential operators defined by

$$\begin{aligned}D_y^pf(y)\cdot g(y)&=(\partial_y-\partial_{y'})^pf(y)g(y')\big|_{y'=y}\\&=\partial_{y'}^pf(y+y')g(y-y')\big|_{y'=0},p\geq 1.\end{aligned} \tag{2.8}$$

Step 2: Suppose

$$f=\frac{p(\eta_1,\eta_2)}{q(\eta_1,\eta_2)}, \tag{2.9}$$

where $p(\eta_1,\eta_2)$ and $q(\eta_1,\eta_2)$ are polynomials and η_1 and η_2 admit, for example,

$$\eta_1''=\frac{d^2\eta_1}{d\xi_1^2}=-\eta_1, \tag{2.10}$$

$$\eta_2''=\frac{d^2\eta_2}{d\xi_2^2}=\eta_2, \tag{2.11}$$

where $\xi_1=k_1x+w_1t+c_1$ and $\xi_2=k_2x+w_2t+c_2$. k_1,k_2,w_1,w_2 can be determined later and c_1 and c_2 are arbitrary constants.

Step 3: Choose appropriate $p(\eta_1,\eta_2)$ and $q(\eta_1,\eta_2)$, we can convert (2.7) into algebra equation involving k_i and w_i. Solving this algebra equation, we will obtain exact complexiton solutions to Eq. (2.1).

In next section, we apply extended transformed rational function method to investigate the complexiton solutions of nonlinear evolution equations referred in abstract.

3. Applications

Example 1. Let us consider new (3+1) dimensional generalized Kadomtsev-Petviashvili equation

$$u_{xxxy}+3(u_x u_y)_x+u_{tx}+u_{ty}+u_{tz}-u_{zz}=0 \tag{3.1}$$

which is introduced and employed to obtain multiple-soliton solutions in (22). One can easily reduce Eq. (3.1) to

$$(D_x^3 D_y+D_t D_x+D_t D_y+D_t D_z-D_z^2)f\cdot f=0, \tag{3.2}$$

by making the transformation $u=2(\ln f)_x$. A simple direct computation shows that the corresponding bilinear equation reads

$$\begin{aligned}P(D_x,D_y,D_z,D_t)f\cdot f&=f_{tx}f-f_tf_x+f_{ty}f-f_tf_y\\&+f_{tz}f-f_tf_z-f_{zz}f+f_z^2+f_{xxxy}f-f_{xxx}f_y-3f_xf_{xxy}+3f_{xx}f_{xy}=0\end{aligned} \tag{3.3}$$

where $f=f(x,y,z,t)$. Suppose that

$$f=A\eta_1+B\eta_2. \tag{3.4}$$

Since Eq.(3.3) is a (3+1) dimensional equation, then we choose

$$\begin{aligned}\xi_1&=k_1x+bk_1y+ck_1z+w_1t+c_1\\ \xi_2&=k_2x+bk_2y+ck_2z+w_2t+c_2\end{aligned} \tag{3.5}$$

where A,B,k_i and w_i are determined later. Substituting (3.4) into (3.3) with the relations (2.10), (2.11) and

$$\eta_1'^2=1-\eta_1^2,\eta_2'^2=1+\eta_2^2, \tag{3.6}$$

we can write the resulting equation in a polynomial form in terms of $\eta_1^2,\eta_2^2,\eta_1\eta_2,\eta_1'\ \eta_2'$. Collecting the coefficients of $\eta_1^2,\eta_2^2,\eta_1\eta_2,\eta'_1\eta'_2$, and equating them to zero we obtain following algebraic equations:

$$\begin{aligned}&-4B^2k_2^4b-B^2w_2k_2+B^2c^2k_2^2-B^2bk_2w_2-B^2ck_2w_2\\&\qquad-A^2w_1bk_1-A^2w_1ck_1-A^2w_1k_1+A^2c^2k_1^2+4A^2k_1^4b=0\\&-Abk_1Bw_2-Bbk_2Aw_1-Ack_1Bw_2-Bck_2Aw_1\\&\quad+2Ac^2k_1Bk_2+4Ak_1^3Bbk_2-4Bk_2^3Abk_1-Ak_1Bw_2-Bk_2Aw_1=0\\&-Aw_1bk_1B-Aw_1ck_1B+Bw_2bk_2A+Bw_2ck_2A-6Ak_1^2Bbk_2^2\\&-Aw_1k_1B+Ac^2k_1^2B+Bw_2k_2A-Bc^2k_2^2A+Bbk_2^4A+Abk_1^4B=0\end{aligned} \tag{3.7}$$

By solving the system of algebraic equations (3.7), we get following solution

$$\begin{aligned}&A=\pm B\frac{k_2}{k_1},\quad w_1=\frac{k_1(k_1^2b+c^2-3bk_2^2)}{b+c+1},\\&w_2=\frac{(-bk_2^2+3k_1^2b+c^2)k_2}{b+c+1},\end{aligned} \tag{3.8}$$

Taking (3.4) into account we can express the solutions of (3.3) as follows:

$$\begin{aligned}f(x,y,z,t)=A\Bigg[&\sin\left(\begin{aligned}&k_1x+bk_1y+ck_1z\\&+\frac{k_1(k_1^2b+c^2-3bk_2^2)}{b+c+1}t+c_1\end{aligned}\right)\\&\pm\frac{k_1}{k_2}\sinh\left(\begin{aligned}&k_2x+bk_2y+ck_2z\\&+\frac{(-bk_2^2+3k_1^2b+c^2)k_2}{b+c+1}t+c_2\end{aligned}\right)\Bigg]\end{aligned} \tag{3.9}$$

or

$$\begin{aligned}f(x,y,z,t)=A\Bigg[&\cos\left(\begin{aligned}&k_1x+bk_1y+ck_1z\\&+\frac{k_1(k_1^2b+c^2-3bk_2^2)}{b+c+1}t+c_1\end{aligned}\right)\\&\pm\frac{k_1}{k_2}\sinh\left(k_2x+bk_2y+ck_2z+\frac{(-bk_2^2+3k_1^2b+c^2)k_2}{b+c+1}t+c_2\right)\Bigg]\end{aligned} \tag{3.10}$$

where A,b,c,c_1,c_2,k_1,k_2 are arbitrary.

Example 2. Now we consider (3+1) dimensional KdV equation which occurs in various areas of physics and is given in the form of

$$u_t+6u_xu_y+u_{2xy}+u_{4xz}+60u_x^2u_z+10u_{3x}u_z+20u_xu_{2xz}=0 \tag{3.11}$$

in the literature (23). With the aid of transformation

$$u=(\ln f)_x, \tag{3.12}$$

Eq. (3.11) is transformed into bilinear equation

$$(D_x^3D_y+D_x^5D_z+D_xD_t)f\cdot f=0 \tag{3.13}$$

Eq. (3.13) can be expressed as

$$\begin{aligned}P(D_x,D_y,D_z,D_t)f\cdot f&=f_{xxxxxz}f-f_{xxxxx}f_z-5f_{xxxxz}f_x\\&+5f_{xz}f_{xxxx}+10f_{xx}f_{xxxz}-10f_{xxx}f_{zxx}+f_{xxxy}f-f_{xxx}f_y\\&-3f_xf_{xxy}+3f_{xx}f_{xy}+f_{tx}f-f_tf_x=0\end{aligned} \tag{3.14}$$

Substituting (3.4) into (3.14) with the relations (2.10), (2.11) and (3.6), we get the following system of algebraic equations from coefficients of $\eta_1^2,\eta_2^2,\eta_1\eta_2,\eta_1'\ \eta_2'$:

$$\begin{aligned}&-4B^2k_2^4b-B^2w_2k_2-16B^2ck_2^6-16A^2ck_1^6\\&-A^2w_1k_1+4A^2k_1^4b=0\\&4Ak_1^3Bbk_2-4Bk_2^3Abk_1+20Ak_1^3Bk_2^3c\\&-6Ak_1^5Bck_2-6Bk_2^5Ack_1-Ak_1Bw_2-Bk_2Aw_1=0\\&-6Ak_1^2Bbk_2^2-15Bk_2^4Ack_1^2+15Ak_1^4Bck_2^2-Ack_1^6B\\&+Bck_2^6A-Aw_1k_1B+Bw_2k_2A+Bbk_2^4A+Abk_1^4B=0\end{aligned} \tag{3.15}$$

Solution of system of algebraic equations (3.15) gives us:

$$
\begin{aligned}
w_1 &= \frac{2ck_1\left(6A^2k_1^6 - B^2k_1^4k_2^2 - 10A^2k_1^4k_2^2 - 15B^2k_2^6\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}, \\
w_2 &= \frac{2ck_2\left(15A^2k_1^6 + 10B^2k_2^4k_1^2 + A^2k_1^2k_2^4 - 6B^2k_2^6\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}, \\
b &= \frac{5c\left(3A^2k_1^4 - B^2k_1^2k_2^2 - A^2k_1^2k_2^2 + 3B^2k_2^4\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)},
\end{aligned} \tag{3.16}
$$

With the aid of (3.16), we get the solution of equation (3.14) in the form of

$$
\begin{aligned}
f(x,y,z,t) &= A\sin\Bigg(k_1x + \frac{5c\left(3A^2k_1^4 - B^2k_1^2k_2^2 - A^2k_1^2k_2^2 + 3B^2k_2^4\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}k_1y \\
&+ck_1z + \frac{2ck_1\left(6A^2k_1^6 - B^2k_1^4k_2^2 - 10A^2k_1^4k_2^2 - 15B^2k_2^6\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}t + c_1\Bigg) \\
&\pm B\sinh\Bigg(k_2x + \frac{5c\left(3A^2k_1^4 - B^2k_1^2k_2^2 - A^2k_1^2k_2^2 + 3B^2k_2^4\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}k_2y + \\
&+ck_2z + \frac{2ck_2\left(15A^2k_1^6 + 10B^2k_2^4k_1^2 + A^2k_1^2k_2^4 - 6B^2k_2^6\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}t + c_2\Bigg)
\end{aligned} \tag{3.17}
$$

or

$$
\begin{aligned}
f(x,y,z,t) &= A\cos\Bigg(k_1x + \frac{5c\left(3A^2k_1^4 - B^2k_1^2k_2^2 - A^2k_1^2k_2^2 + 3B^2k_2^4\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}k_1y \\
&+ck_1z + \frac{2ck_1\left(6A^2k_1^6 - B^2k_1^4k_2^2 - 10A^2k_1^4k_2^2 - 15B^2k_2^6\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}t + c_1\Bigg) \\
&\pm B\sinh\Bigg(k_2x + \frac{5c\left(3A^2k_1^4 - B^2k_1^2k_2^2 - A^2k_1^2k_2^2 + 3B^2k_2^4\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}k_2y \\
&+ck_2z + \frac{2ck_2\left(15A^2k_1^6 + 10B^2k_2^4k_1^2 + A^2k_1^2k_2^4 - 6B^2k_2^6\right)}{3\left(Ak_1 - Bk_2\right)\left(Ak_1 + Bk_2\right)}t + c_2\Bigg)
\end{aligned} \tag{3.18}
$$

where $A, B, c, c_1, c_2, k_1, k_2$ are arbitrary.

4. Conclusion

In this paper, we present complexiton solutions to some (3+1) dimensional nonlinear equations. Hirota derivatives allows us to express given nonlinear equations in bilinear form. Upon this, with appropriate choice of solution form, we obtain complexiton solutions with rich parametric values which is thought useful for further works. Since the existence of hyperbolic and trigonometric type functions in solutions, the phrase "complexiton" arises. Employed method can be generalized to obtain solutions of other nonlinear partial differential equations by taking different differential equations that η_1 and η_2 are supposed to satisfy.

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6. References

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