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A quadratic programming approach to a survey sampling cost minimization problem

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Abstract

An analytical algorithmic methodology developed by Kabe [1-3], and Scobey and Kabe [4] for solving matrix quadratic programming problems (QPP), and for solving matrix linear programming problems (LPP) is utilized here to minimize the cost of conducting a certain census sampling survey. For carrying on the survey, the city is divided into pn blocks, the $(\mathrm{i},\mathrm{j})-\mathrm{th}$ block contains x_{ij} households and the $\mathrm{i}-\mathrm{th}$ census enumerator visits x_{ij} households to be surveyed and the cost of visiting a single household in the $(\mathrm{i},\mathrm{j})-\mathrm{th}$ block is, say, c_{ij}, monetary units. This census survey cost minimization problem is a LPP, and is solved here by using a certain QPP solving methodology. This LPP is exactly similar to the usual standard transportation problem.

Keywords: *Quadratic Programming, Census Sampling, Transportation Problem, Cost Minimization*

Örnekleme maliyetinin minimizasyonu problemine yönelik bir kuadratik programlama yaklaşımı

Özet

Kabe [1-3] ile Scobey ve Kabe [4] tarafından, matris kuadratik programlama problemlerini (QPP) ve matris doğrusal programlama problemlerini (LPP) çözmek üzere geliştirilen analitik algoritmik bir metodolojiden bu çalışmada belirli bir nüfus sayımı örnekleme araştırmasının gerçekleştirilme maliyetini minimize etmede kullanılmaktadır. Araştırmanın gerçekleştirilmesi amacıyla, şehir pn bloklarına ayrılmış, $(\mathrm{i},\mathrm{j})-\mathrm{nci}$ blok x_{ij} hane içermiş ve the $\mathrm{i}-\mathrm{nci}$ sayım görevlisi incelemek üzere x_{ij} hane ziyaret etmiştir ve $(\mathrm{i},\mathrm{j})-\mathrm{nci}$ bloktaki tek bir haneyi ziyaret etmenin maliyeti, c_{ij}, para birimi kabul edilmiştir. Bu nüfus sayımı araştırması maliyet minimizasyon problemi, bir doğrusal programlama problemidir ve bu çalışmada belirli bir kuadratik programlama çözüm metodolojisinden faydalanılarak çözülmüştür. Bu doğrusal programlama problemi, alışılagelmiş standart ulaşım problemi ile tamamiyle benzerdir.

Anahtar Sözcükler: *Kuadratik Programlama, Nüfus Sayımı, Örneklemesi, Ulaşım Problemi, Maliyet Minimizasyonu*

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1. Introduction

Consider a census sampling survey, where for the purpose of the survey the city is divided into pn blocks and the $(i,j)-th$ block contains $x_{ij}, i = 1,\ldots,p; j = 1,\ldots,n$ households, $X = (x_{ij})$. Let the cost of visiting a single household in the $(i,j)-th$ be c_{ij} monetary units, $C = (c_{ij})$. The cost of survey minimization problem now is a matrix linear programming problem (LPP), namely

$$Min\ tr\ CX', \quad subject\ to\ (st), \quad X'J_p = b, \quad XJ_n = q, \quad x_{ij} \geq 0, \tag{1}$$

where J_n denotes an n component (column) vector of unities, and b, and q are known constant vectors.

We write (1) as

$$Min\ c'x, \quad (I \otimes J_p')x = b, \quad (J_n' \otimes I)x = b, \quad x \geq 0, \tag{2}$$

where x is the pn component (column) vector obtained by stacking the columns of X one below the other, and c has similar meaning. Now (2) is the usual LPP, and hence can be solved by the usual methodologies for LPP. However, we solve (2) by using quadratic programming problem (QPP) solving methodology, by writing (2) as

$$Min\ x'cc'x, \ s.t. \ (I \otimes J_p')x = b, \ (J_n' \otimes I)x = q, \ x \geq 0. \tag{3}$$

The next section records Kabe's [1, 2] QPP solving analytical methodology, which solves (3). The methodology is illustrated by five simple numerical examples. Sometimes the same symbol denotes different quantities; however, its meaning is made explicit in the context. Section 3 records (3) in terms of a standard transportation problem.

2. Quadratic Programming Problem

Every QPP can always be written in the standard form

$$Min\ x'Ax, \ s.t. \ Dx = v, \ x \geq 0, \tag{4}$$

where A is an $n \times n$ positive definite symmetric known matrix, and D is a given $q \times n$ matrix of rank $q < n$.

Now to solve (4), Kabe (1991,1992) writes (4) as

$$Min\ x'Ax, \ s.t. \ Dx = v, \ CAx = f, \ x \geq 0, \tag{5}$$

where $(n-q) \times n$ C of rank $(n-q)$ is orthogonal to D and f is an arbitrary $(n-q)$ component vector.

In case A is deficient in rank, then write (5) as

$$Min\ x'(A + D'D)x, \ s.t. \ Dx = v, \ CAx = f, \ x \geq 0, \tag{6}$$

provided $(A + D'D)$ has full rank; otherwise write (6) as

$$Min\ x'(A + \theta D'D + C'C)x, \ s.t. \ Dx = v, \ Cx = 0, \ CAx = f, \ x \geq 0, \tag{7}$$

where $(n-q) \times n$ C orthogonal to D must satisfy $Cx = 0$, as well as $CAx = 0$, and $(A + \theta D'D + C'C)$ must have full rank, where θ is some appropriately chosen constant.

We illustrate (7) by a simple example.

2.1. Example 1

$$Min\ x'Ax, \ s.t. \ (1,0,0)x = 1, \ c = (0\ 1 - 1), \ cx = 0, \ cAx = 0, \tag{8}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ A + D'D = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$A + D'D + c'c = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \qquad (9)$$

where $\theta = 1$. The solution to (8) is

$$x = G^{-1}d\left(d'G^{-1}d\right)^{-1} = \left(1, -\tfrac{1}{2}, -\tfrac{1}{2}\right), G = \left(A + D'D + c'c\right), \qquad (10)$$

Note that there is no loss of generality in assuming D to have full rank, In case D does not have full rank, then write $\mathrm{Dx} = \mathrm{v}$ as $\left(\mathrm{D}_1' \ \ \mathrm{D}_2'\right)\mathrm{x} = \left(\mathrm{v}_1' \ \ \mathrm{v}_2'\right)$, where D_1 has full rank, and replace the linear restrictions $\mathrm{Dx} = \mathrm{v}$ in (7) by the linear restrictions $\mathrm{D}_1\mathrm{x} = \mathrm{v}_1$.

The QPP

$$Min\left(x'Ax - 2\mu'x\right), \ s.t. \ Dx = 0, \ \ x \geq 0, \qquad (11)$$

is the same QPP

$$Min \ y'Ay, \ s.t. \ Dy = v - DA^{-1}\mu, \ \ y \geq 0, \qquad (12)$$

We now solve (5) by a certain Linear Complementary Programming (LPP) methodology, due to Kabe [1, 2]. We first set $\mathrm{x} = \mathrm{A}^{-\frac{1}{2}}\mathrm{y}$, and write (5) as

$$Min \ y'y, \ s.t. \ DA^{-\frac{1}{2}}y = v, \ \ CA^{+\frac{1}{2}}y = f, \qquad (13)$$

and write a solution y to (13) as

$$\begin{pmatrix} DA^{-\frac{1}{2}} \\ CA^{+\frac{1}{2}} \end{pmatrix} y = \begin{pmatrix} v \\ f \end{pmatrix}, \qquad (14)$$

$$y = \begin{pmatrix} DA^{-\frac{1}{2}} \\ CA^{+\frac{1}{2}} \end{pmatrix}' \left[\begin{pmatrix} DA^{-\frac{1}{2}} \\ CA^{+\frac{1}{2}} \end{pmatrix}\begin{pmatrix} DA^{-\frac{1}{2}} \\ CA^{+\frac{1}{2}} \end{pmatrix}'\right]^{-1} \begin{pmatrix} v \\ f \end{pmatrix}$$

$$= A^{-\frac{1}{2}}D'\left(DA^{-1}D'\right)^{-1}v + A^{\frac{1}{2}}C'\left(CA^{-1}C'\right)^{-1}f, \qquad (15)$$

$$x = A^{-1}D'\left(DA^{-1}D'\right)^{-1}v + C'\left(CA^{-1}C'\right)^{-1}f, \qquad (16)$$

$$x = A^{-1}D'\left(DA^{-1}D'\right)^{-1}Dx + C'\left(CA^{-1}C'\right)^{-1}CAx, \qquad (17)$$

$$I = A^{-1}D'\left(DA^{-1}D'\right)^{-1}D + C'\left(CA^{-1}C'\right)^{-1}CA. \qquad (18)$$

Now from (18) note that

$$(A^{-1} - A^{-1}D'\left(DA^{-1}D'\right)^{-1}A^{-1}) = C'\left(CA^{-1}C'\right)^{-1}C, \qquad (19)$$

and hence the solution (17) turns out to be

$$x = A^{-1}D'\left(DA^{-1}D'\right)^{-1}v + (A^{-1} - A^{-1}D'\left(DA^{-1}D'\right)^{-1}A^{-1})t. \qquad (20)$$

$$= x_0 + Mt$$

Again note from (16) that

$$x'Ax = v'\left(DA^{-1}D'\right)^{-1}v + f'\left(CA^{-1}C'\right)^{-1}f$$

$$= x_0'Ax_0 + t'Mt, \qquad (21)$$

and from (20) that

$$t'x = t'x_0 + t'Mt, \;\; x'Ax = x_0'Ax_0 - t'x_0 + t'x. \quad (22)$$

It follows from (22) that $x'Ax$ cannot be a minimum unless $t'x = 0, i.e., t_1x_1 = 0, \ldots, t_nx_n = 0$, which are n quadratic equations in t variables. Each quadratic equation is solved as

$$ax^2 + bx = 0, \;\; i.e., \;\; 2ax + b = \pm b, \quad (23)$$

and hence the context n quadratic equations are solved by 2n simultaneous linear equations of the type (23). We term (23) as the linear complementary programming (LCP) algorithm, Kabe [1, 2].

2.2. Example 2

We illustrate (23) by solving (11), where

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \;\; A^{-1}\mu = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \;\; D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \;\; v = 2. \quad (24)$$

The calculations (23) yield

$$3x = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} + \begin{bmatrix} 2t_1 + t_2 - 3t_3 \\ t_1 + 2t_2 - 3t_3 \\ -3t_1 - 3t_2 + 63t_3 \end{bmatrix}, \quad (25)$$

the three quadratic equations $t_1x_1 = 0, t_2x_2 = 0, t_3x_3 = 0$, yield the six simultaneous linear equations

$$4t_1 + (t_2 - 3t_1 + 6) = \pm(t_2 - 3t_1 + 6), \quad (26)$$

$$4t_2 + (t_1 + 3t_3 + 3) = \pm(t_1 + 3t_3 + 3), \quad (27)$$

$$4t_3 + (1 + t_1 + t_2) = \pm(1 + t_1 + t_2), \quad (28)$$

and the solution is $t_1 = 0, t_2 = 0, 2t_3 = 1; 2x = (3,1,0)$.

To show that our QPP algorithm (23) solves LPP, we solve a trivial LPP

$$Min\,(x_1 + x_2), \;\; s.t. 2x_1 + x_2 = 2, \;\; x \geq 0, \quad (29)$$

$$G = (A + D'D) = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \;\; G^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \quad (30)$$

$$x = x_0 + Mt = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \left[G^{-1} - G^{-1}d\left(d'G^{-1}d\right)^{-1}d'G^{-1}\right]t$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + t_1 - 2t_2 \\ -2 - 2t_1 + 4t_2 \end{bmatrix}, \quad (31)$$

and the solution is $t_1 = 0, 2t_2 = 1, x_1 = 1, x_2 = 0$.

2.3. Example 3

We illustrate (1) by a simple example. Let

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \;\; C = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \quad (32)$$

$$(J_2' \otimes I)x = \begin{pmatrix} x_{11} + x_{12} \\ x_{21} + x_{22} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

$$(I \otimes J_2')x = J_2'x = \begin{bmatrix} x_{11} + x_{21} \\ x_{12} + x_{22} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad (33)$$

$$Min\ c'x = Min\ (2,1,0,0)(x_{11}, x_{21}, x_{12}, x_{22})',\ \ s.t. (33), \tag{34}$$

and $x \geq 0$. Then (3) yields

$$Min\ x' \begin{bmatrix} 6 & 3 & 1 & 0 \\ 3 & 3 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} x,\ \ s.t. \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix},\ \ x \geq 0, \tag{35}$$

where $x'(CC' + (J_2J_2' \otimes I) + (I \otimes J_2J_2'))x$ is the first term in (34). Now (34) is within the framework of (20), and a solution is $x_{11} = 2$, $x_{21} = 2$, $x_{12} = 1$, $x_{22} = 0$, which means the city contains a total of five households, and only four are surveyed, the first enumerator surveys $x_{11} = 2$ households, the second enumerator surveys $x_{21} = 2$ households, one household $x_{12} = 1$ is not surveyed, and $x_{22} = 0$ means there is no household. The cost of not surveying the household is zero. The census wants to survey only four households to minimize the cost. The minimum cost is six monetary units.

3. Standard Transportation Problem

In a standard transportation problem p dealers transport their goods to n destinations, depending on the demands of these destinations. The tabular representation of the problem is

(36)

j destinations / i dealers	1	2	…	J	…	n	Totals
1	x_{11}	x_{12}	…	x_{1j}	…	x_{1n}	q_1
2	x_{21}	x_{22}	…	x_{2j}	…	x_{2n}	q_2
\''	…	…	…	…	…	…	…
i	…	…	…	x_{ij}	…	…	…
…	…	…	…	…	…	…	…
p	x_{p1}	x_{p2}	…	x_{pj}	…	x_{pn}	q_p
Totals	b_1	b_2		b_j		b_n	

The matrix (x_{ij}) (36) is denoted by the $\mathrm{p} \times \mathrm{n}$ matrix $X = (x_{ij})$. Associated with X is also the cost matrix $C = (c_{ij}), i = 1, \dots, p; j = 1, \dots, n$, where c_{ij} is the per unit cost of transporting x_{ij} units of goods from the $\mathrm{i-th}$ dealer to the $\mathrm{j-th}$ destination. the LPP problem is now the same as (1), the vector q denotes the total units of stock of the p dealers, and the vector b of (1) denotes the total demand of the of n destinations. Thus e-g, the $\mathrm{i-th}$ dealer has total number of q_i units of stock and $\mathrm{j-th}$ destination wishes to buy a total of b_j units of goods. Depending on the cost of transportation, the $\mathrm{j-th}$ destination decides how the amount $x_{1j}, x_{2j}, \dots, x_{pj}, x_{1j} + x_{2j} + \cdots + x_{pj} = b_j$ should be purchased, and that is the problem (1).

Note from (35) that

$$x'(CC' + (J_pJ_p' \otimes I) + (I \otimes J_nJ_n'))x = x'Ax, \tag{37}$$

must be a positive definite quadratic form i.e., (36) matrix A must be symmetric positive definite matrix. We shall illustrate (7) by a simple example.

3.1. Example 4

$$Min\ x'Ax,\ \ s.t.\ \ (1\ 0\ 0)x = 1,\ \ d' = (1,0,0), \tag{38}$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, (A + dd') = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{39}$$

and $(A + dd')$ is not a full rank matrix, hence we choose $C = (0,1,-1)$ orthogonal to d, such that $Cx = 0$ implies $CAx = 0$, and choose $\theta = 2$, and find that

$$\begin{aligned} G &= (A + \theta dd' + CC') \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \end{aligned} \tag{40}$$

is a full rank matrix, and (38) now is

$$Min\ x'Gx,\ \ s.t. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \ Dx = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the solution is

$$x = G^{-1}D'(DG^{-1}D')^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now note that in (32) there are two dealers and only one destination which desires four units of items, the first dealer has 3 items, and second dealer has two items. The cost of transportation of first dealer is twice of the second dealer, and so to minimize the cost, the destination buys 2 items from the first dealer and two items from the second dealer.

We mention that in place of G of (40), neither the generalized invers of $(A + dd')$ of (39), nor the Moore-Penrose inverse of $(A + dd')$ of (39), if used in place of G of (40), will give the correct answer to (40). If $Cx = 0$, does not imply $CAx = 0$, then G of (40) becomes

$$G = (A + \theta dd' + CC' + CAAC'),\ \ s.t.\ \ d'x = v, Cx = 0, CAx = 0. \tag{41}$$

We mention that the problem Arthari and Dodge [5 (p.241, problem 5.4.2, and p.251, equation 5.5.2)] are the problems of the type solved in this paper. The problem of the type Arthari and Dodge [5 (p.148, problem 7.3.1)] are solved by using equations (40), (41).

References

[1] D.G. Kabe, Solving Matrix Quadratic Programming Problems. *Journal of Mathematical Analysis and Application,* 161, 212-217, (1991).

[2] D.G. Kabe, Solving Linear and Quadratic Programming Problems. *Industrial Mathematics.* 42, 67-78, (1992).

[3] D.G. Kabe, A Quadratic Programming Approach to a Finite Sampling Problem. *Parisankhyam Samitikha*, 2, 1-8, (1995).

[4] P. Scobey and D.G. Kabe, Direct Solutions to Some Multidimensional Transportation Problems. *Transportation Science*, 15, 1-15, (1981).

[5] T.S. Arthari and Yadolah Dodge, *Mathematical Programming in Statistics*. John Wiley, New York, 1981.