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A LAGRANGEAN RELAXATION APPROACH FOR MULTI PRODUCT, MULTI ECHELON INVENTORY SYSTEMS WITH CAPACITATED DYNAMIC LOTSIZING

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Abstract: This paper focuses on multi-echelon inventory systems having an arborescent structure. In the structure each intermediate facility has exactly one predecessor and possibly several successors. All inventory costs are assumed linear with ordering cost that is independent of the order quantity for each stocking point. The model takes account of dynamic cost structure and dynamic demand pattern as well as capacity limitations. The paper exploits a mixed bivalent programming model to determine what inventory levels, if any, should be maintained at the various stocking points in order to minimise total inventory cost of the entire system. A computationally efficient Lagrangean relaxation-based procedure is developed to decompose the model into submodels by each stocking point and product.

Subject Classifications: Inventory/Production: Multi-Echelon Lot-Sizing. Programming: Integer, Branch and Bound, Lagrangean Relaxation.
Key Words: Inventory, Integer Programming, Modelling, Production.

1. INTRODUCTION

A multi-echelon or organisational hierarchy is a very common type of hierarchy. In reality, multi-echelon structure exists in any complex system. An attribute specific for multi-echelon systems lies in the partially conflicting goals and objectives between decision problems on different echelons. These partial conflicts are not only a result of the composition of the multi-echelon system, but are also necessary for efficient functioning of the overall system. In this sense, most actual inventory problems of significance have multi-echelon aspects.

The most common notion of a multi-echelon inventory system is one involving a number of retail outlets in business to satisfy customer demands for goods and which, in turn, act as customers of higher-level wholesale activities. The wholesale activities themselves may be customers of still higher-level wholesale activities or production facilities. The customer demands occur only at the stocking points in the lowest echelon. Echelon n has its stocks replenished by shipments from the echelon n+1. In this paper we shall consider the problem of the determining optimal purchasing quantities in a multi-echelon system of this type.

A sequence of purchasing decision is made at the beginning of a number of regularly spaced intervals. The cost of purchasing any amount of goods will be a constant value for each product during each period. The model can be extended to handle cases in which there are delivery lead times in moving batches between stocking points. During each period the stock on hand is depleted by an amount equal to the demand during that period, which is known at t=0 (i.e., dynamic deterministic demand).

In addition to the ordering cost, it is customary to charge several other costs during each period. The first of these costs is the holding cost proportional to the stock on hand at the end of the period if it is positive; and the other is the penalty cost proportional to the deficit of available stock also at the end of the period if there is such a deficit. The crucial point about penalty cost is that only the stocking points in the first echelon (i.e., the lowest echelon) incur this cost. The penalty cost is infinite for the other stocking points. In this regard, it is assumed that all excess demand for the first echelon is backlogged. However, the backlogging of the demand for other echelons is not permitted. Otherwise, echelon j+1 supplies echelon j in spite of the fact that echelon j+1 is out of stock. This would permit increase in the amount of inventory in echelon j without receiving goods from echelon j+1. The backlogging in echelons 2,...,M allows this to occur. In many inventory problems the above situations are meaningless because echelon j could not meet the demand without the inputs from echelon j+1.

Another aspect of the model is that it permits capacity limitations for each stocking point and in consequence the trade-off between the storage of different types of products at each stocking point is considered. The capacity limitations at various stocking points may be different from each other and that gives more flexibility to the model.

The paper is organised as follows: In §2 the previous work on deterministic multi-echelon problem is summarised. In §3 the model notation is presented. The mixed bivalent programming formulation of the problem is given in §4 and the reformulation of the problem in terms of echelon stocks is given in §5. §6 is devoted to the decomposition of the model into smaller subproblems by means of the Lagrangean relaxation method. Finally, conclusions and directions for further research are presented in §7.

2. PREVIOUS WORK

From computational point of view, lotsizing problems in multi-stage systems seem to be extremely difficult, mainly due to their complex combinatorial structure. An attempt was made to solve these problems with an integer programming method by McLaren[1]; he used a general mixed integer programming code and failed to solve problems with more than three facilities and twelve periods. This lack of success was in part due to the fact that the formulation used were intractable for conventional Integer Programming techniques. Following that failure, Afentakis et al.[2] presented a linear transformation of that formulation and showed that the lot-scheduling problem can be formulated in terms of its "echelon stocks." In their model, two costs are incurred in the production process: set-up and holding costs. No backlogging is permitted and capacity limitations are not considered. Their new formulation leads to a straightforward decomposition of the problem using Lagrangean relaxation methods. The Lagrangean problem is solved by a subgradient optimisation procedure. The sharp bounds obtained are subsequently used in developing an efficient Branch and Bound algorithm. In this regard, this paper is an extension of the aforementioned study of Afentakis et al.

Some other papers on this topic are presented by Crowston and Wagner [3] and Graves [4]. A detailed literature review of multi-echelon inventory systems is given by Tarim[5].

3. MODEL NOTATION

Consider a particular echelon m of the multi-echelon structure. Let the number of stocking points at this echelon be N_m. For each of these stocking points, we define S^{im}_n to be the set of descending ($n<m$) or immediate ascending ($n=m+1$) stocking points that are connected to the ith ($i:1,\ldots,N_m$) stocking point of the mth echelon. G(i,m) is the set of all successors of the stocking point i in the mth echelon (i.e., $G(i,m)=S^{im}_{\{j|j<m\}}$). V(i,m) is the set of all stocking points that are in the first echelon and originate from stocking point i of the mth echelon. This multi-echelon structure is illustrated in the figure.

The indices, used in modelling process, are i, j, k, and l, which denote stocking point, echelon, period and product respectively. Additional notation associated with the multi-echelon structure includes:

α : a constant value greater than the total demand of each stocking point,
T : total number of planning periods,
r : total number of products stored in the multi-echelon inventory system,
M : total number of echelons,
a_l : volume per item,
β_{ijk} : Lagrange multiplier used for relaxing volume constraint set,

γ^l_{ijk} : Lagrange multiplier used for relaxing nonnegativity of echelon stocks,

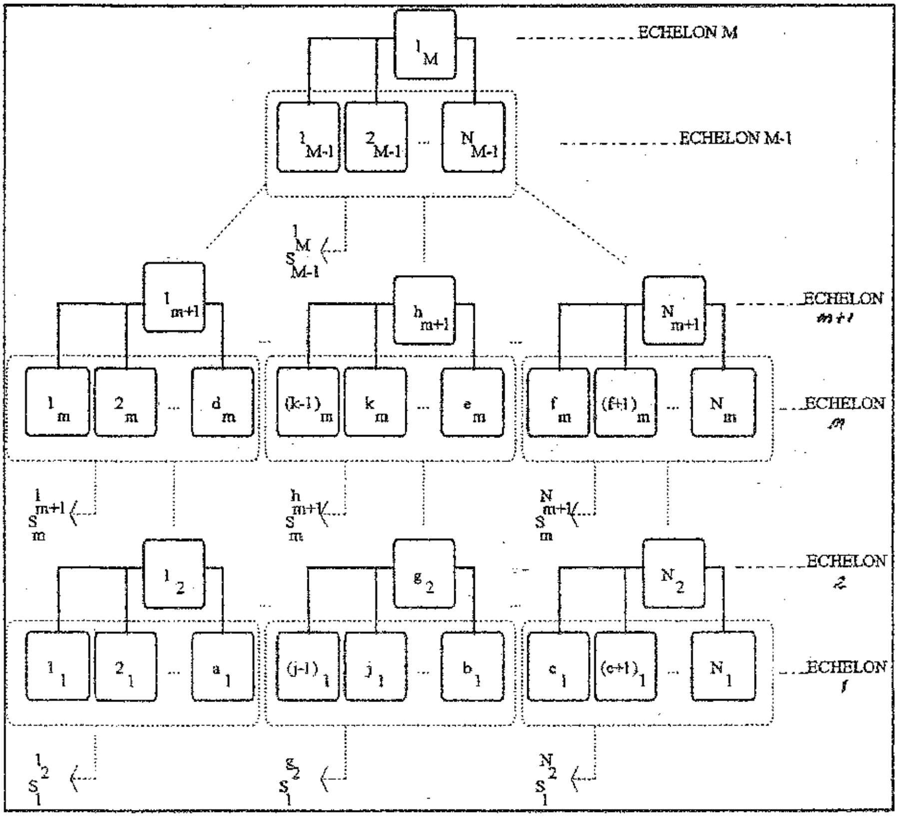

Figure - Multi-Echelon Inventory System

p^l_{ik} : penalty cost per unit of inventory shortage at end of any period,
h^l_{ijk} : cost to carry a positive unit of inventory from period i to period i+1,
e^l_{ijk} : "echelon stock" holding cost (clearly defined in §5),
E^l_{ijk} : "echelon stock" (clearly defined in §5),
K^l_{ijk} : fixed procurement cost per order,
C_{ijk} : capacity limitation in volume,
D^l_{ik} : instantaneous external deterministic demand,
X^l_{ijk} : $-\infty<X^l_{ijk}<\infty$ (j=1); $X^l_{ijk}\geq 0$ (j=2,...,M), inventory level just before

the delivery of the orders (i.e., the inventory level at the end of period k-1),

U^l_{ijk} : amount ordered; from the definitions of U^l_{ijk} and D^l_{ik}, $U^l_{i0k}=D^l_{ik}$,

Z^l_{ik} : $Z^l_{ik}\geq 0$ ($i=1,...,N_1$), level of excess closing inventory,

Y^l_{ik} : $Y^l_{ik}\geq 0$ ($i=1,...,N_1$), amount of inventory shortage at the end of period k,

δ^l_{ijk} : $\delta^l_{ijk}=0$ if no order is placed or $\delta^l_{ijk}=1$ if order is placed at the beginning of period k.

4. THE MIXED BIVALENT PROGRAMMING MODEL

The formulation of the problem is given in four steps. The first step deals with the formulation of the rudimentary constraints of the system. The second step is for the determination of the constraints for the lowest, i.e., first echelon. In the third step, expressions for the echelon 2,...,M are presented. Finally, the fourth step aims at tackling the objective function.

Step 1

At each stocking point, the inventory level at the beginning of period k+1 equals inventory level augmented by the amount of the order delivered at the beginning of period k and depleted by an amount equal to the total demand during the period k.

$$X^l_{ij(k+1)} = X^l_{ijk} + U^l_{ijk} - \sum_{m_{ij}\in S^{ij}_{j-1}} U^l_{m_{ij}k} \tag{1}$$

$$(i=1,...,N_j\ ;\ j=1,...,M\ ;\ k=1,...,T\ ;\ l=1,...,r)$$

At each stocking point, the total inventory volume could not exceed the capacity limitations during the planning period (k=1,...,T). Since the total inventory volume may expand only after the delivery of orders in each period, controlling the total inventory level only after the delivery in each period for each stocking point is sufficient to constraint the volume with the capacity limitations. The constraint set for echelons j=2,...,M is given below; case for echelon 1 is considered in Eq. 6.

$$\sum_{l=1}^{r} a_l (X_{ijk}^l + U_{ijk}^l) \le C_{ijk} \tag{2}$$

$$(i=1,\dots,N_j \ ; \ j=2,\dots,M \ ; \ k=1,\dots,T)$$

As it has been mentioned before, the ordering cost is assumed to be independent of order quantity. In other words, the ordering cost is considered as a constant value. The model allows different ordering costs for different periods at different stocking points. These relations are included in the model by means of the following inequality.

$$\frac{U_{ijk}^l}{\alpha} \le \delta_{ijk}^l \tag{3}$$

$$(i=1,\dots,N_j \ ; \ j=1,\dots,M \ ; \ k=1,\dots,T \ ; \ l=1,\dots,r)$$

From the definition of δ^l_{ijk}, if U^l_{ijk} gets any value different from zero then δ^l_{ijk} equals 1, which means that an order is placed and the ordering cost is incurred. Otherwise, δ^l_{ijk} equals 0, which means that no order is placed so no ordering cost is incurred. It is obvious that α must be at least the total of U^l_{ijk} for k=1 to T, to ensure that the ratio is not greater than 1.

Step 2

It's crucial to distinguish between the amount of out-of-stock and stock on hand at the end of the each period for the stocking points in the first echelon. These two quantities determine the inventory costs for the first echelon except the ordering cost. The two variables, Z^l_{ik} and Y^l_{ik} are used to monitor the aforementioned inventory levels.

$$Z_{ik}^l \ge X_{i1k}^l + U_{i1k}^l - D_{ik}^l \tag{4}$$

$$(i=1,\dots,N_1 \ ; \ k=1,\dots,T \ ; \ l=1,\dots,r)$$

$$Y_{ik}^l \ge D_{ik}^l - X_{i1k}^l - U_{i1k}^l \tag{5}$$

$$(i=1,\dots,N_1 \ ; \ k=1,\dots,T \ ; \ l=1,\dots,r)$$

As it can be seen easily, since Z^l_{ik} and Y^l_{ik} are non-negative variables, at a time only one of them takes a non-zero value and the other is netted out. The following inequality is the capacity limitation for the first echelon:

$$\sum_{l=1}^{r} a_l \left(Z_{i1k}^{l}+U_{i1k}^{l}\right) \le C_{i1k} \qquad (i=1,\dots,N_1 \;;\; k=1,\dots,T) \qquad (6)$$

Step 3

An aforementioned aspect of the model is that it allows backlogging only at the first echelon -in order to be meaningful. The following inequality assures that the inventory levels at the echelons, except the first, is non-negative.

$$\sum_{m_{ij}\in S_{j-1}^{ij}} U_{m_{ij}k}^{l} \le X_{ijk}^{l}+U_{ijk}^{l} \qquad (7)$$

$$(i=1,\dots,N_j \;;\; j=2,\dots,M \;;\; k=1,\dots,T \;;\; l=1,\dots,r)$$

Actually, from the definition of variable X^l_{ijk} $j=2,\dots,M$ in which X^l_{ijk} is forced to be nonnegative, Eq.7 is redundant and can be omitted. However, for the sake of future reference and clarity it is included in the model explicitly.

Step 4

The last step is the determination of the objective function. The following expression comprises three cost components. These are total ordering, holding (for echelons 2,...,M and echelon 1) and penalty costs (for echelon 1) respectively from left to right.

Minimise

$$\sum_{l=1}^{r}\sum_{k=1}^{T}\left\{\sum_{j=1}^{M}\sum_{i=1}^{N_j} K_{ijk}^{l}\delta_{ijk}^{l}+\sum_{j=2}^{M}\sum_{i=1}^{N_j} h_{ijk}^{l}X_{ijk}^{l}+\sum_{i=1}^{N_1} h_{i1k}^{l}Z_{ik}^{l}+p_{ik}^{l}Y_{ik}^{l}\right\}$$

(8)

The objective function completes the model. The entire model is given below for the sake of convenience.

Mixed Bivalent Programming Model

Minimise

$$\sum_{l=1}^{r}\sum_{k=1}^{T}\left\{\sum_{j=1}^{M}\sum_{i=1}^{N_j} K_{ijk}^{l}\delta_{ijk}^{l}+\sum_{j=2}^{M}\sum_{i=1}^{N_j} h_{ijk}^{l}X_{ijk}^{l}+\sum_{i=1}^{N_1} h_{i1k}^{l}Z_{ik}^{l}+p_{ik}^{l}Y_{ik}^{l}\right\}$$

Subject To

$(k=1,\ldots,T;\ l=1,\ldots,r;\ i=1,\ldots,N_j)$

$$(1)\quad X_{ij(k+1)}^{l}=X_{ijk}^{l}+U_{ijk}^{l}-\sum_{m_{ij}\in S_{j-1}^{ij}} U_{m_{ij}k}^{l} \qquad (j=1,\ldots,M)$$

$$(2)\quad \sum_{l=1}^{r} a_l\,(X_{ijk}^{l}+U_{ijk}^{l})\le C_{ijk} \qquad (j=2,\ldots,M)$$

$$(3)\quad \frac{U_{ijk}^{l}}{\alpha}\le\delta_{ijk}^{l} \qquad (j=1,\ldots,M)$$

$$(4)\quad Z_{ik}^{l}\ge X_{i1k}^{l}+U_{i1k}^{l}-D_{ik}^{l} \qquad (i=1,\ldots,N_1)$$

$$(5)\quad Y_{ik}^{l}\ge D_{ik}^{l}-X_{i1k}^{l}-U_{i1k}^{l} \qquad (i=1,\ldots,N_1)$$

$$(6)\quad \sum_{l=1}^{r} a_l\,(Z_{ik}^{l}+U_{i1k}^{l})\le C_{i1k} \qquad (i=1,\ldots,N_1)$$

$$(7)\quad \sum_{m_{ij}\in S_{j-1}^{ij}} U_{m_{ij}k}^{l}\le X_{ijk}^{l}+U_{ijk}^{l} \qquad (j=2,\ldots,M)$$

$$U_{ijk}^{l}\ge 0\ ;\ -\infty> X_{i1k}^{l}>\infty\ ;\ Z_{ik}^{l}\ge 0\ ;$$

$$Y_{ik}^{l}\ge 0\ ;\ \delta_{ijk}^{l}=\{0,1\}\ ;\ \underset{j=2,\ldots,M}{X_{ijk}^{l}}\ge 0$$

5. AN ALTERNATIVE FORMULATION

The essential innovation of this alternative formulation is the interpretation of the inventory system as a nested set of echelons (i.e., in terms of "echelon" stocks and "echelon" holding costs) rather than as individual activities. The model associates, with each activity, an echelon consisting of all stock in the system at that activity and below, including all on-hand and in-transit amounts. With this interpretation, the multi-state variable problem for the system as a whole can be decomposed into a set of interconnected one-state variable problems, one for each echelon in the system. The echelon stock and echelon holding cost concepts are first introduced by Clark and Scarf [6] and used by many authors (see for example Blackburn and Millen [7], Crowston et al.[8,9], Schwarz and Schrage [10]). The echelon stock for product l at stocking point i in the echelon j during the period k is denoted by E^l_{ijk}, and e^l_{ijk} is the corresponding echelon holding cost. The definitions of E^l_{ijk} and e^l_{ijk} are as follows:

$$(j=1,\dots,M\ ;\ i=1,\dots,N_j)$$

$$e^l_{ijk}=h^l_{ijk}-\left\{h^l_{m_{ij}k}\,\middle|\,m_{ij}\in S^{i_j}_{j+1}\right\} \tag{9}$$

$$E^l_{ijk}=\left\{X^l_{ijk}\middle|j>1\,,\,Z^l_{ik}\middle|j=1\right\}+\sum_{m_{ij}\in G(i,j)}\left\{X^l_{m_{ij}k}\middle|m_{ij}\notin V_{(i,j)}\,,\,Z^l_{m_{ij}k}\middle|m_{ij}\in V_{(i,j)}\right\} \tag{10}$$

Using the above linear transformations, without loss of generality, the mixed bivalent programming model of §4 can be written as below. The concept behind this transformation is known in the MRP literature as "explosion" (see Afentakis [11]).

The Eq.(A1) is the immediate result of Eq.1; however, the lowest echelon is not considered as a result of the unboundedness of the inventory levels in the first echelon. This bit is considered in Eqs.(A4), (A5), and (A6). The non-negativity constraint of inventory levels given in Eq.7 yields Eq.(A2). The Eqs.(A3), (A4), and (A5) are the same as the Eqs.3, 4, and 5 respectively except the consideration of the echelon stocks instead of the stocks of the individual stocking points. The last, but certainly not the least, constraint set (i.e., Eq.(A7)) is obtained by substituting Eq.10 in Eqs.2 and 6, and rearranging it. It is clear that the objective function is an immediate result of Eq.8 and Eqs.9 and 10.

Alternative Formulation

Minimise

$$\sum_{l=1}^{r}\sum_{k=1}^{T}\left\{\sum_{j=1}^{M}\sum_{i=1}^{N_j}(K_{ijk}^{l}\delta_{ijk}^{l}+e_{ijk}^{l}E_{ijk}^{l})+\sum_{i=1}^{N_1}p_{ik}^{l}Y_{ik}^{l}\right\}$$

Subject To

$(i=1,\ldots,N_j\ ;\ k=1,\ldots,T\ ;\ l=1,\ldots,r)$

$$E_{ij(k+1)}^{l}=E_{ijk}^{l}+U_{ijk}^{l}-\sum_{m_{ij}\in S_1^{ij}}D_{m_{ij}k}^{l} \qquad (j=2,\ldots,M) \qquad (A1)$$

$$-E_{ijk}^{l}+\sum_{m_{ij}\in S_{j-1}^{ij}}E_{m_{ij}k}^{l}\le 0 \qquad (j=1,\ldots,M) \qquad (A2)$$

$$\frac{U_{ijk}^{l}}{\alpha}\le\delta_{ijk}^{l} \qquad (j=1,\ldots,M) \qquad (A3)$$

$$E_{i1k}^{l}\ge X_{i1k}^{l}+U_{i1k}^{l}-D_{ik}^{l} \qquad (i=1,\ldots,N_1) \qquad (A4)$$

$$Y_{i1k}^{l}\ge D_{ik}^{l}-X_{i1k}^{l}-U_{i1k}^{l} \qquad (i=1,\ldots,N_1) \qquad (A5)$$

$$X_{i1(k+1)}^{l}=X_{i1k}^{l}+U_{i1k}^{l}-D_{ik}^{l} \qquad (i=1,\ldots,N_1) \qquad (A6)$$

$$\sum_{l=1}^{r}a_l(E_{ijk}^{l}-\sum_{m_{ij}\in S_{j-1}^{ij}}E_{m_{ij}k}^{l}+U_{ijk}^{l})\le C_{ijk} \qquad (j=1,\ldots,M) \qquad (A7)$$

6. LAGRANGEAN RELAXATION

One of the most computationally useful ideas of the 1970s is the observation that many hard problems like the alternative model can be viewed as easy problems complicated by a relatively small set of side constraints. Dualizing the side constraints produces a Lagrangean problem that is easy to solve and whose optimal value is a lower bound on the optimal value of the original problem. The Lagrangean problem can thus be used in place of a linear programming relaxation to provide bounds in a Branch and Bound algorithm. The birth of the Lagrangean approach as it exists today occurred in 1970 when Held and Karp [12] used a Lagrangean problem based on minimum spanning trees to devise a dramatically successful algorithm for the travelling salesman problem. Motivated by Held and Karp's success

Lagrangean methods were applied in the early 1970s to scheduling problems (see Fisher [13]) and the general Integer Programming problem (Shapiro [14], Fisher and Shapiro [15]). Lagrangean methods had gained considerable currency by 1974 when Geoffrion [16] coined the perfect name for this approach -"Lagrangean Relaxation." The reader is referred to Geoffrion [16], Fisher [17], [18] and Shapiro [19] for theory and survey of Lagrangean relaxation.

In this section, Lagrangean relaxation is used to decompose the alternative formulation into smaller subproblems. Multiplying the second and last constraint sets by Lagrange multiplier vectors $\gamma^{l}_{ijk}\geq 0$ and $\beta_{ijk}\geq 0$ respectively and adding them to the objective function yields the following relaxed problem.

Lagrangean Relaxed Model

Minimise

$$\sum_{k=1}^{T}\sum_{j=1}^{M}\sum_{i=1}^{N_j}\left\{\sum_{l=1}^{r}(K^{l}_{ijk}\delta^{l}_{ijk}+e^{l}_{ijk}E^{l}_{ijk})+(\beta_{ijk}\sum_{l=1}^{r}U^{l}_{ijk}-\beta_{ijk}C_{ijk}) +\beta_{ijk}\sum_{l=1}^{r}(1-\gamma^{l}_{ijk})(E^{l}_{ijk}-\sum_{m_{ij}\in S^{ij}_{j-1}}E^{l}_{m_{ij}k})\right\}+\sum_{l=1}^{r}\sum_{k=1}^{T}\sum_{i=1}^{N_1}p^{l}_{ik}Y^{l}_{ik}$$

Subject To $(i=1,\ldots,N_j\ ;\ k=1,\ldots,T\ ;\ l=1,\ldots,r)$

$$E^{l}_{ij(k+1)}=E^{l}_{ijk}+U^{l}_{ijk}-\sum_{m_{ij}\in V(i,j)}D^{l}_{m_{ij}k} \qquad (j=2,\ldots,M)$$

$$\frac{U^{l}_{ijk}}{\alpha}\leq\delta^{l}_{ijk} \qquad (j=1,\ldots,M)$$

$$E^{l}_{i1k}\geq X^{l}_{i1k}+U^{l}_{i1k}-D^{l}_{ik}\ , \qquad Y^{l}_{i1k}\geq D^{l}_{ik}-X^{l}_{i1k}-U^{l}_{i1k}$$

$$X^{l}_{i1(k+1)}=X^{l}_{i1k}+U^{l}_{i1k}-D^{l}_{ik}$$

Hence the relaxed problem is decomposed into subproblems of the form SP1 and SP2 given below.

Subproblem SP1

$[l=1,\dots,r\ ;\ j=2,\dots,M\ ;\ i=1,\dots,N_j]$

Minimise

$$\sum_{k=1}^{T}\Big(\beta_{ijk}U^l_{ijk}+K^l_{ijk}\delta^l_{ijk}+e^l_{ijk}E^l_{ijk}+\beta_{ijk}(1-\gamma^l_{ijk})E^l_{ijk}-\beta_{m_{ij}k}(1-\gamma^l_{ijk})E^l_{ijk}\ \Big|_{m_{ij}\in S^{ij}_{j+1}}\Big)$$

Subject To $(k=1,\dots,T)$

$$E^l_{ij(k+1)}=E^l_{ijk}+U^l_{ijk}-\sum_{m_{ij}\in V_{(i,j)}}D^l_{m_{ij}k}$$

$$\frac{U^l_{ijk}}{\alpha}\le\delta^l_{ijk}$$

Subproblem SP2

$[l=1,\dots,r\ ;\ i=1,\dots,N_1]$

Minimise

$$\sum_{k=1}^{T}\Big(p^l_{ik}Y^l_{ik}+\beta_{i1k}U^l_{i1k}+K^l_{i1k}\delta^l_{i1k}+e^l_{i1k}E^l_{i1k}+\beta_{i1k}E^l_{i1k}-\beta_{m_{i1}k}E^l_{m_{i1}k}\ \Big|_{m_{i1}\in S^{i_1}_2}\Big)$$

Subject To $(k=1,\dots,T)$

$$X^l_{i1(k+1)}=X^l_{i1k}+U^l_{i1k}-D^l_{ik}$$

$$\frac{U^l_{i1k}}{\alpha}\le\delta^l_{i1k}$$

$$E^l_{i1k}\ge X^l_{i1k}+U^l_{i1k}-D^l_{ik}$$

$$Y^l_{ik}\ge D^l_{ik}-X^l_{i1k}-U^l_{i1k}$$

Hence, SP1 separates by stocking points (except the ones in the lowest echelon) and by products into a smaller mixed bivalent programming subproblems. Each of these subproblems can be solved efficiently using any minimum cost flow network algorithm. Another, more popular approach in the lotsizing literature is to formulate it as shortest path network flow problem and to solve it by dynamic programming (see Zangwill [20]). SP2 is very similar to SP1 except the consideration of the penalty cost. However, SP2 can still be solved by dynamic programming approach of Zangwill which permits backlogging.

One crucial point that should be made clear is the process of determination of Lagrange multipliers, $\pi=\{\gamma l_{ijk},\beta_{ijk}\}$. It is well known that the optimal value of the relaxed problem is less than or equal to the optimal value of the mixed bivalent programming problem. As mentioned before, this fact allows Lagrangean relaxed problem to be used in place of linear programming relaxation to provide lower bounds in a Branch and Bound algorithm. It is clear that the best choice for Lagrange multipliers would be an optimal solution to the dual problem, Z_D, where $Z_D(\pi)$ is the Lagrangean relaxed problem.

$$Z_D = \max_{\pi} Z_D(\pi)$$

Most schemes for determining π have as their objective finding optimal or near optimal solution to the mixed bivalent programming problem. One of these schemes is the subgradient method. The subgradient method is a brazen adaptation of the gradient method in which gradients are replaced by subgradients. Given an initial value π^0 a sequence $\{\pi\}$ is generated by the rule

$$\pi_{(k+1)} = \pi_k + t_k (Ax^k - b)$$

where x^k is an optimal solution to relaxed problem, t_k is a positive scalar step size, and Ax^k-b is the relaxed constraint set. Because the subgradient method is easy to program and has worked well on many practical problems, it has become the most popular method for maximisation of $Z_D(\pi)$. There have also been many papers, such as Camerini et al.[21], which suggest improvements to the basic subgradient method. Computational performance and theoretical convergence properties of the subgradient method are discussed in Held et al.[22] and their references. The fundamental theoretical result is that

$$(t_k \to 0) \quad \wedge \quad (\sum_{i=0}^{k} t_i \to \infty) \quad \Rightarrow \quad Z_D(\pi) \to Z_D$$

The step size used most commonly in practice is

$$t_k = \frac{\lambda_k (Z^* - Z_D(\pi))}{\|Ax^k - b\|^2}$$

where λ_k is a scalar satisfying $0<\lambda_k\leq 2$ and Z^* is an upper bound on Z_D, frequently obtained by applying a heuristic to the problem under consideration. Justification of this formula is given in Held et al.[22]. Often the sequence λ_k is determined by setting $\lambda_0=2$ and halving λ_k whenever $Z_D(\pi)$ has failed to increase in some fixed number of iterations. Unless we obtain a π for which $Z_D(\pi)$ equals the cost of a known feasible solution, there is no way of proving optimality in the subgradient method. To resolve this difficulty, the method is usually terminated upon reaching an arbitrary iteration limit.

7. CONCLUSIONS

Lagrangean relaxation is an important new computational technique in the operational researcher's arsenal. In this paper we have developed algorithms that generate optimal solutions for multi-product, multi-echelon inventory systems with capacitated dynamic lotsizing using Lagrangean relaxation method. Work is now in progress on incorporating stochastic aspects and other ideas such as control theory into an algorithm which will extend the range of problems that can be solved to optimality and extending the applicability of those procedures to other problem classes. Two research areas that deserve further attention are the development and analysis of Lagrangean heuristics and the analysis (worst-case or probabilistic) of the quality of bounds produced by Lagrangean relaxation.

REFERENCES

[1] McLaren, B.J., A Study of Multiple Level Lot Sizing Techniques for Material Requirements Planning Systems, Unpublished Ph.D. Dissertation, Purdue University, 1976.

[2] Afentakis, P., B.Gavish, and U.Karmarkar, "Computationally Efficient Optimal Solutions to the Lot-Sizing Problem in Multistage Assembly Systems," Management Science, Vol.30, 1984, pp.222-239.

[3] Crowston, W.B., and M.H.Wagner, "Dynamic Lot Size Models for Multi-Stage Assembly Systems," Management Science, Vol.20, 1973, pp.14-21.

[4] Graves, S.C., "Multi-Stage Lot Sizing: An Iterative Procedure," in Schwarz, L.B. (ed.), Multi-Level Production/Inventory Control Systems: Theory and Practice, Studies in the Management Science, Vol.16, North-Holland: Amsterdam, 1981.

[5] Tarım, Ş.A., "A Survey of Multi-Echelon Inventory Models,", Hacettepe Üniversitesi, İktisadi ve İdari Bilimler Fakültesi Dergisi, Vol.11, 1993, pp.115-150.

[6] Clark, A.J., and S.Herbert, "Optimal Policies for a Multi-Echelon Inventory Problem," Management Science, Vol.6, 1960, pp.475-490.

[7] Blackburn, J.D., and R.A.Millen, "Improved Heuristic for Multi Stage Requirements Planning Systems," Management Science, Vol.28, 1982,pp.44-56.

[8] Crowston, W.B., and M.H.Wagner, "Dynamic Lot Size Models For Multistage Assembly Systems," Management Science, Vol.20, 1973, pp.14-21.

[9] Crowston, W.B., M.H.Wagner, and J.F.Williams, "Economic Lot Size Determination in Multi-Stage Assembly Systems," Management Science, Vol.19, 1973, pp.517-527.

[10] Schwarz, L.G., and L.Schrage, "Optimal and System Myopic Policies for Multi-Echelon Production/Inventory Assembly Systems," Management Science, Vol.21, 1975.

[11] Afentakis, P., and B.Gavish, Bill of Material Processor Algorithms -Time and Storage Complexity Analysis, Working Paper, The Graduate School of Management, University of Rochester, Rochester, New York, 1983.

[12] Held, M., and R.M.Karp, "The Travelling Salesman Problem and Minimum Spanning Trees," Operations Research, Vol.18, 1970, pp.1138-1162.

[13] Fisher, M.L., "Optimal Solution of Scheduling Problems Using Lagrange Multipliers: Part 1," Operations Research, Vol.21, 1973, pp.1114-1127.

[14] Shapiro, J.F., "Generalized Lagrange Multipliers in Integer Programming," Operations Research, Vol.19, 1971, pp.68-76.

[15] Fisher, M.L., and J.F.Shapiro, "Constructive Duality in Integer Programming," SIAM Journal of Applied Mathematics, Vol.27, 1974, pp.31-52.

[16] Geoffrion, A.M., "Lagrangean Relaxation for Integer Programming," Mathematical Programming Study, Vol.2, 1974, pp.82-114.

[17] Fisher, M.L., "The Lagrangian Relaxation Methods for Solving Integer Programming Problems," Management Science, Vol.27, 1981, pp.1-18.

[18] Fisher, M.L., "An Applications Oriented Guide to Lagrangian Relaxation," Interfaces, Vol.15, 1985, pp.10-21.

[19] Shapiro, J.F., "A Survey of Lagrangean Techniques for Discrete Optimization," Annals of Discrete Mathematics, Vol.5, 1979, pp.113-118.

[20] Zangwill, W.I., "A Backlogging Model and a Multi-Echelon Model of Dynamic Economic Lot Size Production System -A Network Approach," Management Science, Vol.15, 1969, pp.506-527.

[21] Camerini, P.M., L.Fratta, and F.Maffioli, "On Improving Relaxation Methods by Modified Gradient Techniques," Mathematical Programming Study, Vol.3, 1975, pp.26-34.

[22] Held, M., P.Wolfe, and H.D.Crowder, "Validation of Subgradient Optimization," Mathematical Programming, Vol.6, 1974, pp.62-88.