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A SEMİ-SYMMETRIC CONNECTION ON A RIAMANNIAN MANIFOLD

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ABSTRACT

We define a linear Connection on a Riemannian manifold which is semi-symmetric but non-metric and study some properties of the curvature tensor and Weyl projective curvature tensor with respect to semi-symmetric connection.

ÖZET

Bir Riemann Manifoldu Üzerinde semi-symmetric non-metric lineer konneksiyon tanımlamak suretiyle eğrilik tensörünün ve Weyl projektif eğrilik tensörünün bazı özellikleri incelendi.

1. SEMİ-SYMMETRIC NON-METRİK CONNECTION

Let M be a n-dimensional Riemannian manifold with Riemannian metric g. I define a linear connection $\overset{*}{\nabla}$ on a Riemannian manifold M by

$$\overset{*}{\nabla}_x y = \nabla_x y + w(y)x \qquad (1.1.)$$

where ∇ is the Riemannian connection on M and w is a 1-form associated with the vector field u on M by

$$w(x) = g(u,x) = \langle u,x \rangle. \qquad (1.2)$$

Using (1.1), the torsion tensor $\overset{*}{T}$ of M with respect to connection $\overset{*}{\nabla}$ is given by

$$\overset{*}{T}(x,y) = \overset{*}{\nabla}_x y - \overset{*}{\nabla}_y x - [x,y]$$

$$= \nabla_x y + w(y)x - \nabla_y x - w(x)y - [x,y]$$

$$= w(y)x - w(x)y. \qquad (1.3)$$

A linear connection satisfying (1.3) is called a semi-symmetric connection [1].

Further, Using (1.1), we have,

$$\overset{*}{\nabla}_x(g(y,z) = (\overset{*}{\nabla}_x g)(y,z) + g(\overset{*}{\nabla}_x y,z) + g(y,\overset{*}{\nabla}_x z)$$

$$= (\overset{*}{\nabla}_x g(y,z) + \nabla_x(g(y,z)) + w(y)g(x,z) + w(z)g(x,y)$$

which implies

$$(\overset{*}{\nabla}_x g)(y,z = -w(y)g(x,z) - w(z)g(x,y) \qquad (1.4)$$

for vector fields x, y, z on M.

A linear connection $\overset{*}{\nabla}$ defined by (1.1) satisfies (1.3) and (1.4) and hence we call $\overset{*}{\nabla}$ a semi-symmetric non-metric connection.

Conversely, we will show that a linear connection satisfying (1.3) and (1.4) is defined by (1.1).

Let $\overset{*}{\nabla}$ be a linear connection defined on M by

$$\overset{*}{\nabla}_X y = \nabla_X y + T(x,y) \tag{1.5}$$

where ∇ is the Riemannian connection and T is a tensor of type (1.2) defined on M and $\overset{*}{\nabla}$ satisfies (1.3) and (1.4).

From (1.4) and (1.5), we have

$$\overset{*}{\nabla}_X (g(y,z)) = g(\overset{*}{\nabla}_X y, z) + g(y, \overset{*}{\nabla}_X z) - w(y)g(x,y) -$$

w(z)g(x,y)

$$= \nabla_X (g(y,z)) = g(T(x,y),z) + g(T(x,z),y)$$
$$-w(y)g(x,z) - w(z)g(x,y).$$

which implies

$$g(T(x,y),z) + g(T(x,z),y) = w(y)g(x,z) + w(z)g(x,y) \tag{1.6}$$

On the other hand, from (1.5), we have

$$\overset{*}{T}(x,y) = \overset{*}{\nabla}_X y - \overset{*}{\nabla}_y x - [x,y]$$
$$= T(x,y) - T(y,x). \tag{1.7}$$

From (1.2), (1.6) and (1.7), we have

$$g(\overset{*}{T}(x,y),z) + g(\overset{*}{T}(z,x),y) + g(\overset{*}{T}(z,y),x)$$

$$= g(T(x,y) - T(y,x),z) + g(T(z,x) - T(x,z),y)$$
$$+ g(T(z,y) - T(y,z),x)$$

$$= g(T(x,y),z) - g(T(y,x),z) + g(T(z,x),y) + g(T(x,z),y)$$

$$+g(T(z,y),x) - g(T(y,z),x)$$
$$= 2\{g(T(x,y),z) - w(z)g(x,y)\}$$

$$= 2\{g(T(x,y),z) - g(z,u)g(x,y)\} \quad (1.8)$$

Hence we obtain

$$T(x,y) = \frac{1}{2}\{\overset{*}{T}(x,y) + \overset{*}{T}(x,y) + \overset{*}{T}(y,x)\} + g(x,y)u \quad (1.9)$$

where the tensor $\overset{*}{T}$ of type (1.2) is defined on M by

$$g(\overset{*}{T}(z,x),y) = g(\overset{*}{T}(x,y),z) \quad (1.10)$$

From (1.3) and (1.10), we have

$$g(\overset{*}{T}'(x,y),z) = g(w(x)z - w(z)x, y)$$
$$= w(x)g(z,y) - w(z)g(x,y) \quad (1.11)$$

which implies

$$\overset{*}{T}(x,y) = w(x)y - g(x,y)u. \quad (1.12)$$

Hence from (1.3), (1.9) and (1.12), we have

$$T(x,y) = \frac{1}{2}\{w(y)x - w(x)y + w(x)y - g(x,y)u + w(y)x$$
$$-g(y,x)u\} + g(x,y)u$$
$$= w(y)x. \quad (1.13)$$

on M. Hence from (1.5) and (1.13), we obtain

$$\overset{*}{\underset{X}{\nabla}} y = \underset{X}{\nabla} y + w(y)x.$$

Further, for a 1-form U on M, we have

$$\overset{*}{\underset{X}{\nabla}}(U(y)) = (\overset{*}{\underset{X}{\nabla}}U)y + U(\overset{*}{\underset{X}{\nabla}}y)$$
$$= (\overset{*}{\underset{X}{\nabla}}U)y + U(\underset{X}{\nabla}y) + w(y)U(x)$$

$$= (\overset{*}{\underset{X}{\nabla}}U)y + \underset{X}{\nabla}(U(y) - (\underset{X}{\nabla}U)y + w(y)U(x)$$

which implies

$$(\overset{*}{\underset{X}{\nabla}}U)y = (\underset{X}{\nabla}U)y - w(y)U(x) \tag{1.14}$$

for vector fields x,y on M.

Convariant differentiation of the torsion tensor $\overset{*}{T}$ is given by

$$(\overset{*}{\underset{X}{\nabla}}\overset{*}{T})(y,z) = \overset{*}{\underset{X}{\nabla}}(\overset{*}{T}(y,z)) - \overset{*}{T}(\overset{*}{\underset{X}{\nabla}}y,z) - \overset{*}{T}(y,\overset{*}{\underset{X}{\nabla}}z)$$
$$= ((\overset{*}{\underset{X}{\nabla}}w)z)y - ((\overset{*}{\underset{X}{\nabla}}U)y)z. \tag{1.15}$$

Further we define

$$\overset{*}{T}(x,y,z) = \overset{*}{g}(T(x,y),z) \tag{1.16}$$

From (1.3) and (1.16), we have

$$\overset{*}{T}(x,y,z) + \overset{*}{T}(y,z,x) + \overset{*}{T}(z,x,y) = 0. \tag{1.17}$$

This identify is true for any semi-symmetric connection on M.

2. CURVATURE TENSOR OF M WITH RESPECT TO SEMI-SYMMETRIC CONNECTION

Analogous to the definition of curvature tensor of a Riemannian manifold M with respect to the Riemannian connection ∇, we define the curvature tenior of M with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ by

$$\overset{*}{R}(x,y,z)=\overset{*}{\nabla}_x\overset{*}{\nabla}_y z-\overset{*}{\nabla}_y\overset{*}{\nabla}_x z-\overset{*}{\nabla}_{[x,y]} z. \qquad (2.1)$$

From (1.5) and (2.1), we have

$$\overset{*}{R}(x,y)z=\overset{*}{\nabla}_x(\nabla_y z+w(z)y)-\overset{*}{\nabla}_y(\nabla_x z+w(z)x)$$
$$-\nabla_{[x,y]}z-w(z)[x,y]$$
$$=\nabla_x(\nabla_y z+w(z)y)+w(\nabla_y z+w(z)y)x$$
$$-\nabla_y(\nabla_x z+w(z)x-w(\nabla_x z+w(z)x)y$$
$$-\nabla_{[x,y]}z-w(z)[x,y]$$

$$=R(x,y)z+s(x,z)y-s(y,z)x-s(y,z)x \qquad (2.2)$$

where

$$R(x,y)z=\nabla_x\nabla_y z-\nabla_y\nabla_x z-\nabla_{[x,y]} z.$$

is the curvature tensor of a Riemannion manifold M with respect to the Riemannion Connection ∇ and S is a tensor of type (0,2) defined on M by

$$S(x,y)=(\nabla_x w)y-w(x)w(y)$$
$$=(\overset{*}{\nabla}_x w)y. \qquad (2.3)$$

A relation between the curvature tensors of M with respect to the semi-symmetric non-metric Connection $\overset{*}{\nabla}$ and the Riemannion Connection ∇ is given by (2.2).

Using (2.3), we have

$$\begin{aligned} S(x,y)-S(y,x) &= (\overset{*}{\nabla}_x w)y-(\overset{*}{\nabla}_y w)x \\ &= xw(y)-yw(x)-w[x,y] \\ &= dw(x,y) \end{aligned} \tag{2.4}$$

Thus a tensor S is symmetric iff the 1-form w is closed. Let

$$R(x,y,z,w)=g(R(x,y)z,w)$$

and

$$\overset{*}{R}(x,y;z,w)=g(\overset{*}{R}(x,y)z,w) \tag{2.5}$$

for vector fields x,y,z,w on M. From (2.2) and (2.5),we have

$$\begin{aligned} \overset{*}{R}(x,y,z,w) &= R(x,y,z,w)+S(x,z)g(y,w) \\ &\quad -S(y,z)g(x,w) \end{aligned} \tag{2.6}$$

From (2.1) and (2.5) we have

$$\overset{*}{R}(x,y,z,w)+\overset{*}{R}(x,y,z,w)=0. \tag{2.7}$$

Using (2.2),(2.6) and first Bianchi identify with respect to the Riemannian Connection, we have

$$\begin{aligned} &\overset{*}{R}(x,y)z+\overset{*}{R}(y,z)x+\overset{*}{R}(z,x)y \\ &\quad =\{S(z,y)-S(y,z)\}x+\{S(x,z)-S(z,x\}y \\ &\quad +\{S(y,x)-S(x,y)\}z \end{aligned} \tag{2.8}$$

and hence

$$\overset{*}{R}(x,y,z,w)+\overset{*}{R}(y,z,x,w)+\overset{*}{R}(z,x,y,w)$$

$$= \{S(z,y) - S(y,z)\}g(x,w) + \{S(x,z) - S(z,x)\}g(y,w) + \{S(y,x) - S(x,y)\}g(z,w)$$

We call (2.8) as the first Bianchi identify with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$.

In particular, if the 1-form w is closed, then (2.8) reduces to

$$\overset{*}{R}(x,y)z + \overset{*}{R}(y,z)x + \overset{*}{R}(z,x)y = 0. \quad (2.9)$$

Using (2.6), we have

$$\overset{*}{R}(x,y,z,w) + \overset{*}{R}(x,y,w,z) = S(x,z)g(y,w) + S(x,w)g(y,z) - S(y,z)g(x,w) - S(y,w)g(x,z). \quad (2.10)$$

and

$$\overset{*}{R}(x,y,z,w) - \overset{*}{R}(z,w,x,y) = \{S(x,z) - S(z,x)\}g(y,w) + S(w,x)g(y,z) - S(y,z)g(x,w). \quad (2.11)$$

Using (1.1), (2.7) and the second Bianchi identify for the Riemannion Connection, we obtain the second Bianchi identify associated with semi-symmetric non-metric connection which is given by

$$(\overset{*}{\nabla}_x \overset{*}{R})(y,z) + (\overset{*}{\nabla}_y \overset{*}{R})(z,x) + (\overset{*}{\nabla}_z \overset{*}{R})(x,y)$$

$$= -\overset{*}{R}(\overset{*}{T}(x,y),z) - \overset{*}{R}(\overset{*}{T}(y,z),x) - \overset{*}{R}(\overset{*}{T}(z,x),y)$$

$$= 2\{w(x)\overset{*}{R}(y,z) + w(y)\overset{*}{R}(z,x) + w(z)\overset{*}{R}(x,y)\} \quad (2.12)$$

Analogous to the definition of Ricci tensor of a Riemannion manifold M with respect to the Reimannion connection ∇, we define Ricci tensor of M with respect to semi-symmetric non-metric Connection $\overset{*}{\nabla}$ by.

$$\overset{*}{R}_{ic}(y,z) = \sum_{i=>}^{n} \overset{*}{R}(E_i, y, z, E_i) \quad (2.13)$$

where E_i, ($I \leq i \leq n$) are orthonormal vector fields on M. From (2.5) and (2.13), we have

$$\overset{*}{R}_{ic}(y,z) = \overset{*}{R}_{ic}(y,z) - (n->)S(y,z) \quad (2.14)$$

where

$$\overset{*}{R}_{ic}(y,z) = \sum_{i=1}^{n} R(E_i, y, z, E_i)$$

is Ricci tensor of M with respect to the Riemannian Connection.

A relation between Ricci tensor with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ and the Riemannian Connection ∇ is given by (2.14).

Further, if $\overset{*}{R}_{ic}(x,y) = 0$ on M, then (2.14).

implies is symmetric. From (2.4) and (2.14), we have

$$\overset{*}{R}_{ic}(x,y) - \overset{*}{R}_{ic}(y,x) = -(n->)dw(x,y).$$

Hence, Ricci tensor with respect to semi-symmetric non-metric Connection $\overset{*}{\nabla}$ is symmetric iff the I-form w is closed and hence if S is symmetric.

Using (1.15) and (2.3), we have.

$$(\overset{*}{\nabla}_X \overset{*}{T})(y,x) = S(x,z)y - S(x,y)z. \quad (2.15)$$

In particular, if either the I-form w is closed or Ricci tensor with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ vanishes then from (2.15), we have

$$(\overset{*}{\nabla}_x \overset{*}{T})(y,z)+(\overset{*}{\nabla}_y \overset{*}{T})(z,x)+(\overset{*}{\nabla}_z \overset{*}{T})(x,y)=0. \qquad (2.16)$$

Analogous to the definition of the scalar curvature of a Riemannian manifold M with respect to the Riemannian Connection, we define the scalar curvature of M with respect to semi-symmetric non-metric Connection by

$$\overset{*}{r}=\sum_{i=1}^{n} R_{ic}(E_i,E_i) \qquad (2.17)$$

From (2.14) and (2.17), we obtain a relation between the scalar curvature of M with respect to the Riemmannian Connection and the semi-symmetric non-metric connection which is given by

$$\overset{*}{r}=r-(n-›)\text{traceS} \qquad (2.18)$$

where

$$r=\sum_{i=›}^{n} R_{ic}(E_i,E_i)$$

Is the scalar curvature of M with respect to the Riemannion Connection and s is a tensor of type (1.1) defined on M by

$$s(x,y)=g(Sx,y).$$

3. PROJECTIVE CURVATURE TENSOR OF A RİEMMANNIAN MANIFOLD

Weyl projective curvature tensor of a Riemmanian Manifold M wth respect to the Riemmannian Connection is given by

$$P(x,y)z=R(x,y)z-\frac{1}{n-1}\{\overset{*}{R}_{ic}(y,z)x-\overset{*}{R}_{ic}(x,z)y\} (3.1)$$

(3.1)

$$P(x,y)z = R(x,y)z - \frac{1}{n-1}\{\overset{*}{R}_{ic}(y,z)x - \overset{*}{R}_{ic}(x,z)y\} \quad (3.1)$$

(3.1)

Analogous to fhis definition, we define projective curvature tensor of M with respect to semi-symmetric non-metric connection by

$$\overset{*}{P}(x,y)z = \overset{*}{R}(x,y)z - \frac{1}{n-1}\{\overset{*}{R}_{ic}(y,z)x - \overset{*}{R}_{ic}(x,z)y\}$$

(3.2)

From (2.2), (2.14), (3.1) and (3.2), we have

$$\overset{*}{P}(x,y)z = P(x,y)z \quad (3.3)$$

on M.

Theorem 3.1. If M is a Riemmanian manifold admitting semi-symmetric non-metric connection, then the weyl projective curvature tensor with respect to semi-symmetric non-metric connection is equal to the Weyl projective curvature tensor with respect to Riemannian Connection.

From (3.3), we have, the projective curvature tensor with respect to semi-symmetric non-metric connection satisfies the following algebraic properties

$$\overset{*}{P}(x,y)z + \overset{*}{P}(y,x)z = 0$$

and

$$\overset{*}{P}(x,y)z + \overset{*}{P}(y,z)x + \overset{*}{P}(z,x)y = 0 \quad (3.4)$$

for vector fields x,y,z on M.

In particular, Let M be a Riemmanian Manifold Satisfying

$$\overset{*}{R}(x,y)z = 0 \quad (3.5)$$

which implies

$$\overset{*}{R}_{ic}(y,z)=0 \tag{3.6}$$

on M. From (3.2), (3.3), (3.5) and (3.6), we have

$$P(x,y)z=0$$

on M.

Necessary and sufficient condition for a manifold with a symmetric linear connection to be projectively flat is that the projective curvature tensor with respect to it vanishes identically on a manifold [2].

From (2.3), (2,14) and (3.6), we have

$$(\nabla_x w)y=\frac{1}{n-1}R_{ic}(x,y)+w(x)w(y) \tag{3.7}$$

Using (3.7), we have

$$-w(R(x,y)z)=(\nabla_x\nabla_y w-\nabla_y\nabla_x w-\nabla_{[x,y]}w)z$$

$$=\frac{1}{n-1}((\nabla_x R_{ic})(y,z)-(\nabla_y R_{ic}(x,z)$$

$$+w(y)R_{ic}(x,z)-w(x)R_{ic}(y,z) \tag{3.8}$$

Further, from (3.1), we have

$$w(R(x,y,z)=\frac{1}{n-1}(R_{ic}(x,z)w(x)-R_{ic}(x,z)w(y) \tag{3.9}$$

From (3.8) and (3.9), we have

$$(\nabla_x R_{ic})(y,z)-(\nabla_y R_{ic})(x,z)=0.$$

Theorem 3.2. If M is a Riemmannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M is projektively flat and

$$(\nabla_x R_{ic})(y,z)=(\nabla_y R_{ic})(x,z)$$

on M.

It is well Known that a Riemannian manifold is of constant curvature if it is projectivelly flat and a Riemmanian manifold of constant is Conformally flat [4].

Theorem 3.3. If M is a Riemannian manifold with vanishing curvature tensor, with respect to semi-symmetric non-metric connection, then M is a space of constant curvature and hence is confarmally flat.

From (2.2), (2,15), (3,5) and (3.6), we have

$$R(x,y)z = S(z,y)x - S(z,x)y$$
$$= (\overset{*}{\underset{z}{\nabla}} \overset{*}{T})(x,y). \tag{3.10}$$

Theorem 3.4. If M is a Riemannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M is flat iff

$$(\overset{*}{\underset{x}{\nabla}} T)(y,z) = 0$$

on M.

A Riemannian manifold M is a group manifold [5] with respect to semi-symmetric non-metric connection if

$$\overset{*}{R}(x,y)z = 0$$

and

$$(\overset{*}{\underset{x}{\nabla}} \overset{*}{T})(y,z) = 0 \tag{3.11}$$

on M.

Theorem 3.5. If a Riemannian manifold M is a group manifold with respect to semi-symmetric non-metric connection, then M is flat and consequently M is projectively flat and conformally flat.

In particular if either the 1-form w is closed or Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then from (2.2) and (2.15), we have

$$\overset{*}{R}(x,y)z = R(x,y)z + (\underset{z}{\overset{*}{\nabla}}\overset{*}{T})(y,x). \tag{3.12}$$

Theorem 3.6. Let M be a Riemannian manifold with semi-symmetric non-metric connection. If either the 1-form w is closed or Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then

$$R(x,y)z = \overset{*}{R}(x,y)z + (\underset{z}{\overset{*}{\nabla}}\overset{*}{T})(x,y)$$

on M.

In particular, if M is a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection, then from (3.2), (3.3) and (3.12), we have.

$$P(x,y)z = \overset{*}{R}(x,y)z = R(x,y)z + (\underset{z}{\overset{*}{\nabla}}\overset{*}{T})(y,x)$$

(3.13)

on M.

Theorem 3.7. If a Riemannian manifold M with vanishing Ricci tensor with respect to semi-symmetric non-metric connection is projectively flat, then the curvature tensor with respect to semi-symmetric non-metric connection vanishes.

Theorem 3.8. If M is a Riemmanian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection, then M is projectively flat iff the curvature tensor with respect to semi-symmetric non-metric connection vanishes.

Since a flat manifold is projectively flat, from (3.13), we have

$$\overset{*}{R}(x,y)z = 0 \text{ and } (\underset{z}{\overset{*}{\nabla}}\overset{*}{T})(x,y)$$

on M.

REFERENCES

1. Kobayashi S., Nomizu K. : Foundations of differential Geometry I,II, New York. Interscience publishers , 1969.
2. Weyl H. : Gottingen Nachrichten , 99-112 (1912).
3. Weyl H. : Math Z. 384-411 (1918).
4. Sinha B.B. : An Introduction to Modern differential Geometry, New Delhi. Kalyani Publishers, 1982.
5. Eisenhart L. P. : Continuous Groups of Transformations Princeton University. Press, 1933.
6. Yano K. : Rev. Roumaine Math. Pures Appl. 1579-86 15 (1970).
7. Yano K. : Integral Formulas in Riemannian Geometry, Decker Inc. New York, 1970.
8. Imai T. Tensor N.S. 293 - 96 24 (1972).
9. Hayden H. A. : Proc. London Math. Soc. 27-50 34 (1932).
10. Schouten J. A. : Ricci-Calculus, Springer - Verlug, Berlin, 1954 .