Erc.Ünv.Fen Bil.Derg.11,1-2 (1995) 15-21

ON THE SUBSHEAVES OF THE SHEAF OF THE FUNDAMENTAL GROUPS

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ABSTRACT

Let X be a connected and locally arcwise connected topological space an H be the sheaf of fundamental groups on X. In this paper, any two subsheaves of H is constructed and it is prowen that if stalks of these subsheaves are conjugate subgroups for every $x \in X$, then subsheaves are homomorphic.

ÖZET

X bağlantılı ve lokal eğrisel bağlantılı bir topolojik uzay ve H, X üzerinde esas grupların demeti olsun. Bu makalede, H nın herhangi iki altdemeti teşkil edildi ve her $x \in X$ için bu altdemetlerin sapları eşlenik altgruplar ise altdemetlerin homomorf olduğu ispatlandı.

1.INTRODUCTION

Definition 1. A sheaf of groups on X is a pair (H,π) where

i) H is a topological space.

ii) $\pi : H \to X$ is a local homeomorphism onto X.

iii) Each $H_x = \pi^{-1}(x)$, for $x \in X$, is a group (and is called the stalk of H at x).

iv) The group operations are continuous.

Let X be a connected and locally arcwise connected topological space and H_x be the fundamental group of X based any $x \in X$, that is, $H_x =$

$\pi_1(X,x)$. Let $X = (X,c)$ be a pointed topological spaces, for an arbitrary fixed point $c \in X$. Let us denode by H the disjoint union of the fundamental groups obtained for each $x \in X$. i.e., $H = V_{x \in X} H_x$. Also H is a set over X and the mapping $\varphi : H \to X$ defined by $\varphi(\sigma_x) = ([\alpha]_x) = x$, for any $\sigma_x = [\alpha]_x \in H_x \subset H$, is onto.

We introduce a topology on H as follows : Let H_c be the fundamenral group of X with respect to c, $x_0 \in X$ be an arbitrary fixed point, $W = W(x_0)$ be an arcwise connected open neighboorhood of x_0 and $\sigma_c = [\alpha]_c \in H_c$ be any point of H. Let us define a mapping $s : W \to H$ such that $s(x) = [(\gamma^{-1}\alpha)\gamma]_x$ for every $x \in W$, where $\gamma \in [\gamma]$ is an arc with initial point c and terminal point x. $[\gamma]$ determines s mapping between H_c and H_x. Let us chose the homotopy class $[\gamma]$ arbitrarrily fixed for each $x \in W$. Thus, $s = s(\sigma_c)$ and s a well-defined mapping from W to H such that $\varphi \circ s = I_W$ [1]. Let us denote the totality of the mappings s defined on W by $\Gamma(W,H)$.

If B is a base for X, then $B^* = \{s(W) : W \in B, s \in \Gamma(W,H)\}$ is a base for H. In this topology, the mappings φ and s are continuous and φ is a locally topological mapping. Then (H,φ) is a sheaf over X. (H,φ) (or only H) is called "The Sheaf of the Fundamental Groups" over X[1]. For any open set $W \subset X$, an element s of $\Gamma(W,H)$ is called a section of the sheaf H over W. The set $\Gamma(W,H)$ is a group with the pointwise operation of multiplication. Thus, H is a sheaf of the groups over X [2]. Furthermore, the $H_x = \pi_1(X,x)$ is called the stalk of the sheaf H for any $x \in X$.

2.CHARACTERISTIC FEATURES OF H

i) Let $W \subset X$ be an open set. Then, any section over W can be extented to a global section over X [3]. Furhermore,

$$\Gamma(W,H) = \{s|_W : s \in \Gamma(X,H)\} = \Gamma(X,H)|_W$$

ii) Any two stalks of H are isomorphic with each other [3].

iii) Let $W_1, W_2 \subset X$ be any two open sets, $s_1 \in \Gamma(W_1,H)$ and $s_2 \in \Gamma(W_2,H)$. If $s_1(x_0) = s_2(x_0)$ for any point $x_0 \in W_1 \cap W_2$, then $s_1 = s_2$ over the whole $W_1 \cap W_2$ [5].

iv) Let $W \subset X$ be an open set and $s_1, s_2 \in \Gamma(W,H)$. If $s_1(x_0) = s_2(x_0)$ for any point $x_0 \in W$, then $s_1 = s_2$ over the whole W [5].

v) To each point $\sigma_x \in H_x \subset H$, there is uniquely corresponds a section $s \in \Gamma(W,H)$ such that $s(x)=\sigma_x$. Hence $H_x \cong \Gamma(W,H)$. In particular, $H_x \cong \Gamma(X,H)$ [2].

vi) Let $x \in X$ be any point and $W=W(x)$ be any open set. Then, $\pi^{-1}(W)=V_{i\in I} s_i(W)$, $s_i \in \Gamma(W,H)$ and $\pi|_{s_i(W)} : s_i(W) \rightarrow W$ is a topological mapping for every $i \in I$. Hence, $W = W(x)$ is evenly covered by φ. Then, π is a cover projection and (H,x) is a covering space of X. Moreover (H,π) is regular, because the group T of cover transformations of H is isomorphic to the group H_x, that is, T is transitive on H_x [4].

Definition 2. Let (H,π) be the sheaf of fundamental groups over X and $H' \subset H$ be an open set. Then H' is called a subsheaf of groups, if:

i) $\pi(H') = X$

ii) For each point $x \in X$, the stalk H'_x is subgroup of H_x.

Let $H' \subset H$ be a subsheaf of groups and $W \subset X$ be an open set. Then, the set $\Gamma(W,H') \subset \Gamma(W,H)$ is a subgroup. In particular, if we take $W = X$, then $\Gamma(X,H') \subset \Gamma(X,H)$. Conversely, let us suppose that $\Gamma(X,H)$ is the group of global sections of H over X and $D \subset \Gamma(X,H)$ be a subgroup. Then, the set $\{s_i(x) : s_i \in D\}$ is a subgroup of H_x over X for each $x \in X$. Let us denote $\{s_i(x) : s_i \in D\}$ by H'_x. Then $H' = V_{x\in X} H'_x$ is a set over X with the natural projection $\pi' = \pi|_{H'}$ and $D=\Gamma(W,H')$. One can show that (H',π') is a subsheaf of groups [1].

If $H', H'' \subset H$ be any two subsheaves of groups, then H'_x and H''_x are two subgroups of H_x for each $x \in X$ by definition 2. Moreover, subsheaves H' and H" have a relative topology. If $s \in \Gamma(W_1,H)$ for every $x \in X$ and $W_1 = W_1(x) \subset X$, then whenever $H' \subset H$ and $H'' \subset H$ are open subsets, the sets $s(W_1) \cap H' = s'(W)$ and $s(W_1) \cap H'' = s''(W)$ are open in H' and H", respectively. Therefore the sets

$$B^*_1 = \{s'(W) : W = W(x)\}$$

and

$$B^*_2 = \{s''(W) : W = W(x)\}$$

are base for a topology on H' and H", respectively.

Definition 3. Let (H_1,π_1), $H_2,\pi_2)$ be any two sheaf on X and $\varphi : H_1 \to H_2$ be a mapping.

i) The mapping $\varphi : H_1 \to H_2$ is called stalk preserving, if $\pi_2 o \varphi = \pi_1$ (therefore, $\varphi(H_1)_x \subset (H_2)_x$ for all $x \in X$)

ii) Let $\varphi : H_1 \to H_2$ be a stalk preserving and continuous mapping. Then the mapping φ is called a sheaf morphism between the sheaves H_1 and H_2.

iii) Let $\varphi : H_1 \to H_2$ be a sheaf morphism. If φ is a homomorphism on each stalk, then it is called a sheaf homomorphism between the sheaves H_1 and H_2.

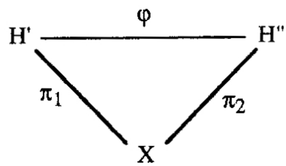
iv) Let $\varphi : H_1 \to H_2$ be a sheaf homomorphism. If φ is a homeomorphism, then it is called a sheaf isomorphism between the sheaves H_1 and H_2.

Theorem 4. Let X be a connected and locally arcwise connected topological space, (H,π) be the sheaf of the fundamental groups over X, (H',π_1) and (H'',π_2) be any two subsheaves of H. If H'_x and H''_x are conjugate subgroups of H_x with an element $\sigma = [\delta]_x$ for every $x \in X$, i.e.,$H''_x = [\delta]_x H' [\delta]^{-1}{}_x$

then, the sheaves H' and H" are homomorphic, where $\pi_1 = \pi|_{H'}$ and $\pi_2 = \pi|_{H''}$.

Proof. Let $x \in X$ be an arbitrarily fixed point and β_1,β_2 be any two closed path based at x. If $\beta_1 \sim \beta_2$, then $(\delta\beta_1)\delta^{-1} \sim (\delta\beta_2)\delta^{-1}$. Thus correspondence

$[\beta]_x \to [(\delta\beta)\delta^{-1}]_x$ is well-defined. Since the point $x \in X$ is arbitrarily fixed, the above correspondence gives us a map $\varphi : H' \to H''$ such that $\varphi([\beta]_x) = ([\delta]_x[\beta]_x)\,[\delta]^{-1}{}_x$ for every $[\beta] \in H'$.



i) **φ preserves the stalks.** In fact, for arbitrarily fixed point $x \in X$ and any $[\beta]_x \in H'$

$$\begin{aligned}
\pi_2 o\varphi)([\beta]_x) &= \pi_2(\varphi[\beta]_x) \\
&= \pi_2(([\delta]_x[\beta]_x)[\delta]^{-1}{}_x) \\
&= \pi_2([\delta\beta]_x[\delta]^{-1}{}_x) \\
&= \pi_2([\delta\beta)\delta^{-1}]_x) \\
&= \pi_2([\rho]_x) \\
&= x \\
&= \pi_1([\beta]_x).
\end{aligned}$$

Since the point $x \in X$ is arbitrarily fixed, we obtain $\pi_2 \text{ o } h = \pi_1$.

i) **φ is continuous.** Let us show that if $U_2 \subset H''$ is any open set, then $\varphi^{-1}(U_2) = U_1 \subset H'$ is an open set. Without loss of generality, we assume that $U_2 = s''(W)$, where $W \subset X$ is an open set and $s'' \in \Gamma(W,H'')$. Thus, $\pi_2(U_2) = \pi_2(s''(W)) = W$. Now let $\sigma_2 = [\rho]_x \in U_2$ be an element. Then, there exists at least one element $\sigma_1 = [\beta]_x \in U_1 = \varphi^{-1}(U_2)$ such that $\varphi(\sigma_1) = \sigma_2$. Since $\pi_1(\sigma_1) = \pi_1([\delta]_x) = x$, there is a section $s' \in \Gamma(W,H')$ such that $s'(x) =$

$[\delta]_x = \sigma_1$ and $s'(W) \subset H'$ is an open. Also $s'(W) \subset U_1$. It is easily seen that $U_1 = \cup_{i \in I} s'_i(W_i)$. Therefore, $U_1 \subset H'$ is an open set, that is, φ is a sheaf morphism.

iii) φ **a sheaf homomorphism.** For every $x \in X$ the map $\varphi|_{(H')_x} : (H')_x \to (H'')_x$ is homomorphism. In fact, if β_1, β_2 are two arcs at $x \in X$ then $(\delta\beta_1)\delta$, $(\delta\beta_2)\delta$ are two arcs at $x \in X$. Hence,

$$\begin{aligned}\varphi([\beta_1]_x)\,\varphi([\beta_2]_x) &= (([\delta][\beta_1])[\delta]^{-1})\,(([\delta][\beta_2])[\delta]^{-1}) \\ &= ([\delta][\beta_1])\,([\beta_2])[\delta]^{-1}) \\ &= [\delta]([\beta_1]\,[\beta_2])[\delta]^{-1} \\ &= [\delta][\beta_1\beta_2][\delta]^{-1} \\ &= \varphi([\beta_1\beta_2]_x).\end{aligned}$$

Theorem 5. Let X be a connected and locally arcwise connected topological space, (H,) be the sheaf of the fundamental groups over X, $(H', {}_1)$ and $(H', {}_2)$ be any two subsheaves of H. If subshaves H' and H" are homomorphic, then the map $\varphi_* : \Gamma(W,H') \to \Gamma(W,H'')$ defined by $\varphi_*(s') = \varphi \circ s'$ is a group homomorphism for any $W \subset X$.

Proof. Let s'_1 ve $s'_2 \in \Gamma(W,H')$. For every $x \in W$,

$$\begin{aligned}(\varphi \circ (s'_1.s'_2))(x) &= \varphi((s'_1.s'_2)(x)) \\ &= \varphi(s'_1(x).s'_2(x)) \\ &= \varphi([\beta_1]_x[\beta_2]_x) \\ &= \varphi([\beta_1\beta_2]_x) \\ &= ([\delta][\beta_1\beta_2])[\delta]^{-1} \\ &= ([\delta]([\beta_1][\beta_2]))[\delta]^{-1} \\ &= ([\delta]([\beta_1][\delta]^{-1}[\delta][\beta_2]))[\delta]^{-1} \\ &= (([\delta][\beta_1])[\delta]^{-1})(([\delta][\beta_2])[\delta]^{-1}) \\ &= \varphi([\beta_1]_x)\,\varphi([\beta_2]_x) \\ &= \varphi(s'_1(x))\,\varphi(s'_2(x)) \\ &= (\varphi \circ s'_1)(x)\,(\varphi \circ s'_2)(x).\end{aligned}$$

Thus, $\varphi_*(s'_1.s'_2) = \varphi_*(s'_1)\,\varphi_*(s'_2)$.

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