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ON THE ULTIMATE BOUNDEDNES RESULT FOR THE SOLUTIONS OF CERTAIN FOURTH ORDER DIFFERENTIAL EQATIONS

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ABSTRACT

The main purpose of this paper is to give sufficient conditions, which that all solutions of (1.1) are ultimately bounded.

BELLİ FORMDA DÖRDÜNCÜ BASAMAKTAN DİFERENSİYEL DENKLEMLERİN ÇÖZÜMLERİNİN SINIRLILIĞI ÜZERİNE

ÖZET

Bu çalışmanın amacı (1.1)'in bütün çözümlerinin sınırlı olmasını sağlayan yeter şartları vermektir.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Consider the equation

$$\varphi(\dot{x},\ddot{x},\dddot{x})\ddddot{x}+f(x,\dot{x},\ddot{x})+g(x,\dot{x})+h(x)=p(t,x,\dot{x},\ddot{x},\dddot{x}) \qquad \textbf{(1.1)}$$

where j, f, g, h and p are continuous functions for the arguments displayed explicitly and are such that the existence and uniqueness of solutions, as well as their continuous dependence on the initial conditions, are guaranteed. The dots as usual indicate differentiation with respect to t.

Further it will be supposed that the derivatives

$$\frac{\partial}{\partial y}\varphi(y,z,u), \frac{\partial}{\partial u}\varphi(y,z,u), \frac{\partial}{\partial x}f(x,y,z), \frac{\partial}{\partial x}g(x,y) \frac{\partial}{\partial y}g(x,y) \quad \text{and}$$

h' (x) exist and are continuous for all x, y, z and u.

Key words: Non-linear differential equations of the fourth order, Boundedness.

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We shall examine here a specific property of solutions of (1.1), namely the strong boundedness property of solutions in which the bounding constant is independent of solutions.

Special cases of the differential equation (1.1) have been treated in Abou-El-Ela ([1], [3]). Asmussen [4], Ezeilo& Tejumola [7] , Harrow [8], Tejumola [9] and others.

In [3], Abou-El-Ela dealt with the equation of the form

$$x^{(4)} + f_1(\dot{x},\ddot{x})\dddot{x} + f_2(\dot{x},\ddot{x}) + f_3(x,\dot{x}) + f_4(x) = p(t,x,\dot{x},\ddot{x},\dddot{x})$$

and presented sufficient conditions for the ultimately boundedness of solutions for that equation.

This work is the generalization of Abou-El-Ela [3].

We shall prove here;

Theorem . In addition to the basic assumptions on φ, f, g, h and p suppose that:

(I) There are positive constants α_1 , α_2 , α_3 , α_4 an δ_0 such that

$$\left[\alpha_1\alpha_2\alpha_3 - \alpha_3\frac{\partial}{\partial y}g(x,y) - \alpha_1\alpha_4\,\varphi(y,z,0)\right] \geq \delta_0$$

for all x, y and z

(II) $g(x, 0) = 0$ and $\frac{\partial}{\partial y} g(x, y) \quad \alpha_3$ for all x and y.

(III) $\varphi\,(y, z, 0) \quad \alpha_1$ for all y, z and u.

(IV) $0 \quad 0 \leq \left(\frac{\partial}{\partial y}g(x,y) - \frac{g(x,y)}{y}\right) \leq \delta_1$ for all x and y $\quad$ 0, where

d1 is a positive constant satisfying

$$\delta_1 < \frac{2\alpha_4\,\delta_0}{\alpha_1\alpha_3^2}$$

(V) $\left(\frac{1}{z}\right)\int_0^z \varphi(y,\zeta,0)d\zeta - \varphi(y,z,0) \leq \delta_2$ for all y and z $\quad$ 0, where δ_2

is a positive constant such that $\delta_2 < \dfrac{2\delta_0}{\alpha_1^2\alpha_3}$

(VI) $y\frac{\partial}{\partial\, y}\varphi(y,z,0) \leq 0$ and $z\frac{\partial}{\partial\, y}\varphi(y,z,0) \leq 0$ for all y and z.

(VII) $f(x,y)=0$,

$$\left|\left(\frac{1}{z}\right)\int_0^z \frac{\partial}{\partial\, y} f(x,y,\zeta)d\zeta\right| \leq \frac{\delta_0}{4\alpha_3}, 0 \leq \frac{f(x,y,z)}{z} - \alpha_2 \leq \frac{\varepsilon_0\alpha_3^3}{\alpha_4^2}$$

for all x, y and z $\quad$ 0, and

$y\int_0^z \frac{\partial}{\partial x} f(x,y,\zeta)\, d\zeta \le 0$ for all x, y and z, where ε_0 is a positive constant such that

(1, 2)

$$\varepsilon_0<\varepsilon=\min\left[\frac{1}{\alpha_1},\frac{\alpha_4}{\alpha_3},\frac{\delta_0}{16\alpha_1\alpha_3 d_0},\frac{\alpha_3}{4\alpha_4 d_0}\left(\frac{2\alpha_4\delta_0}{\alpha_1\alpha_3^2}-\delta_1\right),\frac{\alpha_1}{4d_0}\left(\frac{2\delta_0}{\alpha_1^2\alpha_3}-\delta_2\right)\right]$$

with $d_0 = \left(\alpha_1\alpha_2 + \frac{\alpha_2\alpha_3}{\alpha_4}\right)$

(VIII) $\left[\frac{\partial}{\partial x} g(x,y)\right]^2 \le \frac{\alpha_1\delta_0\left(\varepsilon-\varepsilon_0\right)}{16}$ for all x and y, and

$\left(\frac{1}{y}\right)\int_0^y \frac{\partial}{\partial y} g(x,\eta)\,d\eta \le \frac{\alpha_3\left(\varepsilon-\varepsilon_0\right)}{4}$ for all x and y   0.

(IX) h(0) = 0, 0 α_4 - h' (x) εd_0 α_1^2 for all x, and h(x) sgn x ——> as |x| ——> .

(X) $z\frac{\partial}{\partial u}\varphi(y,z,u)+d_y\frac{\partial}{\partial u}\varphi(y,z,u)\ge 0$ for all y, z and u, where

$$d_2 = \frac{\alpha_4}{\alpha_3} = \varepsilon_3 \quad , \qquad \textbf{(1,3)}$$

$\varepsilon > 0$ being the constant in (1,2).

(XI) |p (t, x, y, z, u) | < for all values of t, x, y, z and u. Then there exists a constant D_0 which depends only on , φ, f, g and h such that every solution x(t) of (1.1) satisfies:

$$|x(t)| \le D_0, |\dot{x}(t)| \le D_0, |\ddot{x}(t)| \le D_0, |\dddot{x}(t)| \le D_0 \qquad (1,4)$$

for all sufficiently large t.

Remark 1. When

$\varphi(\dot{x},\ddot{x},\dddot{x}) = \alpha_1$, $f(x,\dot{x},\ddot{x}) = \alpha_2$, $\ddot{x}$, $g(x,\dot{x}) = \alpha_3$, $\dot{x}$ and $h(x) = \alpha_4 x_3$

equation (1.1) reduces to the linear constant coefficient differential equation and conditions (I) - (X) of the theorem reduce to the corresponding Routh-Hurwitz criterion.

Remark 2. When we take $f(x,\dot{x},\ddot{x},\dddot{x}) = \varphi(\dot{x},\ddot{x},\dddot{x})$ in [6], our conditions far less restrictive than those obtained in [6].

Notation. In what follows the capitals D, D_0, D_1, ... denote finite positive constants whose magnitudes depend only on the functions φ, f, g, h and p as well as on the constants α_1, α_2, α_3, α_4, δ_0, δ_1, δ_2, ε_0, and ε; but are independent of solutions of the differential equation under consideration. The D's are not necessarily the same each time they occur, but each D_i, $i = 1, 2, 3, \ldots$ retains its identity throughout.

2. THE FUNCTION W(x, y, z, u)

It will be convenient in what follows to use the equivalent system:

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u$$

$$\dot{u} = -\varphi(y, z, u)\,u - f(x, y, z) - g(x, y) - h(x) + p(t, x, y, z, u) \qquad (2.1)$$

which is obtained from (1.1) by setting $y = \dot{x}$, $z = \ddot{x}$ and $u = \dddot{x}$.

In the proof of the theorem, our main tool is the piecewise continuously differrentiable function W, which is defined as follows

$$W = W_0 + W_1, \qquad (2,2)$$

where

$$2W_0(x,y,z,u)=2d_2\int_0^x h(\xi)d\xi+\left(d_2\alpha_2-d_1\alpha_4\right)y^2+ \tag{2.3}$$

$$2\int_0^y g(x,\eta)\,d\eta+2d_1 zh(x)+2yh(x)\ +2\int_0^z\left[d_1 f(x,y,\zeta)\ -\ d_2\zeta\right]d\zeta+$$

$$2\int_0^z \varphi(y,\zeta,0)\,\zeta d\zeta\ +\ d_1\ u^2\ +2\ d_2\ y\int_0^z \varphi\,(y,\zeta,0)\,d\zeta\ +2d_2\ yu\ +\ 2zu\ ,$$

$$d_1=\varepsilon+\frac{1}{\alpha_1}, \tag{2.4}$$

and

$$W_1(x,u)=\begin{cases} x\ \mathrm{sgn}\ u, & \text{if } |u|\ \geq\ |x|, \\ u\ \mathrm{sgn}\ x, & \text{if } |u|\ \leq\ |x|, \end{cases} \tag{2,5}$$

ε and d_2 being the constant defined by (1,2), (1.3), respectively.

The function W_0 here is the same as the function W_0 of [3] except that we have φ, f, g, h in place of $f_1, f_2, f_3,$ and f_4 respectively and φ, f are the generalized of $f_1, f_2,$ respectively.

Firstly we discuss some important inequalities. From (2.4) and (III) it is clear that

$$d_1 - \frac{1}{\varphi(y,z,0)} \quad \text{for all y and z.} \tag{2.6}$$

It is also easy to show that the functions φ and g satisfy

$$\alpha_2 - d_1 \frac{\partial}{\partial y} g(x, y) - d_2\, \varphi(y, z, 0) \ge \frac{\delta_0}{\alpha_1 \alpha_2} - \varepsilon d_0 \qquad (2,7)$$

for all x, y and z.

Let ϕ_1 be the function defined by

$$\phi_1(y, z, 0) = \begin{cases} \left(\frac{1}{z}\right) \int_0^z \varphi(y, \zeta, 0)\, d\zeta, & z \neq 0 \\ \varphi(y, 0, 0), & z = 0 \end{cases} \qquad (2,\ 8)$$

Then from conditions (III) and (V) we obtain.

$$\phi_1(y, z, 0) \quad \alpha_1 > 0 \text{ for all y and z,} \qquad (2,9)$$

$$\phi_1(y, z, 0) - \varphi_1(y, z, 0) \quad \delta_2 \text{ for all y and z,} \qquad (2,10)$$

We get also from (2.4) and (2.9)

$$d_1 - \frac{1}{\phi_1(y,z,0)} \quad \varepsilon \quad \text{for all y and z} \qquad (2,11)$$

Since $\phi_1(y, z, 0) = \varphi(y, \tilde{z}, 0), \tilde{z} = \theta z, 0 \le \theta \le 1$, then

$$\alpha_2 d_1 \frac{\ }{y} g(x,y) - d_2 \Phi_1(y,z,0) \ge \frac{\delta_0}{\alpha_1 \alpha_3} - \varepsilon d_0 \qquad (2.12)$$

for all x, y and z by the mean value theorem.

The properties of the function W, which are required for the proof of the theorem are summarized in Lemma 1 and Lemma 2.

Lemma 1. Suppose that conditions (I) - (X) of the theorem are hold. Then there is a constant D_1 such that

$$W(x, y, z, u) \quad -D1 \text{ for all x, y, z, and u,} \qquad (2,13)$$

$$W(x, y, z, u) \longrightarrow \quad \text{as } x^2 + y^2 + z^2 + u^2 \longrightarrow \quad . \qquad (2.14)$$

Proof. Clearly, from (2.5), $|W_1|$ $|u|$ for all x and u, then it follows

$$|W_1| \quad -|u| \text{ for all x and u.} \tag{2.15}$$

It can be seen from the similar estimates arising in the course of the [2; Lemma 2] and [1] that

$$2W_0 \quad 2\varepsilon\int_0^x h(\xi)d\xi+\left(\frac{1}{4}\right)\left(\frac{2\alpha_4\delta_0}{\alpha_1\alpha_3^2}-\delta_1\right)y^2+\left(\frac{1}{4}\right)\left(\frac{2\delta_0}{\alpha_1\alpha_3^2}-\delta_2\right)z^2+\varepsilon u^2 \tag{2.16}$$

Summing up (2.15) and (2.16) we obtain

$$2W_0 \quad 2\varepsilon\int_0^x h(\xi)d\xi+\left(\frac{1}{4}\right)\left(\frac{2\alpha_4\delta_0}{\alpha_1\alpha_3^2}-\delta_1\right)y^2+\left(\frac{1}{4}\right)\left(\frac{2\delta_0}{\alpha_1\alpha_3^2}-\delta_2\right)z^2+\varepsilon u^2-2|u|$$

$$= 2\varepsilon\int_0^x h(\xi)d\xi+\left(\frac{1}{4}\right)\left(\frac{2\alpha_4\delta_0}{\alpha_1\alpha_3^2}-\delta_1\right)y^2+\left(\frac{1}{4}\right)\left(\frac{2\delta_0}{\alpha_1\alpha_3^2}-\delta_2\right)z^2$$

$$+\,\varepsilon\left(|u|-\frac{1}{\varepsilon}\right)^2-\left(\frac{1}{\varepsilon}\right). \tag{2.17}$$

From (IX) it follows that the integral on the right-hand side is nonnegative and tends to infinity when $|x| \longrightarrow$. Also, it is obvious that the coefficients of y^2, z^2, and ε are positive. The refore we deduce that (2.13) and (2.14) are verified, where

$D_1 = \left(\frac{1}{\varepsilon}\right)$.

Lemma 2. Let (x(t), y(t), z(t), u(t)) be any solution of (2.1). Define $w(t) = W(x(t), y(t), z(t), u(t))$. Then the limit

$$\omega(t) = \limsup_{h \to 0^{+}} \left[\frac{W(x(t+h), y(t+h), z(t+h), u(t+h)) - W(x(t), y(t), z(t), u(t))}{h} \right]$$

exists and there is a constant D_3 such that

$$\overset{.+}{\omega}(t) \quad -1 \text{ whenever } x^2(t) + y^2(t) + z^2(t) + u^2(t) \quad D_3. \qquad \textbf{(2.18)}$$

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Proof. The existence of ω is immediate, because of the fact that W_0 has continuous first partial derivatives and W_1 can easily be shown to be locally Lipschitzian in x and u so that the composite function $W = W_0 + W_1$ is at least locally Lipschitzian in x, y, z and u.

From (2.3) and (2.1) we have

$$\frac{d}{dt} W_0(x,y,z,u) = \left[d_2\alpha_2 - d_1\alpha_4 \right] yz + y \int_0^y \frac{\ }{x} g(x,\eta)\,d\eta + d_1 y \int_0^z \frac{\ }{x} f(x,y,\zeta)$$

$$+ d_1 z \int_0^z \frac{\ }{x} f(x,y,\zeta)\,d\zeta + z \int_0^z \zeta \frac{\ }{y} \varphi(y,\zeta,0)\,d\zeta - d_1 u^2 \varphi(y,z,u) + y^2 h'(x) + d_1 h'(x) yz$$

$$+ d_2 z \int_0^z \varphi(y,\zeta,0)\,d\zeta + d_2 z \int_0^z y \frac{\ }{y} \varphi(y,\zeta,0)\,d\zeta - d_1 yz \frac{\ }{x} g(x,y) + d_1 z^2 \frac{\ }{y} g(x,y)$$

$$- d_2 y f(x,y,z) - d_2 y g(x,y) + u^2 - z f(x,y,z) - \left[\varphi(y,z,u) - \varphi(y,z,0) \right] zu$$

$$- d_2 \left[\varphi(y,z,0) \right] yu + \left(d_2 y + z + d_1 u \right) p(t,x,y,z,u) .$$

From (VI) we have

$$z\int_0^z \zeta\frac{}{y}\varphi(y,\zeta,0)\,d\zeta \quad 0,\; z\int_0^z y\frac{}{y}\varphi(y,\zeta,0)\,d\zeta \quad 0\,.$$

Thus it follows that

$$\dot{\omega}_0(t) \quad -(V_3+V_4+V_5+V_6+V_7+V_8)+(d_2\,y+z+d_1\,u)\,p\,(t,x,$$
$$y,z,u)$$

where

$$V_3 = d_2\,yg(x,y) - \alpha_4 y^2 - y\int_0^y \frac{}{x}g(x,\eta)d\eta - d_1 yz\frac{}{x}g(x,y)\,,$$

$$V_4 = \left[\alpha_2 - d_1\frac{}{y}g(x,y)\right]z^2 - d_2 z\int_0^z \varphi(y,\zeta,0)\,d\zeta - d_1 z\int_0^z \frac{}{y}f(x,y,\zeta)d\zeta$$

$$V_5 = [d_1 + \varphi\,(y,z,u) - 1]\,u^2$$

$$V_6 = zf(x,y,z) - \alpha_2\,z^2 + d_2\,yf\,(x,y,z) - d_2\,\alpha_2\,yz,$$

$$V_7 = \alpha_4\,y^2 - \;y^2\;h'(x) - d_1\,h'(x)\,yz + \alpha_4\,d_1\;yz,$$

$$V_8 = [\varphi\,(y,z,u) - \varphi\,(y,z,0)]\,zu + d_2\,[\varphi\,(y,z,u) - \varphi\,(y,z,0)]\,yu.$$

From (XI) we obtain

$$\dot{\omega}_0(t) \quad - V_3 + V_4 + V_5 + V_6 + V_7 + V_8\,) + D_4\;(|y| + |z| + |u|)\,, \qquad \textbf{(2.19)}$$

where $D_4 = \max\,(d_2, 1, d_1)$. The function V_3 and V_7 can be estimated as in [2]. In fact the estimates there show that

$$V_3 \quad (\varepsilon\alpha_3)\,y^2 - y\int_0^y \frac{}{x}g(x,\eta)d\eta - d_1 yz\,\frac{}{x}g(x,y)\,, \qquad \textbf{(2.20)}$$

and

$$V_7 \quad -(\varepsilon d_0)\,y^2 \qquad \textbf{(2.21)}$$

Consider the expression.

$$V_4 = \left[\alpha_2 - d_1\frac{}{y}g(x,y) - d_2\phi_1(y,z,0)\right]z^2 - d_1 z\int_0^z \frac{}{y}f(x,y,\zeta)d\zeta\,.$$

We have by using (2.12)

$$V_4 \quad \left[\frac{\delta_0}{\alpha_1\alpha_3} - \varepsilon d_0 \frac{d_1}{z}\int_0^z \frac{}{y} f(x,y,\zeta)d\zeta \,.\right] z^2$$

Further we find from (VII) and (2.4) for z 0

$$\left[\frac{d_1}{z}\int_0^z \frac{}{y} f(x,y,\zeta)d\zeta\right] \quad \left(\varepsilon+\frac{1}{\alpha_1}\right)\frac{\delta_0}{4\alpha_3} = \left(\varepsilon\alpha_1+1\right)\frac{\delta_0}{4\alpha_1\alpha_3} \quad \frac{\delta_0}{2\alpha_1\alpha_3}, \text{ since } \varepsilon \quad \frac{1}{\alpha_1}$$

by (1.2) .Thus we get for z 0, $V_4 \geq \left(\frac{\delta_0}{2\alpha_1\alpha_3} - d_0\varepsilon\right) z^2$, but $V_4 = 0$ when z = 0, therefore we obtain.

$$V_4 \geq \left(\frac{\delta_0}{2\alpha_1\alpha_3} - \varepsilon d_0\right) z^2 \quad \text{for all x, y and z.} \qquad \textbf{(2.22)}$$

From (III) and (2.4) we get

$$V_5 \quad (\alpha_1 \varepsilon)\, u^2$$

We now consider the expression

$$V_6 = zf(x,y,z) - \alpha_2 z^2 + d_2\, yf(x, y, z) - d_2\, \alpha_2\, yz.$$

By similar estimation, using condition (VII) and (1.3) we have for z 0

$$V_6 = \left[\frac{f(x,y,z)}{z} - \alpha_2 \;\mathbf{I}\; z + \frac{d_2}{2}y\right]^2 - \frac{d_2^2}{4}\left[\frac{f(x,y,z)}{z} - \alpha_2\right]y^2 \quad -(\varepsilon_0\alpha_3)y^2,$$

since $\varepsilon \left(\frac{\alpha_4}{\alpha_3}\right)$ by (1.2).

Thus it follows that $V_6 \quad -(\varepsilon_0 + \alpha_3)\, y^2$ for all x, y and z 0,

but $V_6 = 0$ when z = 0, hence we have

$$V_6 \quad -(\varepsilon_0 + \alpha_3)\, y^2 \quad \text{for all x, y and z.} \qquad \textbf{(2.23)}$$

By reasoning as in the proof of [2; Lemma 2], it can be shown

$$V_3+V_6 \quad \left(\frac{1}{2}\right)\left(\varepsilon-\varepsilon_0\right)\alpha_3 y^2-\left(\frac{\varepsilon_0}{4\alpha_1\alpha_3}\right)y^2, \qquad \textbf{(2.24)}$$

by using (VIII) and (2.4). From (X) for $x \quad 0$ we obtain

$V_8 = [z\varphi_u\,(y,z,\theta u) + d_2\, y\varphi_u\,(y, z,\theta u)]\, u^2 \quad 0,\ 0 \quad \theta \quad 1$ **(2.25)**

but $V_8 = 0$ when $u = 0$. Therefore $V_8 \quad 0$ for all y, z and u. Further it is obvious from (2.5) and (2.1) that

$$\dot{\omega}_1^{+} = \begin{cases} y \operatorname{sgn} u, & \text{if } |u| \geq |x|, \\ h(x)\operatorname{sgn}x - \left[\varphi(y,z,u)u + f(x,y,z) + g(x,y) + h(x) - p(t,x,y,z,u)\right]sgnx, & \text{if } |u| \leq |x| \end{cases}$$

From conditions (VII) and (XI) we obtain

$$\dot{\omega}_1^{+} \quad \begin{cases} |y|\ , & \text{if } |u| \geq |x|, \\ -\ h(x)\operatorname{sgn}x + |g(x,y)| + D_5\,(1++|z| + |u|\ , & \text{if } |u| \quad |x|\ ; \end{cases} \qquad \textbf{(2.26)}$$

where in obtaining (2.26) we also used the fact, arising from (I) and (III), that

$\varphi\,(y, z, u) < \alpha_2\alpha_3\alpha_4^{-1}$ for all y, z and u.

As can be shown from (2.19) and (2.21) - (2.26) $\dot{\omega}^{+} = \dot{\omega}_0 + \dot{\omega}_1^{+}$ necessarily satisfies:

$$\dot{\omega}^{+} \quad -\left(\frac{1}{2}\right)\left(\varepsilon-\varepsilon_0\right)\alpha_3 y^2-\left(\frac{\delta_0}{8\alpha_1\alpha_3}\right)z^2-\varepsilon\alpha_1 u^2+D_6\left(|y|+|z|+|u|\right),\ \text{if } |u| \quad |x|$$

(2.27)

or

$$\dot{\omega}^{+} \quad -\left[V_3 - |g(x,y)|\right] - V_6 - \left(\frac{\delta_0}{2\alpha_1\alpha_3} - 2\varepsilon d_0\right) z^2 - \varepsilon\alpha_1 u^2 - h(x)\,\mathrm{sgn}\,x$$

$$+D_7\,(1 + |y| + |z| + |u|)\,, \text{ if } |u| \quad |x|. \qquad \textbf{(2.28)}$$

Proceeding as in [3; Lemma 2] it can be shown that

$$\dot{\omega}^{+} \quad -1 \text{ if } x^2 + y^2 + z^2 + u^2 + \quad D_3\ ,$$

which verifies (2.18), and concludes the proof of Lemma 2.

3.COMPLETION OF THE PROOF

We proved throught Lemma 1 and Lemma 2, that the function $W = W_1 + W_0$ has the following properties:

$$W(x, y, z, u) \quad -D_1 \quad \text{for all } x, y, z \text{ and } u, \qquad \textbf{(3,1)}$$

$$W(x, y, z, u) \longrightarrow \quad \text{as } x^2 + y^2 + z^2 + u^2 \longrightarrow \qquad \textbf{(3,2)}$$

$$\frac{d}{dt} W(x, y, z, u) \quad -1 \text{ whenever } x^2 + y^2 + z^2 + u^2 \quad D_3 \qquad \textbf{(3.3)}$$

The usual Yoshizawa-type argument, Theorem 1 in [5], applied to (3.1) - (3.3) would then show that: For any solution (x(t), y(t), z(t), u(t) of (2.1) we have

$$|x(t)| \quad D_0\,,\ |y(t)| \quad D_0\,,\ |z(t)| \quad D_0\,,\ |u(t)| \quad D_0\,,$$

for all sufficiently larget, which are equivalent to (1.4).

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