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DERIVATION IN THE GROUP RING

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ABSTRACT

In this research, according to Wirtinger representation, acting from the relationships between the relations and the generators of the knot group, by appliying concept of derivation on these relations, we aimed at reaching some conclusions.

In addition, we found out some corollories by extending the formal Taylor series of these relations.

Key Words:Group Ring,Free Derivation,Knot presentation.

GRUP HALKASINDA TÜREV

ÖZET

Bu çalışmada,Düğüm grubunun bağıntılarını ve doğurayları arasındaki ilişkilerden hareketle, Wirtinger temsiline göre bu bağıntılar üzerinde türev kavramını uygulayarak, bazı sonuçları çıkarmayı amaçladık. Buna ek olarak bu bağıntıların biçimsel Taylor serisine açarak bazı sonuçları bulduk.

INTRODUCTION

1.THE DERIVATIVES IN A GROUP RING

For any multiplicative group G there is associated its group ring ZG with respect to the ring Z of rational integers. An element of ZG is a sum $\Sigma a_g g$, g ranging over the elements of G, where the integer a_g is equalto zero for all but a finite number of g. Addition and multiplication in ZG are defined by $\Sigma a_g g + \Sigma b_g g = \Sigma(a_g+b_g)g$ and $(\Sigma a_g g)(\Sigma b_g g) = \Sigma(a_{hg} h^{-1} b_h)g$. The element a of Z is identified with the element a.1 of ZG and the element g of G is identified with the element 1.g of ZG, so that Z and G are to be regarded as subsets of ZG [1,2,5,6].

Definition_1.1_: A mapping D : ZG ———> ZG is said to be a derivative if and only if

(1) $D(f+g) = D(f) + D(g)$

(2) $D(f.g) = D(f)\varepsilon(g) + f.D(g)$ for $f,g \in ZG$.

Where is the augmentation homomorphism for all f and g in ZG as given in following

$\varepsilon : ZG \longrightarrow Z$

$\varepsilon(\Sigma n_g g) = \Sigma n_g$

Note that if g belongs to G, then (2) reads

$D(f.g) = D(f) + f.D(g)$ [3,4].

Consequences_of_axioms_1.2_:

(1) $D(\Sigma n_g g) = \Sigma n_g D(g)$

(2) $D(n) = 0 \quad n \in Z$

(3) $D(g^{-1}) = -g^{-1} D(g)$

(4) $D(g^n) = (1+g+g^2+....+g^{n-1}) D(g)$

(5) $D(g^{-n}) = -(g^{-1}+g^{-2}+....+g^{-n}) D(g)$ for $g \in G$

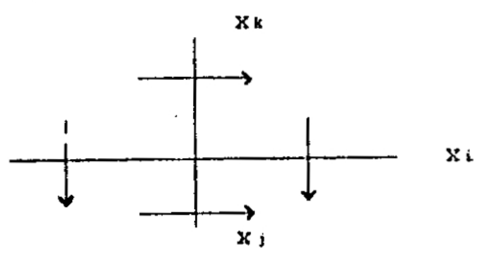
Definition_1.3_: The derivations

$$\frac{d}{dx_i} : ZF \longrightarrow ZF,\ x_j \longrightarrow \begin{cases} 1 \text{ for } i=j \\ 0 \text{ for } i \neq j \end{cases}$$

of the group ring of free group $F = \langle x_i \rangle$ are called partial derivations[3].

<u>Definition 1.4 (Wirtinger Representation) :</u> If we consider a knot K with the directed, the n double points of the regular projection. In this diagram , there are n numbered pieces of curve, counterporting the overpasses. The homotopy classes of closed curves, starting at $p \in S^3-K$ point and ending at p and encirling the overpases simply are iltusrated by the little arrows put on the pieces of curve. We fix a relation at every C_i double point in the following way. Representative curves of x_i, x_j, x_k generators belonging to the overpass at C_i, are directed so as to from the left hand system according to the direction of the knot. Around C_i, we choise a reading direction starling with the one of x_i, x_j, x_k generators, if the direction of each generator is the same as the chosen direction, we take generator it self, if it is the inverse of it we take the inverse of it and equate this multiply with one.

As, seen in the figure 1 the relation at C_i double point is illustrated in the following way $x_i x_j x^{-1}_i x^{-1}_k = 1$



<u>Defination_1.5_(_Knot_Group_)_:</u> Let K be a knot and $G = \pi_1(S^3-K, p) = \langle x_1, x_2, \ldots\ldots, x_n \mid r_1, r_2, \ldots\ldots\ldots, r_n \rangle$ a wirtinger presentation of G. Where $x_1, x_2, \ldots\ldots\ldots, x_n$ are the generators of G and $r_1, r_2, \ldots\ldots, r_n$ are the relations of it

Let $r_i = x_i x_j x^{-1}_i X^{-1}_k = 1$ be a relation of G at C_i double point. Then, we have the following corollaries.

<u>Corollory_1.1_:</u>

$$\left(\frac{dr_i}{dx_i}^{\theta\phi}\right) + \left(\frac{dr_i}{dx_j}^{\theta\phi}\right) + \left(\frac{dr_i}{dx_k}^{\theta\phi}\right) = 0$$

Where θ and ϕ are homomorphism such that

$$\begin{array}{ccccc} F & \xrightarrow{\phi} & G & \xrightarrow{\theta} & H \\ ZF & \xrightarrow{\phi} & ZG & \xrightarrow{\theta} & ZH \end{array}$$

<u>Proff_:</u> If $r_i = x_i x_j x^{-1}_i x^{-1}_k$ is a relation of at C_i double point. Then

$$\left(\frac{dr_i}{dx_i}^{\theta\phi}\right) = \left(1 - x_i x_j x^{-1}_i\right)^{\theta\phi} = 1-t$$

$$\left(\frac{dr_i}{dx_j}^{\theta\phi}\right) = \left(x_i\right)^{\theta\phi} = t$$

$$\left(\frac{dr_i}{dx_k}^{\theta\phi}\right) = \left(-x_i x_j x^{-1}_i x^{-1}_k\right)^{\theta\phi} = -1$$

and

$$\left(\frac{dr_i}{dx_i}^{\theta\phi}\right) + \left(\frac{dr_i}{dx_j}^{\theta\phi}\right) + \left(\frac{dr_i}{dx_k}^{\theta\phi}\right) = 1-t+t-1 = 0$$

<u>Corollory_1.2_:</u> i) $\dfrac{d^n r_i}{dx^n_i} = (-1)^{n+1} \dfrac{dr_i}{dx_i}$

$$ii)\ \frac{d^n r_i}{dx^n{}_j} = 0$$

$$iii)\ \frac{d^n r_i}{dx^n{}_k} = (-1)^{n+1}\frac{dr_i}{dx_k}$$

Prof : It can be proved.

Corollory_1.3_: Let $r_i = x_i x_j x^{-1}{}_i x^{-1}{}_k$ be a relation according to wirtinger representation and the augmentation homomorphism, then

$$\varepsilon\left(\frac{dr_i}{dx_i}\right) + \varepsilon\left(\frac{dr_i}{dx_j}\right) + \varepsilon\left(\frac{dr_i}{dx_k}\right) = 0$$

Corollory_1.4_: $\dfrac{dr^{-1}{}_i}{dx_i} = -r^{-1}\dfrac{dr_i}{dx_i}$, $\dfrac{dr^{-1}{}_i}{dx_j} = -r^{-1}\dfrac{dr_i}{dx_j}$

$$\frac{dr^{-1}{}_i}{dx_k} = -r^{-1}\frac{dr_i}{dx_k}$$

Corollory_1.5_: $\left(\dfrac{d^2 r_i}{dx^2{}_i}\theta\phi\right) + \left(\dfrac{d^2 r_i}{dx^2{}_j}\theta\phi\right) + \left(\dfrac{d^2 r_i}{dx\ y}\theta\phi\right)$

$$\left(\frac{d^2 r_i}{dydx}\theta\phi\right) + \left(\frac{d^2 r_i}{dxdz}\theta\phi\right) + \left(\frac{d^2 r_i}{dydz}\theta\phi\right) + \left(\frac{d^2 r_i}{dz^2}\theta\phi\right) +$$

$$+\left(\frac{d^2 r_i}{dzdy}\theta\phi\right) + \left(\frac{d^2 r_i}{dzdx}\theta\phi\right) = 0$$

That is $\displaystyle\sum_{a+b+c=2}^{2}\left(\frac{d^2 r_i}{dx^a{}_i dx^b{}_j dx^c{}_k}\theta\phi\right) = 0$

Where a+b+c=2, a,b,c= 0,1,2

Corollory.1.6.: $\displaystyle\sum_{a+b+c=n}^{n}\left(\frac{d^n r_i}{dx^a{}_i dx^b{}_j dx^c{}_k}\theta\phi\right) = 0$

Where a+b+c=n, a,b,c=0,1,2,...,n

Corollory_1.7_: Let $r_i = x_i x_j x^{-1}{}_i x^{-1}{}_k$ be a relation accordi... to wirtinger representation at C_i double point, and ε the augmentation homomorphism, then

$$\varepsilon\left(\frac{d^3 r_i}{dx_i dx_j dx_k}\right) + \varepsilon\left(\frac{d^3 r_i}{dx_i dx_k dx_j}\right) + \varepsilon\left(\frac{d^3 r_i}{dx_j dx_k dx_i}\right) +$$

$$+ \varepsilon\left(\frac{d^3 r_i}{dx_k dx_j dx_i}\right) + \varepsilon\left(\frac{d^3 r_i}{dx_k dx_i dx_j}\right) =$$

$$= \varepsilon\left[\left(\frac{dr}{dx_i}\right)\left(\frac{dr}{dx_j}\right)\left(\frac{dr}{dx_k}\right)\right]$$

Similary corollories, when the relations of the knot group reduce to single relation,are valid.

Corollory 1.8 : For $G = \langle x.y \mid r = 1 \rangle$, we obtain

a) (1) $\left(\frac{dr}{dx}\right)^{\theta\phi} + \left(\frac{dr}{dy}\right)^{\theta\phi} = 0$

(2) $\left(\frac{d^2 r}{dx^2}\right)^{\theta\phi} + \left(\frac{d^2 r}{dxdy}\right)^{\theta\phi} + \left(\frac{d^2 r}{dydx}\right)^{\theta\phi} + \left(\frac{d^2 r}{dy^2}\right)^{\theta\phi} = 0$

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(n) $\sum_{i=n}^{1} \frac{d^n r}{dx^i dy^{n-i}} + \sum_{j=1}^{n} \frac{d^n r}{dy^j dx^{n-j}} + \sum_{i+j+k=n} \frac{d^n r}{dx^i dy^j dx^k} +$

$$+ \sum_{i+j+k=n} \frac{d^n r}{dy^i dx^j dy^k} = 0$$

b) (1) $\varepsilon\left(\frac{dr}{dx}\right) + \varepsilon\left(\frac{dr}{dy}\right) = 0$

(2) $\varepsilon\left(\frac{d^2 r}{dx^2}\right) + \varepsilon\left(\frac{d^2 r}{dxdy}\right) + \varepsilon\left(\frac{d^2 r}{dydx}\right) + \varepsilon\left(\frac{d^2 r}{dy^2}\right) = 0$

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(n) $\varepsilon\left(\sum_{i=n}^{1} \frac{d^n r}{dx^i dy^{n-i}}\right) + \varepsilon\left(\sum_{j=n}^{1} \frac{d^n r}{dy^j dx^{n-j}}\right) +$

$$+\ \varepsilon\Big(\sum_{i+j+k=n}^{1} \frac{d^n r}{dx^i dy^j dx^k}\Big) + \varepsilon\Big(\sum_{i+j+k=n} \frac{d^n r}{dy^i dx^j dy^k}\Big) = 0$$

<u>Example 1.1 :</u> A wirtinger presentation of the group of trefoil is;

$$G = \langle\, x, y, z \mid r_1 = yz\bar{y}\bar{x},\ r_2 = zx\bar{z}\bar{y},\ r_3 = xy\bar{x}\bar{z} \,\rangle$$

$$\text{or } G = \langle\, x, y \mid xyx\bar{y}\bar{x}\bar{y} = 1 \,\rangle$$

Where $\bar{x}$ is the inverse x, $(x=x^{-1})$

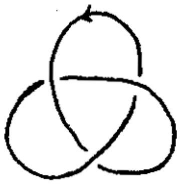

Figure 2

Now, we calculate

$$\frac{dr}{dx} = 1+xy-xyx\bar{y}\bar{x}, \quad \frac{dr}{dy} = x-xyx\bar{y}-xyx\bar{y}\bar{x}\bar{y}$$

$$\frac{d^2r}{dx^2} = -xy+xyx\bar{y}\bar{x}, \quad \frac{d^2r}{dxdy} = xyx\bar{y}, \quad \frac{d^2r}{dydx} = -2xy-1+xyx\bar{y}\bar{x}$$

$$\frac{d^2r}{dy^2} = -2x+2xyx\bar{y}+xyx\bar{y}\bar{x}\bar{y}$$

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Then, we obtain

$$1)\ \Big(\frac{dr}{dx}\Big)^{\theta\phi} + \Big(\frac{dr}{dy}\Big)^{\theta\phi} = t^2-t+1-t^2-t-1 = 0$$

$$2)\ \Big(\frac{d^2r}{dx^2}\Big)^{\theta\phi} + \Big(\frac{d^2r}{dxdy}\Big)^{\theta\phi} + \Big(\frac{d^2r}{dydx}\Big)^{\theta\phi} + \Big(\frac{d^2r}{dy^2}\Big)^{\theta\phi} = -t^2+t+t^2$$

$$=2t^2-1+t-2t+2t^2+1 = 0$$

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(n) . . . .

Similarly, we obtain

$$1)\ \varepsilon\left(\frac{dr}{dx}\right) + \varepsilon\left(\frac{dr}{dy}\right) = 1+1-1+1-1-1 = 0$$

$$2)\ \varepsilon\left(\frac{d^2r}{dx^2}\right) + \varepsilon\left(\frac{d^2r}{dxdy}\right) + \varepsilon\left(\frac{d^2r}{dydx}\right) + \varepsilon\left(\frac{d^2r}{dy^2}\right) = -1+1 +1-2-1-1-2+2+1 = 0$$

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n) . . . .

<u>Corollary 1.9</u> : Let $G = \langle x,y \mid r(xy) = 1 \rangle$ be a wirtinger presentation of the group of each knot, then.

$$1)\ r(xy) = \varepsilon(r(xy)) + \varepsilon\left(\frac{dr}{dx}\right)(x-1) + \varepsilon\left(\frac{dr}{dy}\right)(y-1) + \varepsilon\left(\frac{d^2r}{dx^2}\right)(x-1)^2 + \varepsilon\left(\frac{d^2r}{dxdy}\right)(x-1)(y-1) + \varepsilon\left(\frac{d^2r}{dydx}\right)(y-1)(x-1) + \varepsilon\left(\frac{d^2r}{dy^2}\right)(y-1)^2 + \ldots$$

$$2)\ [r(xy)]^{\Theta\phi} = [\ (r(xy))]^{\Theta\phi} + \left[\varepsilon\left(\frac{dr}{dx}\right)(x-1)\right]^{\Theta\phi} + \left[\varepsilon\left(\frac{dr}{dy}\right)(y-1)\right]^{\Theta} + \left[\varepsilon\left(\frac{d^2r}{dx^2}\right)(x-1)^2\right]^{\Theta\phi} + \left[\varepsilon\left(\frac{d^2r}{dxdy}\right)(x-1)(y-1)\right]^{\Theta\phi} + \left[\varepsilon\left(\frac{d^2}{dydx}\right)(y-1)(x-1)\right]^{\Theta\phi} + \left[\varepsilon\left(\frac{d^2r}{dy^2}\right)(y-1)^2\right]^{\Theta\phi} + \ldots$$

<u>Example 1.2</u> : $G = \langle x.y : r = xyx\bar{y}\bar{x}\bar{y} \rangle$ is the group of trefoil knot.

(1) $xyx\bar{y}\bar{x}\bar{y} = 1+1(x-1)-1(y-1)+0(x-1)^2+1(x-1)(y-1)-(y-1)(x-1)+$

$+1(y-1)^2+\dots\dots\dots$

(2) $1 = 1+1(t-1)-1(t-1)+0(t-1)^2+1(t-1)^2-2(t-1)^2+1(t-1)^2+\dots$

$\longrightarrow 1 = 1$

If we take paranthesis $(t-1), (t-2)^2, \dots (t-1)^n, \dots$ from Corollory 1.7 We have the followings

$$\varepsilon\left(\frac{dr}{dx}\right) + \varepsilon\left(\frac{dr}{dy}\right) = 0$$

$$\varepsilon\left(\frac{d^2r}{dx^2}\right) + \varepsilon\left(\frac{d^2r}{dxdy}\right) + \varepsilon\left(\frac{d^2r}{dydx}\right) + \varepsilon\left(\frac{d^2r}{dy^2}\right) = 0$$

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<u>Corollory_1.10_:</u> If $G = \langle x.y \mid r(x,y) = 1 \rangle$, then

$$\varepsilon\left(\frac{d^2r}{dxdy}\right) + \varepsilon\left(\frac{d^2r}{dydx}\right) = \varepsilon\left(\frac{dr}{dx}\right) \varepsilon\left(\frac{dr}{dy}\right)$$

$$= \varepsilon\left(\frac{dr}{dx} \cdot \frac{dr}{dy}\right)$$

<u>Example_1.3_:</u> Let $r = x^n$. Then, we have the followings

for $n=1$ $\quad x = 1+(x-1)$

for $n=2$ $\quad x^2 = 1+2(x-1)+1(x-1)^2$

for $n=3$ $\quad x^3 = 1+3(x-1)+3(x-1)^2+1(x-1)^3$

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for $n=n$ $\quad x^n = \binom{n}{0}+\binom{n}{1}(x-1)+\binom{n}{2}(x-1)^2+\dots+\binom{n}{n-1}(x-1)^{n-1}+$

$+\binom{n}{n}(x-1)^n$

<u>Example_1.4_:</u> Let $r = x^{-n}$ be, then we have the followings

for n=1 icin $x^{-1} = 1-(x-1)+(x-1)^2+(x-1)^3+\ldots+(-1)^n(x-1)^n+\ldots$

for n=2 icin $x^{-2} = 1-2(x-1)+3(x-1)^2+\ldots+(-1)^n(n+1)(x-1)^n+\ldots$

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for n=n icin $x^{-n} = 1-n(x-1) + \sum_{\substack{k=0\\(a_0=0)}}^{n} (a_k+k)(x-1)^2 + \sum_{\substack{k=0\\(b_0=0)}}^{n} b_k +$

$$+ \sum_{\substack{k=0\\(c_0=1)}}^{n} [c_k+(k+1)(x-1)^3] + \ldots\ldots$$

Then a_k, b_k and c_k are sequences.

2. DERIVATIONS OF SOME COMMUTATORS

<u>Definition_2.1_:</u> Let $[x,y] = xyx^{-1}y^{-1}$ be a commutator. Then we have the following.

(1) $\frac{d}{dx}[x,y] = \frac{d}{dy}(xyx^{-1}y^{-1}) = 1-[x,y]y$

(2) $\frac{d}{dy}[x,y] = x-[x,y]$

(3) $\frac{d}{dx}[x,y]^n = \{ 1+[x,y]+[x,y]^2+\ldots+[x,y]^{n-1}\} \frac{d}{dx}[x,y]$

(4) $\frac{d}{dy}[x,y]^n = \{ 1+[x,y]+[x,y]^2+\ldots+[x,y]^{n-1}\} \frac{d}{dy}[x,y]$

(5) $\frac{d^n}{dx^n}[x,y] = (-1)^{n+1}\frac{d}{dx}[x,y]$

(6) $\frac{d^n}{dy^n}[x,y] = (-1)^{n+1}\frac{d}{dy}[x,y]$

(7) $\frac{d}{dx}[x^n,y^m] = \{ 1-[x^n,y^m]y^m\} \frac{d}{dx}(x^n)$

(8) $$\frac{d}{dy}[x^n, y^m] = (x^n - y^m x^{-n} y^{-m}) \frac{d}{dy}(y^m)$$

(9) If $z=[x,y]$, then $zyx=xy$

$$\frac{dz}{dx} + z.\frac{d(y.x)}{dx} = 1$$

$$\frac{dz}{dx} + z.y = 1 \longrightarrow \frac{dz}{dx} = 1-zy = 1-xyx^{-1}y^{-1}y = 1-xyx^{-1}$$
$$= 1-[x,y]y$$

Similarly

$$\frac{dz}{dy} + z.\frac{d(yx)}{y} = x$$

$$\frac{dz}{dy} + z.1 = x \longrightarrow \frac{dz}{dy} = x-z = x-xyx^{-1}y^{-1} = x-[x,y]$$

(10) If $w = [x,yz]$ then $wyzx = xyz$ we have the followings

$$\frac{dw}{dx} = 1-xyz\bar{x}$$

$$\frac{dw}{dy} = x-[x,yz]$$

$$\frac{dw}{dz} = xy-[x,yz]y$$

(11) a) $$\varepsilon\left(\frac{d}{dx_i}[x_i, x^j]\right) = 0$$

b) $$\varepsilon\left(\frac{d}{dx_j}[x_i, x_j]\right) = 0$$

c) $$\varepsilon\left(\frac{d}{dx_i}[x_i, x_j]^n\right) = 0$$

d) $\varepsilon\left(\frac{d}{dx_j} [x_i, x_j]^n \right) = 0$

e) $\varepsilon\left(\frac{d^n}{dx^n_i} [x_i, x_j] \right) = 0$

f) $\varepsilon\left(\frac{d^n}{dx^n_j} [x_i, x_j] \right) = 0$

<u>Corollory 2.2 :</u> If $[x,y] = xyx^{-1}y^{-1}$, the formal Taylor formula of $[x,y]$ is

$$[x,y] = 1-(x-1)(y-1)+(y-1)(x-1)+(x-1)^2(y-1)-(x-1)(y-1)(x-1)-$$
$$-(y-1)^2(x-1)+(y-1)(x-1)(y-1)+\ldots\ldots$$

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