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ON KNOT GRAPHS

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Abstract: In this paper, The Reidemeister moves applied on knots are adapted on knot graphs and it is proved by graphical approach mentioned in [1]. Also, it is introduced the results obtained by unions of two knot 3_1.

Key words knot, graph, Reidemeister moves, knot graphs, unions of knots.

DÜĞÜM GRAFLARI ÜZERINE

Özet: Bu çalışmada, düğümlere uygulanan Reidemeister hareketleri düğüm graflarına uygulanmış ve [1] de bahsedilen yaklaşımla ispatlanmıştır. Ayrıca, iki 3_1 düğümünün birleşimi ile oluşan durumlar sunulmuştur.

Anahtar kelimeler: Düğüm, Graf, Reidemester hareketleri, Düğüm Grafları, Düğümlerin birleşimleri

1. Introduction

A *graph* is a pair $G = (V, E)$ of sets satisfying $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V . We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. Throughout this paper, every graph is assumed to be finite, simple and connected (See [2] for the basic terminology of graph theory). A *knot* is a simple closed curve in a space S^3 . (See [3],[4] for many standard terminologies in knot theory).

2. Reidemeister Moves

Two knot diagrams are called equivalent, if they are connected by a finite sequence of Reidemeister moves [5] Ω_i ,i =1,2,3, or their inverses Ω_i^{-1} . The moves are described in Figure 1.

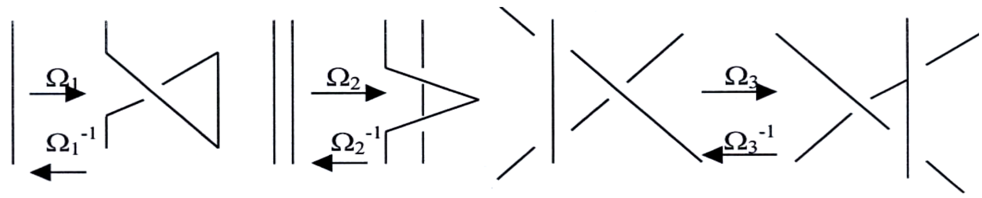
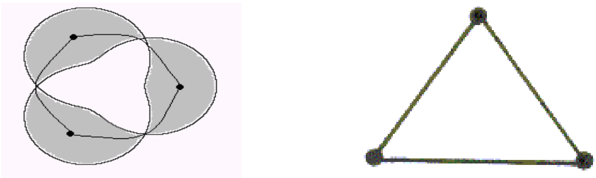


Figure 1

3. Knot graphs

A projection of a knot on a 2-dimensional plane divides the plane into several domains. It is a frequently used method to separate these domains into two classes, white domains and black domains in the study of knot theory. Using this method, C. Bankwitz [6] introduced the notion of knot graph. Let D be a regular projection of a knot on a 2-dimensional sphere S. If D has n double points $D_1,D_2,...,D_n$, then it divides S into $n + 2$ domains, each of which is homeomorphic to an open disk. Now, separate these domains into two classes α and β (Fig. 2(a)).



(a) (b)

Figure 2.

Starting with the outermost domain, we can color the domains either black or white (or inner and outer, or shaded-unshaded). Now, we shall color the outermost domain black (or white). We can color the domains so that neighboring domains are never the same color. Let $W_1,W_2,...,W_v$ be the domains of class α. Take points $c_i \in W_i$ (i = 1,2,...,v) and connect these points by n non-intersecting arcs $d_1,d_2,...,d_n$ in such a way that each d_r corresponds to D_r (r = 1,2,...,n) and c_i and c_j are connected by d_r, if and only if, W_i and W_j have a common double point D_r on their boundaries. The vertices of graph are the centers of the white domains (Figure 2(b)). The domains of class α in all can be considered as a projection of a surface spanning K which is twisted 180^0 at each double point of D. However, in order for the plane graph to embody some of the characteristics of the knot, we need to used the regular diagram rather than the projection. So, we need to consider the under - and over - crossing at a crossing. To this end, in Figure 3 is shown a way of assigning to each edge of graph either the sign + or - . A + sign is assigned to an edge e if the domains are colored in the manner of Figure 3(a), and – sign if they are as in Figure 3(b). A signed plane graph that has been formed by means of the above process is said to be the *graph of K* and denote it by $G(K)$. From the same consideration about the class β, we get another graph $G'(K)$. We call this the dual graph of $G(K)$.

(a) (b)

Figure 3

For a given projection of knot or a link, we have two graphs G and G^d . Conversely, for a given graph, whichever G or G^d, there is a uniquely determined projection of knot. Conversely, we can construct from an arbitrary signed plane graph G a knot (or link) diagram (Figure 4) [1].

Therefore, A signed plane graph that has been formed by means of the above process is said to be the *graph* of K [3],[7].

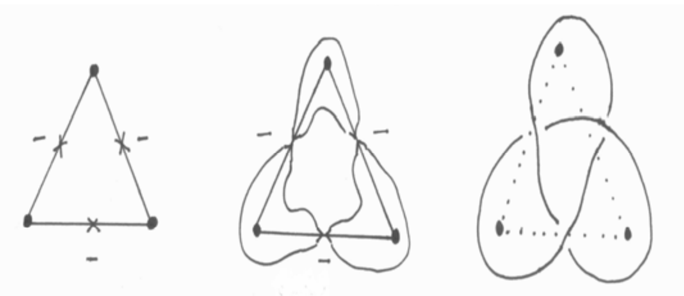


Figure 4

4. Graphs of Unions of Knots

If two knots K and K' with a common arc α, of which K lies inside a cube Q and K' outside of it, α lying naturally on the boundary of Q, are joined together along α, that is, if α is deleted to obtain a single knot out of K and K' ,then we have by definition the product of K and K'. If a knot cannot be the product of any two non trivial knots, then K is said to be prime. It is Schubert [8] who showed that the genus of the product of two knots is equal to the sum of their genera and that every non trivial knot is decomposable in a unique way into prime knots. Thus 8_5 – 9_{45} of Alexander-Briggs table are all composed of two trefoil knots 3_1 and 4_1 in the following way (Figure 5 indicates the composition of 8_{19} out of two 3_1): First Join the trefoil knots K and K' together along their arcs AB and $A'B'$ and as the usual product making and then wind them together in the neighborhood of the arcs CD and $C'D'$. Likewise 9_{22}, 9_{25}, 9_{30} and 9_{45} are composites of the trefoil knot 3_1 and the knot 4_1.

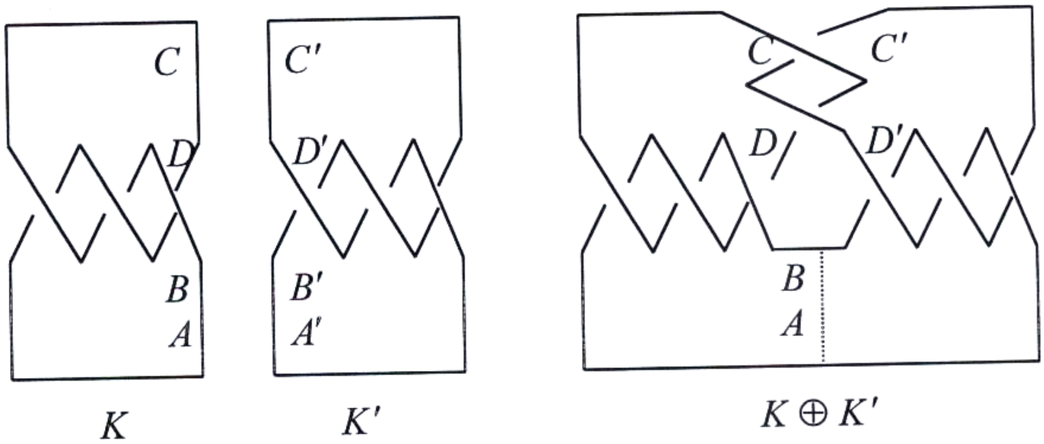
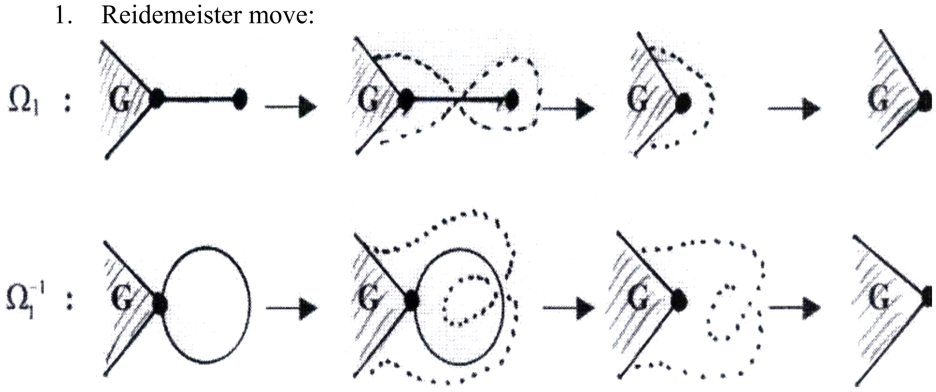


Figure 5

Such a composition of knots will best be described if we make use of the graphs of knots [9].

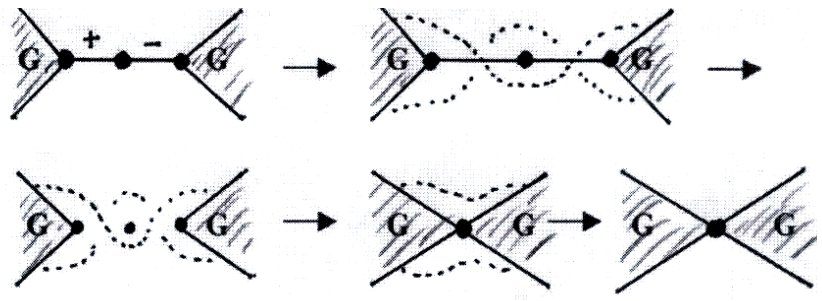
The Reidemeister moves, applied on knots, can be modified on graphs. Since The projection of a knot has two graphs, The inverses of Reidemeister moves can be applied on graphs.

1. Reidemeister move:

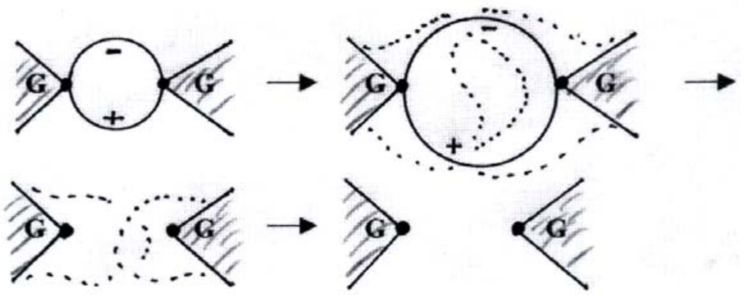


2. Reidemeister move :

Ω_2 :

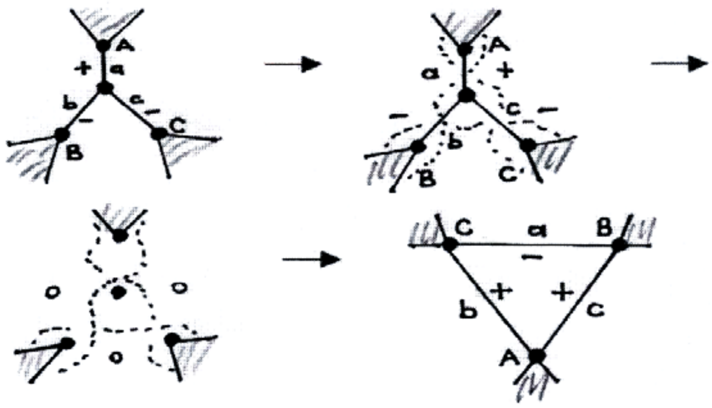


Ω_2^{-1} :



3. Reidemeister move :

$\Omega_3 \equiv \Omega_3^{-1}$:



As by depending on these moves, that is, applied Reidemeister moves on knot graphs, we can obtain many conclusions. For example, we can obtain some knots between 8_5 and 8_{21} (particularly $8_{19(n)}$, $8_{20(n)}$, $8_{21(n)}$) from the unions of two knot 3_1. From union of graphs of the knots, the following possible cases occur. We can prove these cases by Reidemeister moves applied on knot graphs.

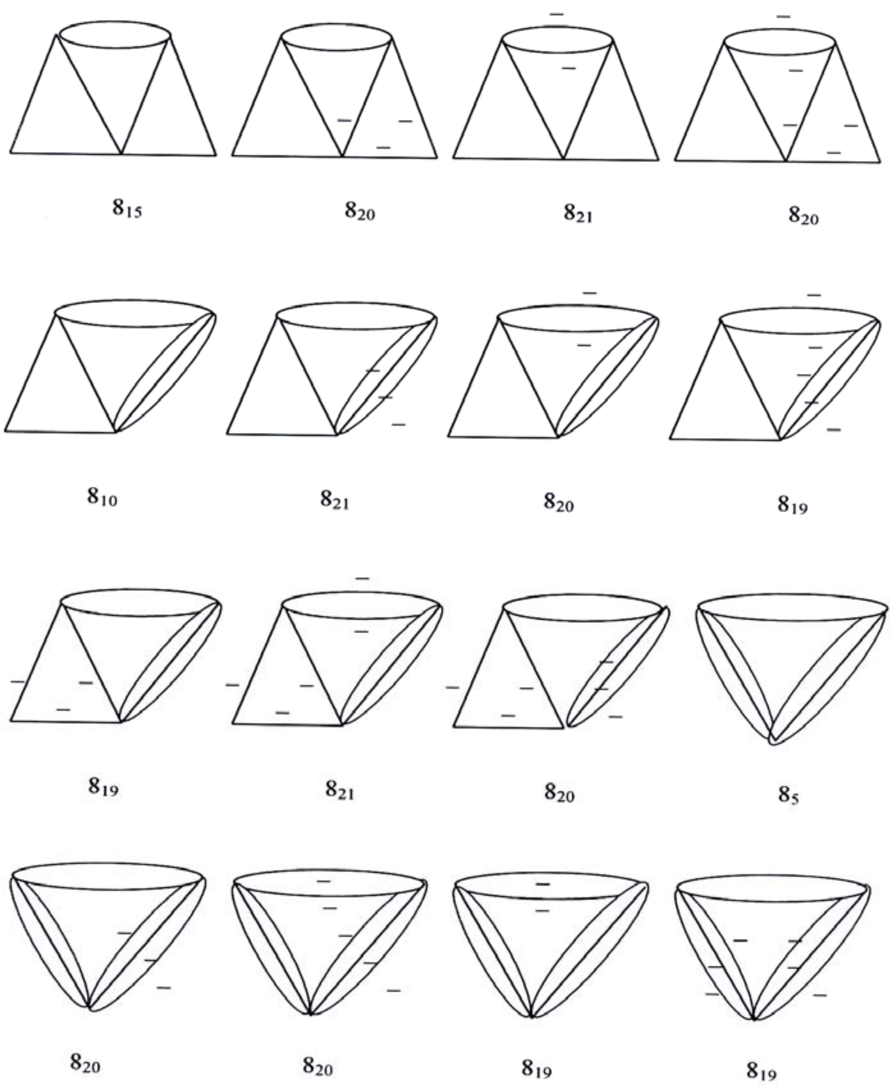


Figure 6

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