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Laguerre series solutions of fredholm integral equations

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ABSTRACT
A matrix method for approximately solving linear Fredholm integral equations of the second kind is presented. The solution involves a truncated Laguerre series approximation. The method is based on first taking the truncated Laguerre series expansions of the functions in the equation and then substituting their matrix forms into the equation. Thereby the equation reduces to a matrix equation, which corresponds to a linear system of algebraic equations with unknown Laguerre coefficients. In addition, some equations considered by other authors are solved in terms of Laguerre polynomials and the results are compared.

Fredholm integral denklemlerin laguerre seri çözümleri

Anahtar Kelimeler
Laguerre Seriler
Fredholm integral denklemleri

ÖZET
İkinci tip lineer Fredholm integral denklemlerin yaklaşık çözümü için bir matris metotu sunulmuştur. Bu çözüm kesilmiş bir Laguerre seri yaklaşımı içerir. Belirtilen yöntem denklemdeki fonksiyonların kesilmiş Laguerre seri açılımlarının matris formlarının denklemde yerine konması esasına dayanır. Böylece denklem bilinmeyen Laguerre katsayılı lineer cebirsel denklemler sistemine karşılık gelen bir matris denklemine indirgenir. Buna ek olarak, diğer araştırmacılar tarafından incelenen bazı denklemler Laguerre polinomları açısından çözülmüş ve sonuçlar tartışılmıştır.

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1. INTRODUCTION

Many mathematical formulation of physical phenomena contain Fredholm integral equations, this equations arise in fluid dynamics, biological models and chemical kinetics. Fredholm integral equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution.As we know, much work has been done on developing and analyzing numerical methods for solving linear Fredholm integral equations [7-9]. Several numerical methods were used such as the successive approximation method [9].

The subject of the presented paper is applying the Laguerre method for solving linear Fredholm integral equations.

On the other hand, in recent years, the matrix method has been developed for solving the linear Fredholm integral equations. For example in [4] this method is used for solving Fredholm integral equation.

For In this paper we consider the Fredholm integral equations of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt \qquad (1)$$

where $y(x)$ is the function to be determined. The constant λ, the kernel function $K(x,t)$ and the function $f(x)$ are given. We assume that the range of the variables is $a \le x, t \le b$

The solution of equation (1) is expressed as the truncated Laguerre series

$$y(x) = \sum_{r=0}^{N} c_r L_r(x) \qquad (2)$$

where $L_r(x)$ is the Laguerre polynomial and of degree r [3], or in the matrix form

$$[y(x)] = \mathbf{L_x C} \qquad (3)$$

Where

$$\mathbf{L_x} = [L_0(x) L_1(x) L_2(x) \ldots L_N(x)]$$
$$\mathbf{C} = [c_0 c_1 c_2 \ldots c_N]^T$$

and c_r, $r = 0,1,\ldots,N$ are coefficients to be determined.

2. METHOD FOR SOLUTION

To obtain the solution of equation (1) in the form of expression (2) we can first deduce the following matrix approximations corresponding to the Laguerre series expansions of the functions $f(x)$, $K(x,t)$ and $y(t)$.

Let the function $f(x)$ be approximated by a truncated Laguerre series

$$f(x) = \sum_{r=0}^{N} f_r L_r(x). \qquad (4)$$

Then we can put series (4) in the matrix form

$$[f(x)] = \mathbf{L_x F} \quad (5)$$

Where

$$\mathbf{F} = [f_0 f_1 \ldots f_N]^T.$$

We now consider the kernel function $K(x,t)$. If the function $K(x,t)$ can be approximated by double Laguerre series of degree N in both x and t of the form [2,7]

$$K(x,t) = \sum_{r=0}^{N} \sum_{s=0}^{N} k_{r,s} L_r(x) L_s(t) \qquad (6)$$

then we can put series (6) in the matrix form

$$[K(x,t)] = \mathbf{L_x K L_t^T}$$

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where

$$\mathbf{L_t} = [L_0(t) L_1(t) \ldots L_N(t)] \qquad (7)$$

$$\mathbf{K} = \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix}.$$

On the other hand, for the unknown function $y(t)$ in integrand, we write from expressions (2) and (3)

$$[y(t)] = \mathbf{L_t C}. \qquad (8)$$

Substituting the matrix forms (3), (5), (7) and (8) corresponding to the functions $y(x)$, $f(x)$, $K(x,t)$ and $y(t)$, respectively, into equation (1), and then simplifying the result equation, we have the matrix equation

$$\mathbf{C} = \mathbf{F} + \lambda \mathbf{K} \left\{ \int_a^b \mathbf{L_t^T L_t} dt \right\} \mathbf{C}$$

or briefly

$$(\mathbf{I} - \lambda \mathbf{KQ})\mathbf{C} = \mathbf{F} \qquad (9)$$

where

$$\mathbf{Q} = \int_a^b L_t^T L_t dt = [q_{ij}] \qquad i, j = 0,1,\ldots,N \quad (10)$$

and $\mathbf{I}$ is the unit matrix; the elements of the fixed matrix $\mathbf{Q}$ are given by [1,7]

$$q_{ij} = \int_a^b L_i(t) L_j(t) dt.$$

In equation (9), if $D(\lambda) = |\mathbf{I} - \lambda \mathbf{KQ}| \neq 0$ we get

$$\mathbf{C} = (\mathbf{I} - \lambda \mathbf{KQ})^{-1} \mathbf{F} \qquad \lambda \neq 0. \qquad (11)$$

Thus the unknown coefficients c_r, $r = 0,1,\ldots,N$ are uniquely determined by equation (11) and thereby the integral equation (1) has a unique solution. This solution is given by the truncated Laguerre series (2) .

3. ACCURACY OF SOLUTION

We can easily check the accuracy of the method. Since the truncated Laguerre series in (2) is an approximate solution of Eq.(1), it must be approximately satisfied this equation.

Then for each $x_i \in [a,b]$

$$E(x_i) = \left| y(x_i) - f(x_i) - \lambda \int_a^b K(x_i,t) y(t) dt \right| \cong 0$$

or

$$E(x_i) \leq 10^{-k_i} \qquad (\ k_i \text{ is any positive integer})$$

If

$$\max\left(10^{-k_i}\right) = 10^{-k} \qquad (\ k \text{ is any positive integer})$$

is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k}.

On the other hand, the error function can be estimated by

$$E(x) = y(x) - f(x) - \lambda \int_a^b K(x,t) y(t) dt.$$

4. NUMERICAL ILLUSTRATIONS

We show the efficiency of the presented method using the following examples. (In all figures different line shows the exact solution and the numerical solution.)

Example 1. Let us first consider the linear Fredholm integral equation [6,7,9]

$$y(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)y(t)dt \qquad (12)$$

and seek the solution $y(x)$ in Laguerre series

$$y(x) = \sum_{r=0}^{N} c_r L_r(x)$$

so that

$$f(x) = x^2 + 2x + 1 \ , \ K(x,t) = (xt + x^2t^2) \ , \ \lambda = 1 \ , \ N = 2 .$$

By using the expansions for the powers x^r in terms of the Laguerre polynomials $L_r(x)$ [3], we easily find the representations

$$f(x) = x^2 + 2x + 1 = 5L_0(x) - 6L_1(x) + 2PL_2(x)$$

and

$$K(x,t) = (xt + x^2t^2) = 5L_0(x)L_0(t) - 9L_0(x)L_1(t) + 4L_0(x)L_2(t) - 9L_1(x)L_0(t)$$
$$+17L_1(x)L_1(t) - 8L_1(x)L_2(t) + 4L_2(x)L_0(t) - 8L_2(x)L_1(t) + 4L_2(x)L_2(t)$$

and hence, from relations (5) and (7), the matrices

$$F = \begin{bmatrix} 5 \\ -6 \\ 2 \end{bmatrix} \quad , \ K = \begin{bmatrix} 5 & -9 & 4 \\ -9 & 17 & -8 \\ 4 & -8 & 4 \end{bmatrix}.$$

If we use expression (10) for $i, j = 0,1,2$ we obtain the fixed matrix

$$Q = \begin{bmatrix} 2 & 2 & 7/3 \\ 2 & 8/3 & 11/3 \\ 7/3 & 11/3 & 163/30 \end{bmatrix}.$$

Next, we substitute these matrices in equation (11) and then simplify to obtain

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 29/9 & 2/9 & -10/9 \\ -40/9 & -13/9 & -16/9 \\ 20/9 & 20/9 & 35/9 \end{bmatrix} \begin{bmatrix} 5 \\ -6 \\ 2 \end{bmatrix}.$$

The solution of this equation is

$c_0 = 113/9, c_1 = 154/9, \ c_2 = 50/9.$

By substituting the obtained coefficients in (2) the solution of (12) becomes

$$y(x)=\frac{113}{9}L_0(x)+\frac{154}{9}L_1(x)+\frac{50}{9}L_2(x)$$

or

$$y(x)=\frac{25}{9}x^2+6x+1$$

which is the exact solution.

Example 2. Second we can study the following linear Fredholm integral equation [5,8,9]

$$y(x)=0.9x^2+0.5\int_0^1 x^2t^2y(t)dt$$

and seek the solution $y(x)$ in Laguerre series

$$y(x)=\sum_{r=0}^{N}c_rL_r(x)$$

so that

$$f(x)=0.9x^2\ ,\ K(x,t)=0.5(x^2t^2)\ ,\ \lambda=1\ ,\ N=3\,.$$

By using the expansions for the powers x^r in terms of the Laguerre polynomials $L_r(x)$ [3], we easily find the representations

$$f(x)=0.9x^2=1.8L_0(\mathrm{x})-3.6L_0(\mathrm{x})+1.8L_2(\mathrm{x})$$

and

$$\begin{aligned}K(x,t)=0{,}5(x^2t^2)&=2L_0(x)L_0(t)-4L_0(x)L_1(t)+2L_0(x)L_2(t)-4L_1(x)L_0(t)\\&+8L_1(x)L_1(t)-4L_1(x)L_2(t)+2L_2(x)L_0(t)-4L_2(x)L_1(t)+2L_2(x)L_2(t)\end{aligned}$$

and hence, from relations (5) and (7), the matrices

$$\mathbf{F}=\begin{bmatrix}1.8\\-3.6\\1.8\end{bmatrix}\quad,\quad K=\begin{bmatrix}2&-4&2\\-4&8&-4\\2&-4&2\end{bmatrix}.$$

If we use expression (10) for $i,j=0,1,2$ we obtain the fixed matrix

$$\mathbf{Q}=\begin{bmatrix}1&1/2&1/6\\1/2&1/3&5/24\\1/6&5/24&13/60\end{bmatrix}$$

Next, we substitute these matrices in equation (11) and then simplify to obtain

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 37/27 & 5/54 & 2/27 \\ 20/27 & 22/27 & 4/27 \\ 10/27 & 5/54 & 25/27 \end{bmatrix} \begin{bmatrix} 1.8 \\ -3.6 \\ 1.8 \end{bmatrix}.$$

The solution of this equation is

$$c_0 = 2 \ , c_1 = -4 \ , c_2 = 2 \ .$$

Substituting these values in equation (2) we obtain

$$y(x) = 2L_0(x) - 4L_2(x) + 2L_3(x)$$

or

$$y(x) = x^2$$

which is the exact solution.

Example3. Let us now take the equation

$$y(x) = x^{11} + 2x^7 - x^6 + 5x^3 + 44x^2 - \frac{2557}{117}x + \frac{113}{9} + \int_{-1}^{1} 7(x^2 + xt + t^2)y(t)dt \ .$$

.

Next, we substitute these matrices in equation (11) and then simplify to obtain

If we use expression (10) for $i, j = 0,1,2$ we obtain the fixed matrix

Following the previous procedures, we get the exact solution of linear Fredholm integral equation for

$N = 11$ as

$$y(x) = x^{11} + 2x^7 - x^6 + 5x^3 + x - 3$$

which is the exact solution.

Example 4. We consider the problem

$$y(x) = e^{3x} - \frac{1}{9}(2e^3 + 1)x + \int_0^1 xty(t)dt \ .$$

We give numerical analysis for $N = 8,10$ in Table 1 and Figure 1.

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Table 1. Comparing the solutions and error analysis which has been found for $N = 8,10$ at Example 4.

Present method: Laguerre Method						
i	x_i	Exact Solution	$N = 8$		$N = 10$	
		$y(x_i) = e^{3x}$	$y(x_i)$	$E(x_i)$	$y(x_i)$	$E(x_i)$
0	0	1	1	0	1	0
1	0.1	1.3498588076	1.3498588075	5.5 E-11	1.3498588076	6.5 E-14
2	0.2	1.8221188004	1.8221187709	2.9 E-08	1.8221188003	8.5 E-11
3	0.3	2.4596031112	2.4596019391	1.1 E-06	2.4596031028	8.3 E-09
4	0.4	3.3201169227	3.3201007911	1.6 E-05	3.3201167163	2.0 E-07
5	0.5	4.4816890703	4.4815647656	1.2 E-04	4.4816865957	2.4 E-06
6	0.6	6.0496474644	6.048983534	6.6 E-04	6.0496285489	1.8 E-05
7	0.7	8.1661699126	8.1634153481	2.7 E-03	8.1660638118	1.0 E-04
8	0.8	11.023176381	11.013674403	9.5 E-03	11.022701699	4.7 E-04
9	0.9	14.879731725	14.851256761	2.8 E-02	14.877944531	1.7 E-03
1	1	20.085536923	20.00915	7.6 E-02	20. 079663	5.8 E-03

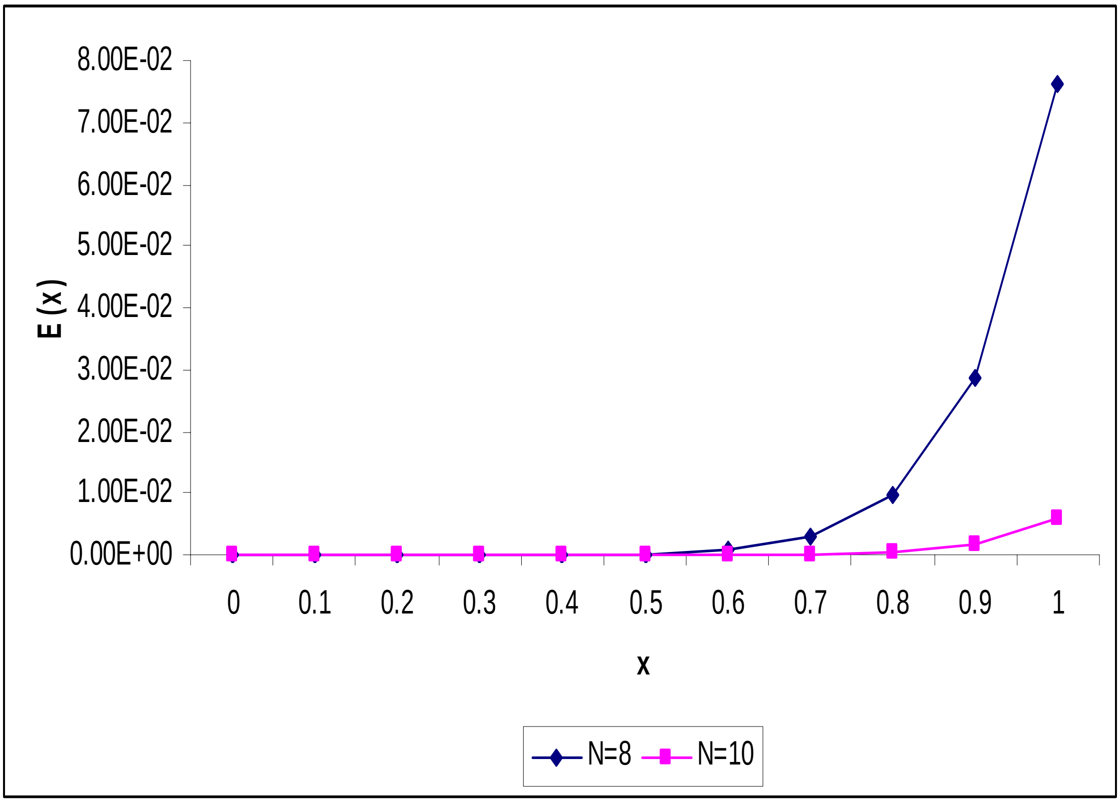

Figure 1. Numerical results of Example 4 for $N = 8,10$.

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Example 5. Consider the following the integral equation in [4]

$$y(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t} y(t)dt .$$

We give numerical analysis for various N values in Table 2 and Figure 2, 3.

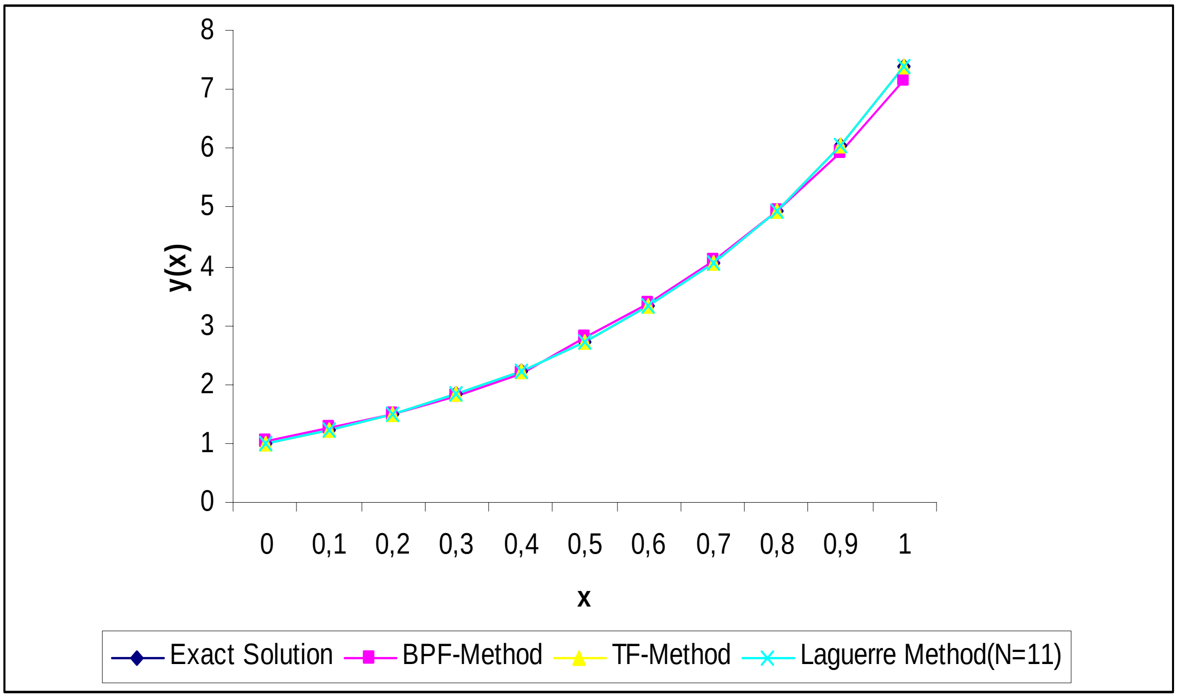

Figure 2. Comparing the solutions with the other methods which has been found for $N = 11$ at Example 5.

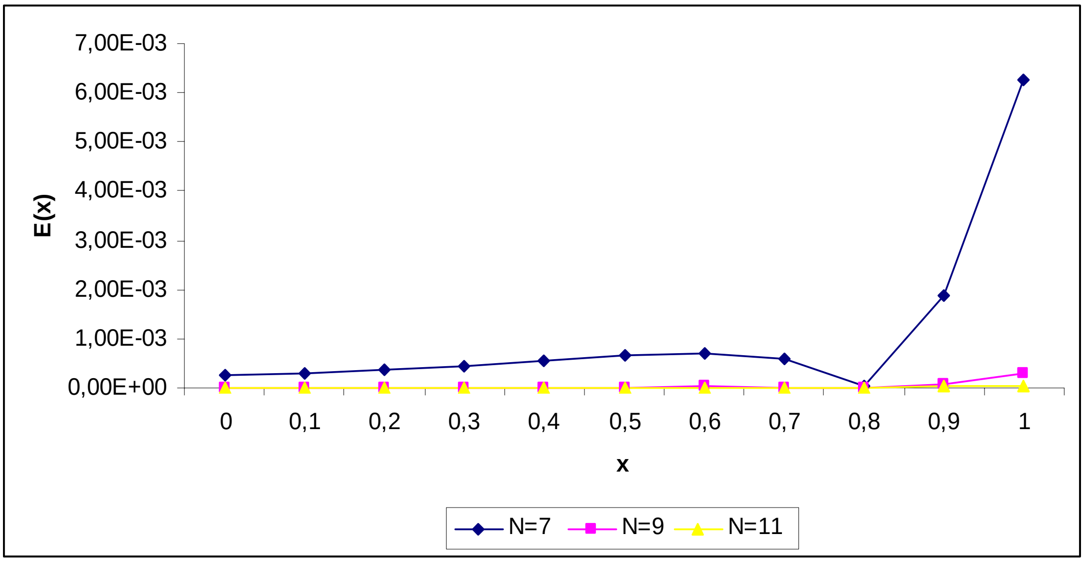

Figure 3. Numerical results of Example 5 for $N = 7,9,11$.

Table 2. Error analysis of Example 5 for the x values and comparison of present method, exact and the methods in [4] $(N = 7,9,11)$.

		Present method : Laguerre Method						
i	x_i	Exact Solution $y(x_i) = e^{2x}$	$N = 7$		$N = 9$		$N = 11$	
			$y(x_i)$	$E(x_i)$	$y(x_i)$	$E(x_i)$	$y(x_i)$	$E(x_i)$
0	0	1	1. 00025	2.5 E-05	1. 00001	1.0 E-05	1	0
1	0.1	1.22140	1. 22171	3. 0 E-04	1. 22141	1. 1 E-05	1. 2214	3. 2 E-08
2	0.2	1.49182	1. 49220	3. 7 E-04	1. 49184	1. 2 E-05	1. 49182	2. 6 E-07
3	0.3	1.82212	1. 82257	4. 5 E-04	1. 82213	1. 4 E-05	1. 82212	8. 6 E-07
4	0.4	2.22554	2. 22609	5. 5 E-04	2. 22556	1. 6 E-05	2. 22554	2. 0 E-06
5	0.5	2.71828	2. 71893	6. 5 E-04	2. 7183	1. 8 E-05	2. 71828	3. 8 E-06
6	0.6	3.32012	3. 32083	7. 0 E-04	3. 32014	1. 9 E-05	3. 32011	6. 5 E-06
7	0.7	4.05520	4. 05578	5. 8 E-04	4. 05522	1. 6 E-05	4. 05519	1. 0 E-05
8	0.8	4.95303	4. 95298	4. 9 E-05	4. 95303	4. 8 E-06	4. 95302	1. 5 E-05
9	0.9	6.04965	6. 04777	1. 8 E-03	6. 04957	8. 0 E-05	6. 04962	2. 2 E-05
10	1	7.38906	7. 3828	6. 2 E-03	7. 38876	2. 9 E-04	7. 38902	3. 6 E-05

i	x_i	BPF Method in [4] (Block Pulse Fnc.)	TF Method in [4] (Triangular Fnc.)
		$m = 32$	
0	0	1.031832	0.999844
1	0.1	1.244627	1.221598
2	0.2	1.501307	1.492294
3	0.3	1.810922	1.822684
4	0.4	2.184388	2.225880
5	0.5	2.804810	2.717857
6	0.6	3.383247	3.320648
7	0.7	4.080975	4.056474
8	0.8	4.922595	4.954570
9	0.9	5.937783	6.050568
10	1	7.162334	7.387901

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Example 6. Our last example is the equation

$$y(x) = e^{2x} - x + \int_0^1 (xe^{-2t})y(t)dt .$$

The comparison of solutions (for $N = 7, 9, 11$) with exact solution e^{2x} is given in Table 3 and Fig. 4.

Table 3. Error analysis of Example 6 for the x values.

			Present method: Laguerre Method		
i	x_i	Exact Solution $y(x_i) = e^{2x}$	$N = 7$ $y(x_i)$	$N = 9$ $y(x_i)$	$N = 11$ $y(x_i)$
0	0	1	1	1	1
1	0.1	1.221402758	1. 220964379 39	1. 22138590267	1. 22140272519
2	0.2	1.491824698	1. 490947923 22	1. 49179098664	1. 49182443738
3	0.3	1.8221188	1. 820803218 17	1. 82206823216	1. 82211793560
4	0.4	2.225540928	2. 223782850 89	2. 22547347465	2. 22553891520
5	0.5	2.718281828	2. 716062074 73	2. 71819724813	2. 71827797504
6	0.6	3.320116923	3. 317363876 34	3. 32001387690	3. 32011040308
7	0.7	4.055199967	4. 051699242 40	4. 05507286033	4. 05518978734
8	0.8	4.953032424	4. 948235426 23	4. 95286220449	4. 95301720650
9	0.9	6.049647464	6. 042305014 51	6. 04937847363	6. 04962464895
10	1	7.389056099	7. 376568593 90	7. 38854396716	7. 38901959700

i	x_i	$N = 7$ $E(x_i)$	$N = 9$ $E(x_i)$	$N = 11$ $E(x_i)$
0	0	0	0	0
1	0.1	4. 3 E-04	1. 6 E-05	3. 2 E-08
2	0.2	8. 7 E-04	3. 3 E-05	2. 6 E-07
3	0.3	1. 3 E-03	5. 0 E-05	8. 6 E-07
4	0.4	1. 7 E-03	6. 7 E-05	2. 0 E-06
5	0.5	2. 2 E-03	8. 4 E-05	3. 8 E-06
6	0.6	2. 7 E-03	1. 0 E-04	6. 5 E-06
7	0.7	3. 5 E-03	1. 2 E-04	1. 0 E-05
8	0.8	4. 7 E-03	1. 7 E-04	1. 5 E-05
9	0.9	7. 3 E-03	2. 6 E-04	2. 2 E-05
10	1	1. 2 E02	5. 1 E-04	3. 6 E-05

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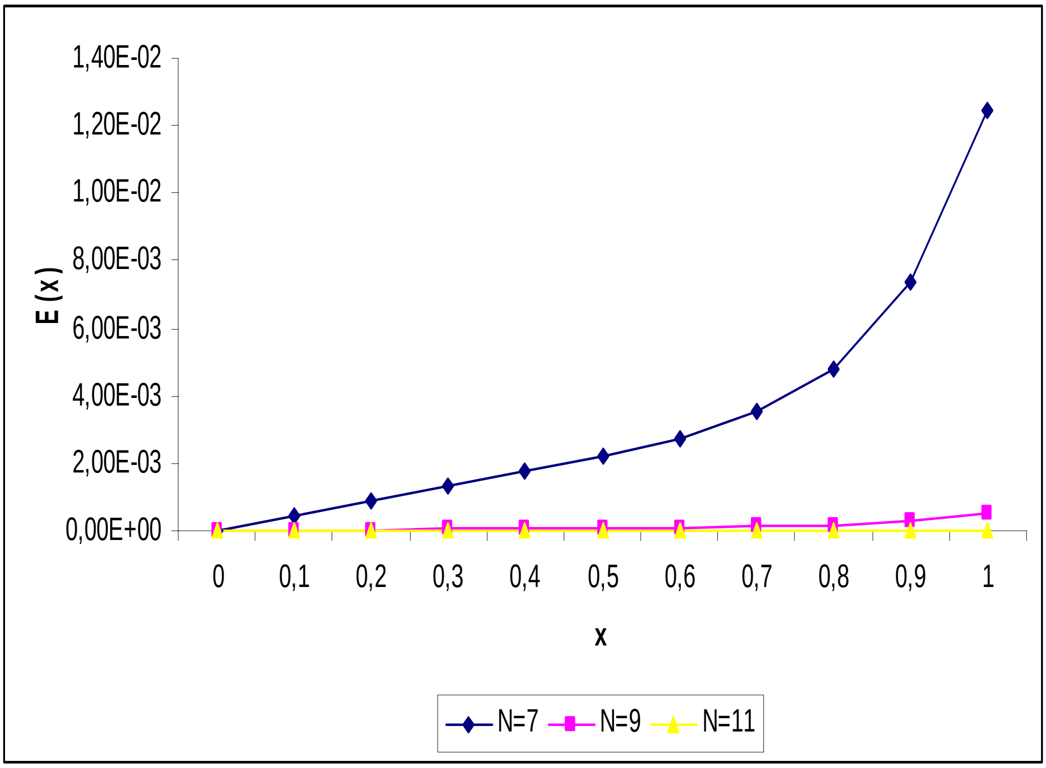

Figure 4. Numerical results of Example 6 for $N = 7, 9, 11$.

5. CONCLUSIONS AND DISCUSSIONS

In this paper, the usefulness of the method presented for the approximate solution of Fredholm integral equation (1) is demonstrated. To show the accuracy of the method, five integral equations are chosen. A considerable advantage of the method is that the solution is expressed as a truncated Laguerre series. This means that, after calculation of the Laguerre coefficients, the solution $y(x)$ can be easily evaluated for arbitrary values of x at low computation effort. If the functions $f(x)$ and $K(x,t)$ can be expanded to the Laguerre series in $a \le x,t \le b$, then there exists the solution $y(x)$; otherwise, the method cannot be used in. On the other hand, it would appear that our method shows to best advantage when the known functions $f(x)$ and $K(x,t)$ have Taylor series about the origin which converge rapidly. To get the best approximating solution of the equation, we must take more terms from the Laguerre expansions of functions, especially when they converge slowly. Briefly, for computational efficiency, the truncation limit N must be chosen sufficiently large.

REFERENCES

1. L. Fox, I. Parker, *Chebyshev polynomials in Numerical Analysis*, Clarendon Press, Oxford, 1968.
2. N. K. Basu, SIAM J. Numer. Anal., 10, 496-505, 1973.
3. Arfken,G., Laguerre Functions, Mathematical Methods for Physicists, 1985.
4. E. Babolian, H.R. Marzban, M. Salmani, *Using triangular orthogonal functions for solving Fredholm integral equations of the second kind*, App. Math. and Comput., 201, 452-464, 2008.
5. R. Farnoosh, M. Ebrahimi, *Monte Carlo method for solving Fredholm integral equations of the second kind*, App. Math. and Comput., 195, 309-315, 2008.

Erciyes Üniversitesi Fen Bilimleri Enstitüsü Dergisi 26(2): 131-142 (2010)

6. R. P. Kanwal, and K. C. Liu, *A Taylor expansion approach for solving integral equations*, Int. J. Math. Educ. Sci. Technol., 20, 411-414, 1989.
7. M. Sezer, S. Dogan, *Chebyshev series solutions of Fredholm integral equations*, Int. J. Math. Educ. Sci. Technol., 27, 5, 649-657, 1996.
8. K. Maleknejad, M. Karami, *Using the WPG method for solving integral equations of the second kind*, App. Math. and Comput., 166, 123-130, 2005.
9. S. Yalçınbaş, M. Aynigül, T. Akkaya, *Legendre Series Solutions of Fredholm Integral Equations*, Math. and Comput. App. 15, 3, 371-381, 2010.