

RELATION BETWEEN CROSSED SQUARE AND CROSSED CORNER

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ABSTRACT

The term crossed corner was introduced by Alp in (Alp , 1999) and its examples were also given in (Alp , 1999). In this paper we who defined the crossed corner morphism and also gave an important proposition which is established the equivalence between crossed corner and crossed square.

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ÖZET

Bu makalede (Alp , 1999 ; Alp , 2000 ; Alp , 1997 ; Alp and Wensley, 2000) deki çalışmaların ışığı altında çaprazlanmış kare ile çaprazlanmış köşe arasındaki ilişki incelenmiş olup , bu kategorilerin birbirlerine denk oldukları gösterilmiştir.

1.Introduction

The term of crossed module was introduced by J.H.C. Whitehead in (Whitehead , 1949). A computer programming package XMOD (Alp and Wensley, 2000) has been developed by C.D. Wensley and M. Alp, written using the GAP (Schönert , 1993) group theory programming language to calculate crossed modules, their morphism and derivations; Cat^1 -groups, their morphism and sections. The study of bi-relative Steinberg groups has led to the definition of a type of 2-dimensional crossed module which is called crossed square in (Guin Walery and Loday , 1981). The term crossed corner which is a pair of crossed modules was defined and its some examples were given by Alp in (Alp , 1999) and (Alp , 2000) respectively. Section 2 contains some basic definitions such as crossed modules, crossed square, crossed corner and their standard examples. Section 3 includes a main theorem which gives the equivalent relation between crossed corner and crossed square.

2. Crossed Modules, Crossed Square and Crossed Corner

A *crossed module* (Whitehead , 1949). $X = (\partial : S \rightarrow R)$ consist of a group homomorphism ∂ , called the *boundary* of X , together with an action $\alpha : R \rightarrow \text{Aut}(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

$$\text{CM1: } \partial(s^r) = r^{-1}(\partial s)r$$

$$\text{CM2: } s^{\partial s'} = s'^{-1}ss'$$

The standard examples of crossed modules are :

1. Any homomorphism $\partial : S \rightarrow R$ of abelian groups with R acting trivially on S may be regarded as a crossed module.
2. A *conjugation crossed module* is an inclusion of a normal subgroup $S \leq R$, where R acts on S by conjugation.
3. A *central extension crossed module* has as boundary a surjection $\partial : S \rightarrow R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.
4. An *automorphism crossed module* has as range a subgroup R of the automorphism group $\text{Aut}(S)$ of S which contains the inner automorphism group of S . The boundary maps $s \in S$ to the inner automorphism of S by s .
5. An *R-Module crossed module* has an R -module as source and ∂ is the zero map.
6. The direct product $X_1 \times X_2$ of two crossed modules has source $S_1 \times S_2$, range $R_1 \times R_2$ and boundary $\partial_1 \times \partial_2$, with R_1, R_2 acting trivially on S_2, S_1 respectively.
7. An important motivating topological example of crossed module due to Whitehead (Whitehead , 1946) is the boundary $\partial : \pi_2(X, A, x) \rightarrow \pi_1(A, x)$ from the second relative homotopy group of a based pair (X, A, x) of topological spaces, with the usual action of the fundamental group $\pi_1(A, x)$.

A morphism between two crossed modules $X = (\partial : S \rightarrow R)$ and $X' = (\partial' : S' \rightarrow R')$ is a pair (σ, ρ) , where $\sigma : S \rightarrow S'$ and $\rho : R \rightarrow R'$ are homomorphisms satisfying

$$\partial' \sigma = \rho \partial, \quad \sigma(s^r) = (\sigma s)^{\rho r}$$

A *crossed square* (Ellis . 1984 ; Guin Walery and Loday . 1981 ; Loday . 1982) consists of a commutative diagram of groups

$$\begin{array}{ccc}
 L & \xrightarrow{\mu} & M \\
 \downarrow \nu & & \downarrow \lambda \\
 N & \xrightarrow{\lambda'} & P
 \end{array}$$

together with an action of P on L , M and N (hence M acts on L and N via λ' and N acts on L and M via λ) and a function $h : M \times N \rightarrow L$ such that:

CS1 Each of maps $\mu, \nu, \lambda, \lambda'$ and the composite $\lambda' \nu$ are crossed modules:

CS2 The maps μ, ν preserve the actions of P :

CS3 $h(mm', n) = {}^m h(m', n) \cdot h(m, n)$

$h(m, nn') = h(m, n) \cdot {}^n h(m, n')$

CS4 ${}^p h(m, n) = h({}^p m, {}^p n)$

CS5 $\nu h(m, n) = {}^m n n^{-1}$

$\mu h(m, n) = m^n m^{-1}$

CS6 $h(m, \nu l) = {}^m l l^{-1}$

$h(\mu l, n) = l^n l^{-1}$

for all $l \in L$, $m, m' \in M$, $n, n' \in N$, $p \in P$.

The standard examples of crossed square are the following (Ellis . 1984) :

1. If M, N are normal subgroups of the group P , then the diagram of inclusion

$$\begin{array}{ccc}
 M \cap N & \longrightarrow & N \\
 \downarrow \nu & & \downarrow \lambda \\
 M & \longrightarrow & P
 \end{array}$$

together with the actions of P on M, N and $M \cap N$ given by conjugation, and the function $h: M \times N \rightarrow M \cap N$, $(m, n) \alpha [m, n]$ is a crossed square.

2. If M, N are ordinary P -modules and A is arbitrary abelian group on which P is assumed to act trivially, then the diagram

$$\begin{array}{ccc} A & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & P \end{array}$$

in which each map is a zero map, together with the zero map $0: M \times N \rightarrow A$ is a crossed square:

3. The diagram

$$\begin{array}{ccc} M & \xrightarrow{x} & \text{Inn}M \\ \downarrow x & & \downarrow i \\ \text{Inn}M & \longrightarrow & \text{Aut}M \end{array}$$

where X^m is the inner automorphism determined by $m \in M$ and where i is the inclusion of the inner automorphism subgroup, together with the standard actions and the function $h: \text{Inn}M \times \text{Inn}M \rightarrow M$, $(X^m, X^{m'}) \alpha [m, m']$ is a crossed square:

4. If U, V are subspaces of X with a point x_0 in common, the diagram of boundary maps (Brown and Loday, 1984).

$$\begin{array}{ccc} \pi_3(X; U, V, x_0) & \longrightarrow & \pi_2(V, U \cap V, x_0) \\ \downarrow & & \downarrow \\ \pi_2(U, U \cap V, x_0) & \longrightarrow & \pi_1(U \cap V, x_0) \end{array}$$

in which $\pi_3(X; U, V, x_0)$ is the triad homotopy group, together with the standard actions and the triad Whitehead product $h: \pi_2(U, U \cap V, x_0) \times \pi_2(V, U \cap V, x_0) \rightarrow \pi_3(X; U, V, x_0)$ is a crossed square.

5. Let M, N be groups, acting on the left of a group L . Each actions itself by 'conjugation' ${}^m m' = mm' m^{-1}$, ${}^n n' = nn' n^{-1}$

A *crossed corner* is a pair of crossed modules

$$\begin{array}{ccc} L & \xrightarrow{\mu} & M \\ \downarrow \nu & & \\ N & & \end{array}$$

(with the given actions of M, N on L) together with a function $h : M \times N \rightarrow L$ such that

$$\text{CC1} \quad h(mm', n) = {}^m h(m', n) h(m, n)$$

$$\text{CC2} \quad h(m, nn') = h(m, n) {}^n h(m, n')$$

$$\text{CC3} \quad h(\mu l, n) = l {}^n l^{-1}$$

$$\text{CC4} \quad h(m, \nu l) = {}^m l l^{-1}$$

$$\text{CC5} \quad {}^m n l = {}^{mm^{-1}} l$$

$$\text{CC6} \quad {}^n m l = {}^{nn^{-1}} l$$

The left actions of M on N , N on M by the equation

$$\text{ACT1} \quad {}^n m = (\mu h(m, n))^{-1} m$$

$$\text{ACT2} \quad {}^m n = (\nu h(m, n)) n$$

for $l \in L$, $m, m' \in M$, $n, n' \in N$.

The standard examples of crossed corner are the following (Alp, 2000):

1. The crossed corner diagram of groups

$$\begin{array}{ccc} L & \xrightarrow{\mu} & M \\ \downarrow \nu & & \\ N & & \end{array}$$

together with the function $g : N \times M \rightarrow L$. $g(n, m) = h(m, n)^{-1}$ is a crossed corner.

2. The diagram

$$\begin{array}{ccc} M \cap N & \xrightarrow{\mu} & M \\ \downarrow \nu & & \\ N & & \end{array}$$

the function $h : M \times N \rightarrow M \cap N$. $(m, n) \alpha [m, n]$ is a crossed corner together with action in **ACT1** . **ACT2** and M, N acting L by conjugation.

A morphism of crossed corner is a family of homomorphisms such that

$$\begin{array}{ccccc} L & \xrightarrow{\mu} & M & & \\ \downarrow \nu & \searrow I_l & \searrow I_m & & \\ N & & L' & \xrightarrow{\mu'} & M' \\ & \searrow I_n & \downarrow \nu' & & \\ & & N' & & \end{array}$$

satisfying $I_n \nu = \nu' I_l$ and $\mu' I_l = I_m \mu$.

Proposition 2.1 Crossed corner together with the action **ACT1** and **ACT2** satisfies the following equations.

$$h(m, n)^{nm} h(m', n') = {}^{nm} h(m', n') h(m, n) \quad (1)$$

$${}^n h(m, n^{-1}) = h(m, n)^{-1} = {}^m h(m^{-1}, n) \quad (2)$$

$$h(m, 1) = h(m, 1) = 1 \quad (3)$$

$$h({}^m m', {}^m n) = {}^m h(m', n) \quad (4)$$

$$h({}^n m, {}^n k) = {}^n h(m, k) \quad (5)$$

Proof: The proofs of all equations can be found in (Alp , 1999).

3.Main Theorem

Theorem 3.1 Suppose that M, N, L are groups and $h: M \times N \rightarrow L$ is a function. Then the following statements are equivalent.

1. Crossed corner with the six equations ;
2. Equation CS1-CS6 hold.

Proof: The conditions CS3 and CS6 are satisfied by the conditions CC1,CC2,CC3 and CC4 of crossed corner. The defined crossed corner action ACT1 and ACT2 satisfy the CS5. At the same time the numbered items (4) and (5) Proposition 2.1 satisfy the CS4. CS1 is clear since μ and ν are crossed modules. The maps μ and ν preserve the action of P , since $P = 1$ in the case of crossed corner. Hence CS2 is satisfied. Inversely, CC1, CC2,CC3 and CC4 are clear from the crossed square definition. We can get $vh(m, n)n = mnm^{-1}$ from the numbered item (1) of Proposition 2.1. So CC5 and CC6 are satisfied very easily. Therefore, the proof is completed.

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