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*RESEARCH ARTICLE*

# SOLUTIONS OF LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS OF ORDER $n-1<nq<n$

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**ABSTRACT**

In this study, the linear Caputo fractional differential equation of order $n-1<nq<n$ is investigated. First, the solution of the equation of order $0<q<1$, with variable coefficients, is obtained by using the solution of differential equation of integer order which is the least integer greater than fractional order. Moreover, the solution of linear fractional differential equations of order $n-1<nq<n$ is considered. The solutions of the equation are presented in terms of Mittag-Leffler function with three parameters. The main goal of this study is to present a closed-series form of the solutions. To demonstrate the accuracy and the effectiveness of the proposed approach, some numerical solutions are given.

**Keywords:** *Fractional Differential Equation, Mittag-Leffler function, Three parameters*

## 1. INTRODUCTION

The fractional derivative is very appropriate in modelling of physical pneumonias which involves past memory. Since fractional differential equations hold memory and are non-local in nature, it is known that mathematical models with fractional derivatives are more convenient and economical (See [1-8] and the references there in for more details). Due to this fact, they have been drawing increasing importance from scientist in past three decades (See [9-15] for some applications).

In this study, the Caputo fractional derivative is used. Since properties of Caputo fractional derivative are closed to properties of integer derivative, enormous studies on Caputo fractional differential equation and their applications can be found in the literature.

The linear Caputo fractional differential equation of order $0<q<1$, with variable coefficients, is considered and its solution is obtained as a series form. Next, the solution of linear sequential Caputo fractional differential equations of order $n-1<nq<n$ is examined. The solutions of the equation are presented in terms of Mittag-Leffler function with three parameters.

## 2. PRELIMINARY DEFINITIONS

***Definition 1.*** The derivative operator $D^q$ of order $q>0$ is defined as follows

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s) ds \quad for \quad n-1 < q < n, \tag{1}$$

where $n$ is the smallest integer that exceeds $q$, $t \in [t_0, t_0 + T]$ and $u^{(n)}(t) = \frac{d^n u(t)}{dt}$.

The expression (1) is definition of Caputo (left-sided) fractional derivative of $u(t)$ for $n-1 < q < n$ [7].
Moreover, it is clear that Caputo fractional derivative of order $0 < q < 1$ is defined as follows:

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds. \tag{2}$$

***Definition 2.*** The Mittag-Leffler function is defined by the series expansion as shown below

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}, \tag{3}$$

where $q > 0$ [8].

In [7], some properties of Mittag-Leffler function and Caputo derivative are given as follows:

*(i)* $D^{nq} u(t) = D^{(n-1)q}(D^q) u(t)$ for $n-1 < nq < n$.
*(ii)* If $q = 1$, $E_{1,1}(t) = e^t$, then it can be said that Mittag-Leffler function is generalization of usual exponential function.
*(iii)* $D^q(E_q(t^q)) = E_q(t^q)$.
*(iv)* $D^{nq}(E_q(rt^q)) = r^n E_q(rt^q)$ where $0 < q < 1$, $r$ is a constant and $n \in \mathbb{N}$.

***Definition 3.*** The Mittag-Leffler function with two parameters is defined by the series expansion as shown below

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{4}$$

where $\alpha, \beta > 0$ [8].

***Definition 4.*** The Mittag-Leffler function with three parameters is defined by the series expansion as shown below

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\,\Gamma(\alpha k + \beta)}, \tag{5}$$

where $\alpha, \beta, \gamma > 0$ and $(\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1)$ [8].

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## 3. METHODOLOGY

In this study, fractional power series expansion method is used for linear Caputo fractional differential equation. The method is presented in [16] to obtain approximate solutions of fractional partial differential equations. The numerical results show that the method is significant method for fractional differential equation because of its simplicity and accuracy.

The main idea of the method is based on solving the differential equation of integer order which is the least integer greater than fractional order. Then, this solution is transformed the solution of fractional differential equation by using following series expansion:

$$u(t) \cong T\big(v(t)\big) = \sum_{k=0}^{m-1} \frac{v^{(k)}(0)t^k}{k!} + \sum_{k=1}^{\infty}\sum_{i=0}^{m-1} \frac{v^{(mk+i)}(0)t^{kq+i}}{\Gamma(kq+1+i)}, \tag{6}$$

where $u(t)$ and $v(t)$ are solutions of fractional and ordinary differential equations respectively, $m$ is the least integer greater than fractional order $q$. For $0 < q < 1$, transformation (6) is expressed as follows:

$$u(t) \cong T\big(v(t)\big) = \sum_{k=0}^{\infty} \frac{v^{(k)}(0)t^{kq}}{\Gamma(kq+1)}. \tag{7}$$

## 4. SOLUTION OF THE LINEAR FRACTIONAL DIFFERENTIAL EQUATION

### 4.1. Fractional Differential Equation of Order $0 < q < 1$

Let us consider the following linear differential equation with initial condition:

$$\begin{aligned} D^q u(t) &= p(t,q)u(t) + f(t,q), \quad t\epsilon[0,T], \quad T>0 \\ u(0) &= u_0, \end{aligned} \tag{8}$$

where $p(t,q)$ and $f(t,q)$ are continuous on $t\epsilon[0,T]$ for $0 < q \leq 1$ and $p(t,1)$, $f(t,1) \in C^\infty([0,T],\mathbb{R})$.

In order to obtain the solution $u(t)$ of the problem (8) by using fractional power series expansion method, we need to have the solution of following ordinary differential equation

$$\begin{aligned} v'(t) &= \tilde{p}(t)v(t) + \tilde{f}(t), \quad t\epsilon[0,T], \quad T>0 \\ v(0) &= u_0, \end{aligned} \tag{9}$$

where $\tilde{p}(t) = p(t,1)$ and $\tilde{f}(t) = f(t,1)$.

The solution of problem (9) is well known and as follows:

$$v(t) = u_0 e^{\int \tilde{p}(t)dt} + e^{\int \tilde{p}(t)dt} \int_0^t \tilde{f}(\tau) e^{\int \tilde{p}(\tau)d\tau} d\tau. \tag{10}$$

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By plugging the following derivatives of the solution $v(t)$ at $t = 0$

$$v(0) = u_0$$
$$v^{(n)}(0) = \sum_{k=0}^{n-1} \binom{n-1}{k} \tilde{p}^{(k)}(0) v^{(n-1-k)}(0) + \tilde{f}^{(n-1)}(0) \tag{11}$$

into transformation (7), the solution $u(t)$ of the problem (8) is obtained as follows:

$$u(t) \cong u_0 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \left( \binom{n-1}{k} \tilde{p}^{(k)}(0) v^{(n-1-k)}(0) + \tilde{f}^{(n-1)}(0) \right) \frac{t^{nq}}{\Gamma(nq+1)} . \tag{12}$$

If $p(t,q) = \lambda$ is a constant in problem (8), the solution of problem (9) is obtained as follows:

$$v(t) = u_0 e^{\lambda t} + e^{\lambda t} \int_0^t \tilde{f}(\tau) e^{\lambda \tau} d\tau . \tag{13}$$

Moreover, derivatives of $v(t)$ are formed as follows:

$$v^{(n)}(0) = \lambda^n u_0 + \sum_{k=1}^{n} \lambda^{n-k} \tilde{f}^{(k-1)}(0). \tag{14}$$

Thus, by using transformation (7), the solution $u(t)$ of the problem (8) for $p(t,q) = \lambda$ (constant) is obtained in the following form:

$$u(t) \cong u_0 E_q(\lambda t^q) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left( \lambda^{n-k} \tilde{f}^{(n-1)}(0) \right) \frac{t^{nq}}{\Gamma(nq+1)} . \tag{15}$$

Additionally, for $p(t,q) = \lambda$, the problem (8) is investigated in [5,6] and its solution is presented in the following explicit form

$$u(t) = u_0 E_q(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s,q) ds . \tag{16}$$

Hence, because of existence and uniqueness of the solution, the solution (15) equals to the solution (16) and this equality implies the following proposition.

***Proposition 1.*** Let $f(t,q)$ be continuous on $t \epsilon [0,T]$ for $0 < q \leq 1$ and $\lambda \in \mathbb{R}$. The following equality is satisfied

$$\int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s,q) ds \cong \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\left( \lambda^{n-k} \tilde{f}^{(n-1)}(0) \right) t^{nq}}{\Gamma(nq+1)} , \tag{17}$$

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where $\tilde{f}(t) = f(t,1) \in C^{\infty}([0,T],\mathbb{R})$ .

In order to show efficiency of the method, Proposition 1 is used. In Figure 1 and Figure 2, numerical simulations of the difference of both side of the equality (17) are given for some values of $q$ and functions $f(t,q) = \frac{t^q}{\Gamma(q+1)}$ and $(t,q) = \frac{t^{2q}}{\Gamma(2q+1)}$ .

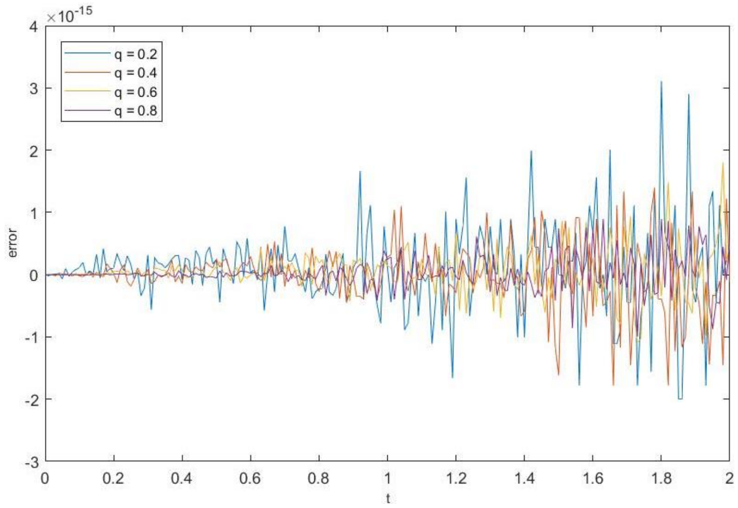


**Figure 1:** When $q = 0.2, 0.4, 0.6, 08$, the graph of difference of both side of the equality (17) for function $f(t,q) = \frac{t^q}{\Gamma(q+1)}$ and for 10 iteration

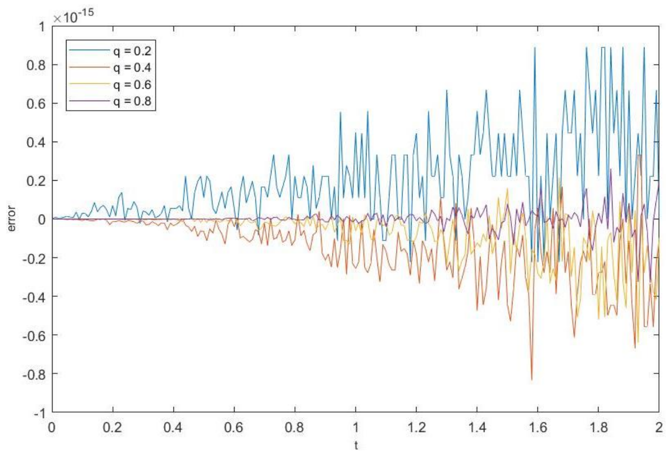


**Figure 2:** When $q = 0.2, 0.4, 0.6, 08$, the graph of difference of both side of the equality (17) for function $f(t,q) = \frac{t^{2q}}{\Gamma(2q+1)}$ and for 10 iteration

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**4.2. Fractional Differential Equation of Order $m-1<mq<m$**
The following linear sequential fractional differential equation with initial conditions is investigated in this section.

$$\begin{aligned} &D^{mq}u(t)+b_{m-1}D^{(m-1)q}u(t)+\cdots+b_1D^q u(t)+b_0u(t)=0, \quad t\epsilon[0,T], \quad T>0 \\ &\qquad u(0)=u_0, \quad D^{iq}u(0)=u_{i,0}, \quad i=1,2,\cdots,m-1 \end{aligned}, \tag{18}$$

where $(m-1)<mq<m$ for $m\in\mathbb{N}$ and $b_i\in\mathbb{R}$, $i=1,2,\cdots,m-1$.

It is well known that the solutions of equation (18) are given in the form of $u=E_q(rt^q)$ where $r$ is the solution of the corresponding characteristic equation of the following form:

$$P(r)=r^m+b_{m-1}r^{m-1}+\cdots+b_1r+b_0=0\,. \tag{19}$$

If equation (19) has $k$ distinct roots $r_i$ for $i=1,2,\cdots,k$ and $k\le m$, a solution of the equation (18) is obtained as follows:

$$u(t)=\sum_{i=1}^{k}c_iE_q(r_it^q)\,. \tag{20}$$

Moreover, if equation (19) has $k$ coincident roots $r_0$ for $k\le m$, the solutions $U_i(t)$ of the equation (18) can be constructed by the following recursive relation which is obtained from equation (16).

$$\begin{aligned} U_0(t)&=E_{q,1}(r_0t^q) \\ U_i(t)&=u_0E_{q,1}(r_0t^q)+U_{i-1}(0)\int_0^t(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)U_{i-1}(s)ds\,, \end{aligned} \tag{21}$$

where $i=1,2,\cdots,k-1$. Hence, the solution of the equation (18) is formed as follows:

$$u(t)=\sum_{i=1}^{k}c_iU_i(t)\,. \tag{22}$$

In the following subsection, the closed forms of the solutions $U_i(t)$ related with the coincident roots of characteristic equation are presented by using fractional power series method.

**4.2.1. The closed forms of the solutions related with the m coincident roots**
If the characteristic equation (19) has $m$ coincident roots $r_0$, sequential differential equation (18) reduced to

$$(D^q-r_0)(D^q-r_0)^{m-1}u=0\,. \tag{23}$$

Letting $(D^q-r_0)^{m-1}u(t)=U_0(t)$, the following fractional differential equation is obtained

$$(D^q-r_0)U_0(t)=0$$

and by using the solution (15) with $f(t) = 0$, the solution is obtained as follows:

$$U_0(t) = E_q(r_0 t^q)\,. \tag{24}$$

Moreover, letting $(D^q - r_0)^{m-2}u(t) = U_1(t)$, the following fractional differential equation is obtained

$$\begin{gathered}(D^q - r_0)U_1(t) = U_0(t) \\ \Rightarrow D^q U_1(t) - r_0 U_1(t) = E_q(r_0 t^q)\end{gathered} \tag{25}$$

Now, we solve the equation (25) by using fractional power series method. Let us consider the following ordinary differential equation which is obtained from equation (25) by equaling q to 1

$$V'_1(t) - r_0 V_1(t) = e^{r_0 t}. \tag{26}$$

The solution of equation (26) is $V_1(t) = te^{r_0 t}$. Then, applying transformation (7) to $V_1(t) = te^{r_0 t}$, it is obtained that

$$U_1(t) = \frac{t^q}{q} E_{q,q}(r_0 t^q)\,, \tag{27}$$

which is exact solution of (25). Similarly, letting $(D^q - r_0)^{m-3}u(t) = U_2(t)$, the following fractional differential equation is obtained

$$\begin{gathered}(D^q - r_0)U_2(t) = U_1(t) \\ \Rightarrow D^q U_2(t) - r_0 U_2(t) = \frac{t^q}{q} E_{q,q}(r_0 t^q)\,.\end{gathered} \tag{28}$$

Also, the solution of the equation (28) is obtained in the form of Mittag-Leffler function with three parameters as follows:

$$U_2(t) = \frac{t^{2q}}{2q} E^2_{q,2q}(r_0 t^q) \tag{29}$$

which is exact solution of (28). After applying m times these calculations recursively, the following solutions are obtained

$$\begin{aligned} U_1(t) &= \frac{t^q}{q} E^1_{q,q}(r_0 t^q) \\ U_2(t) &= \frac{t^{2q}}{2q} E^2_{q,2q}(r_0 t^q) \\ U_3(t) &= \frac{t^{3q}}{3q} E^3_{q,3q}(r_0 t^q)\,. \\ &\vdots \end{aligned}$$

Moreover, these solutions can be written as follows:

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$$U_n(t) = \frac{t^{nq}}{nq} E^n_{q,nq}(r_0 t^q), \quad n = 1,2,\cdots,m-1\,. \tag{30}$$

Consequently, if characteristic equation (19) has $m$ coincident roots $r_0$, there exists $m$ linearly independent solutions as follows:

$$\begin{aligned} U_0(t) &= E_q(r_0 t^q), \\ U_n(t) &= \frac{t^{nq}}{nq} E^n_{q,nq}(r_0 t^q), \quad n = 1,2,\cdots,m-1\,. \end{aligned} \tag{31}$$

**5. CONCLUSION**

The approximate solution of linear Caputo fractional equation of order $0 < q < 1$ with variable coefficients is constructed as a series form. In this construction, the solution of differential equation of integer order which is the least integer greater than fractional order and its derivatives are used. Numerical results show that the approximation is significantly efficient when components of $f(t,q)$ are formed as $\frac{t^{nq}}{\Gamma(nq+1)}$ and $p(t,q)$ is constant in equation (8). Moreover, this result yields to obtain the solution of linear sequential fractional differential equation of higher order with constant coefficient. Specifically, when characteristic equation has coincident root, the solutions are presented in terms of Mittag-Leffler function with three parameters.

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