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Exactness of Proximal Groupoid Homomorphisms

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Abstract

This article introduces proximal algebraic structures in descriptive proximity spaces. A descriptive proximity space is an extension of an Efremovič proximity space that contains non-abstract points describable with feature vectors. Various types of groupoids is such spaces are considered. A groupoid is a nonempty set equipped with a binary operation. A groupoid A is descriptively near a groupoid B, provided there is at least one pair of points $a \in A, b \in B$ with matching descriptions. This leads to a consideration of mappings on groupoid A into groupoid B that are descriptive homomorphisms.

Keywords: Proximity relation, descriptive proximity space, proximal groupoid, descriptive homomorphism.

Proksimal Grupoid Homomorfizmalarının Tamlığı

Özet

Bu çalışmada tanımsal proksimiti uzayda proksimal cebirsel yapılar tanıtıldı. Tanımsal proksimiti uzay, özellik vektörleri ile nitelendirilebilen ve soyut olmayan noktaları içeren Efremovič proksimiti uzayının bir genelleştirilmişidir. Grupoidlerin farklı türleri böyle düşünülen uzaylardır. Grupoid, bir ikili işlem ile donatılmış boş olmayan bir kümedir. A ve B iki grupoid olmak üzere, eşleşen tanımlamalar ile en az bir $a \in A, b \in B$ nokta çifti varsa

A grupoidi B grupoidine tanımsal yakındır. Bu kavram, A grupoidinden B grupoidine dönüşümleri ve özellikle tanımsal homomorfizmaları göz önünde bulundurmamıza yol açar.

Anahtar Kelimeler: Proksimiti bağıntı, tanımsal proksimiti uzay, proksimal grupoid, tanımsal homomorfizma.

Introduction

This article introduces exactness of homomorphisms on groupoids in proximity and descriptive proximity spaces. A descriptive proximity space [1, 2] is an extension of an Efremovič proximity space [3]. This extension is made possible by the introduction of feature vectors that describe each point in a proximity space. Sets A, B in a proximity space X are near, provided there is at least one pair of points $a \in A, b \in B$ with matching descriptions. The focus is on descriptive groupoids (a groupoid is a set with binary operation "$*$") that can be found in such spaces. Groupoids $A(*), B(*)$ in a descriptive proximity space are near each other, provided the A and B are descriptively near.

1. Preliminaries

Let X be a nonempty set endowed with an Efremovič proximity relation [3]. $\mathcal{P}(X)$ denotes the collection of all subsets of X. In an ordinary metric closure space [4, §14A.1] X, the closure of $A \subset X$ (denoted by $cl(A)$) is defined by

$$cl(A) = \{x \in X : d(x, A) = 0\}, \text{ where}$$

$$d(x, A) = inf\,\{d(x, a) : a \in A\},$$

i.e., $cl(A)$ is the set of all points x in X that are close to A ($d(x, A)$ is the Hausdorff distance [5, §22, p.128] between x and the set A where $d(x, a) = |x - a|$ (standard distance)). Subsets $A, B \in \mathcal{P}(X)$ are spatially near (denoted by $A\ \delta\ B$), provided the intersection of closure of A and the closure of B is nonempty, i.e., $cl(A) \cap cl(B) \neq \emptyset$. That is, nonempty sets are spatially near, provided the sets have at least one point in common.

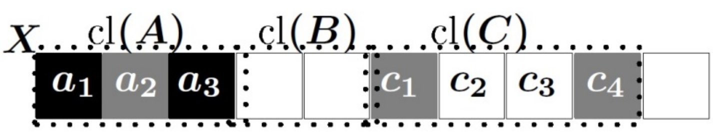

Figure 1. $cl(A) \cap cl(B) \neq \varnothing$ implies A is close to B

Example 1.1 (Spatially Near Sets) Let the set of points X be represented by the weave cells in Fig. 1 and let the closures of sets $A, B, C \in \mathcal{P}(X)$ be represented by $cl(A), cl(B), cl(C)$ in Fig. 1. The boundary points for A, B, C are represented by dotted lines in Fig. 1. Since A and B have common boundary points, we have $cl(A) \cap cl(B) \neq \varnothing$. Hence $A \, \delta \, B$.

A spatial nearness relation δ (called a *discrete proximity*) is defined by

$$\delta = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : cl(A) \cap cl(B) \neq \varnothing\}.$$

The following proximity space axioms are given by J. M. Smirnov [6] based on what V. Efremovič introduced during the first half of the 1930s [3]:

EF.1 If the set A is close to B, then B is close to A.

EF.2 $A \cup B$ is close to C, if and only if, at least one of the sets A or B is close to C.

EF.3 Two points are close, if and only if, they are the same point.

EF.4 All sets are far from the empty set $\varnothing$.

EF.5 For any two sets A and B which are far from each other, there exists C and D, $C \cup D = X$, such that A is far from C and B is far from D (*Efremovič axiom*).

The pair (X, δ) is called an EF-proximity space. In a proximity space X, the closure of A in X coincides with the intersection of all closed sets that contain A. From Smirnov, $\delta(A, B) = 0$ indicates that A is close to B.

Theorem 1.1 [6] The closure of any set A in the proximity space X is the set of points $x \in X$ that are close to A.

Descriptively near sets were introduced as a means of solving classification and pattern recognition problems arising from disjoint sets (i.e., sets with empty spatial intersections) that resemble each other. Descriptively near sets were introduced in 2007 [7, 8]. Recently, the connections between spatially near sets and descriptively near sets have been explored in [1, 2, 9].

Let X be a nonempty set of non-abstract points, x a member of X, $\Phi = \{\phi_1, \ldots, \phi_n\}$ a set of probe functions that represent features of each x. Points as locations with features lead, for example, to a proximal view of sets of picture points in digital images [10]. A probe function $\phi : X \to \mathbb{R}$ is real-valued and represents a feature of an object such as a sample point (pixel) in a picture. Let $\Phi(x)$ denote a feature vector for the object x, i.e., a vector of feature values that describe x. A feature vector provides a description of a point x in X. To obtain a descriptive proximity relation (denoted by δ_Φ), one first chooses a set of probe functions, which provides a basis for describing points in a set. Let $A, B \in \mathcal{P}(X)$. Let $\mathcal{Q}(A), \mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. That is,

$$\mathcal{Q}(A) = \{\Phi(a) : a \in A\},$$

$$\mathcal{Q}(B) = \{\Phi(b) : b \in B\}.$$

The expression $A \ \delta_\Phi \ B$ reads A is descriptively near B. The relation δ_Φ is called a *descriptive proximity relation*. Similarly, $A \ \underline{\delta}_\Phi \ B$ denotes that A is descriptively far (remote) from B. The descriptive proximity of A and B is defined by

$$A \ \delta_\Phi \ B \Leftrightarrow \mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset.$$

The descriptive intersection $\underset{\Phi}{\cap}$ of A and B is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \mathcal{Q}(A) \ and \ \Phi(x) \in \mathcal{Q}(B)\}.$$

That is, $x \in A \cup B$ is in $A \underset{\Phi}{\cap} B$, provided $\Phi(x) = \Phi(a) = \Phi(b)$ for some $a \in A, b \in B$.

Example 1.2 (Descriptive Intersection of Disjoint Sets) Choose Φ to be a set of probe functions representing weave cell greylevel intensities (from black to shades of grey to white) in Fig. 1. Let the set of cells X in the sample weave strip be endowed with δ_Φ. Sets $A, C \in \mathcal{P}(X)$ are disjoint but descriptively close. Let $a_2 \in A, c_4 \in C$ be a pair of weave cells. Observe that $\Phi(a_2)$ in $\mathcal{Q}(A)$ is descriptively near $\Phi(c_4)$ in $\mathcal{Q}(C)$, since $\Phi(a_2) = \Phi(c_4)$. Also observe that $\Phi(a_2) = \Phi(c_1)$. Except for a_2, the cells in A do not have descriptions that match the description of any cell C. Hence we have $A \underset{\Phi}{\cap} C = \{a_2, c_1, c_4\}$.

The descriptive proximity relation δ_Φ is defined by

$$\delta_\Phi = \left\{ (A,B) \in \mathcal{P}(X) \times \mathcal{P}(X) : cl(A) \underset{\Phi}{\cap} cl(B) \neq \varnothing \right\}.$$

Whenever the sets A and B have no points with matching descriptions, the sets are descriptively far from each other (denoted by $A \underline{\delta}_\Phi B$), where

$$\underline{\delta}_\Phi = \mathcal{P}(X) \times \mathcal{P}(X) \setminus \delta_\Phi .$$

A binary relation δ_Φ is a descriptive EF-proximity, provided the following axioms are satisfied for $A, B, C \in \mathcal{P}(X)$:

dEF.1 If the set A is descriptively close to B, then B is descriptively close to A.

dEF.2 $A \cup B$ is descriptively close to C, if and only if, at least one of the sets A or B is descriptively close to C.

dEF.3 Two points $x, y \in X$ are descriptively close, if and only if, the description of x matches the description of y.

dEF.4 All nonempty sets are descriptively far from the empty set $\varnothing$.

dEF.5 For any two sets A and B which are descriptively far from each other, there exists C and D, $C \cup D = X$, such that A is descriptively far from C and B is descriptively far from D (*descriptive Efremovič axiom*).

The pair (X, δ_Φ) is called a descriptive EF-proximity space.

In a descriptive proximity space X, the descriptive closure of A in X contains all points in X that are descriptively close to the closure of A. Let $\delta_\Phi(A,B) = 0$ indicate that A is descriptively close to B. The descriptive closure of a set A (denoted by $cl_\Phi(A)$) is defined by

$$cl_\Phi(A) = \{ x \in X : \Phi(x) \in \mathcal{Q}(cl(A)) \}.$$

That is, $x \in X$ is in the descriptive closure of A, provided $\Phi(x)$ (description of x) matches $\Phi(a) \in \mathcal{Q}(cl(A))$ for at least one $a \in cl(A)$.

Example 1.3 (Descriptive Closure of a Set) Choose X to be the set of weave cells shown in Fig. 1 and let Φ contain probe functions representing weave cell greyscale intensities. Since cells c_1, c_4 in $cl(C)$ are descriptively near a_2 in $cl(A)$, then $cl_{\Phi}(C) = \{a_2\} \cup cl(C)$. Observe that each $c \in cl(C)$ matches the description of itself, i.e., $\Phi(c) \in \mathcal{Q}(C)$. Consequently, $cl(C) \subseteq cl_{\Phi}(C)$. This is true in general (see Lemma 1 in [2, §3]).

Theorem 1.2 [11] The descriptive closure of any set A in the descriptive proximity space X is the set of points $x \in X$ that are descriptively close to A.

2. Descriptive Mappings and Homomorphisms

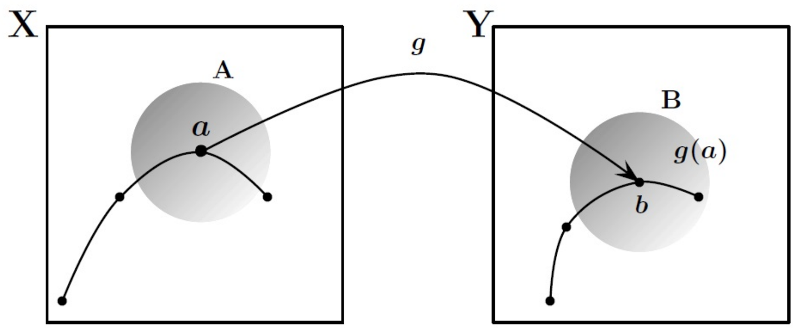

Figure 2. $g(a) = b$ such that $\Phi(a) = \Phi(b)$.

Let (X, δ_{Φ}), (Y, δ_{Φ}) be descriptive EF-proximity spaces and $A \subseteq X$, $B \subseteq Y$. A mapping $g : A \to B$ is defined by

$$g(a) = \begin{cases} b & , \quad \textit{if } \Phi(a) = \Phi(b) \quad \textit{for some } b \in B \\ y & , \quad \textit{if } \Phi(a) \neq \Phi(y) \quad \textit{for any } y \in B \end{cases}.$$

The mapping g is called a Φ*-descriptive mapping*. Hence we can observe that if there is a Φ-descriptive mapping of A to B, then $A\delta_{\Phi} g(A)$ or $A\underline{\delta}_{\Phi} g(A)$.

Example 2.1 (Φ-Descriptive Mapping Based on Gradient Orientation) Let X, Y in Fig. 2 be endowed with a descriptive proximity relation δ_{Φ} such that Φ contains a probe function that represents the gradient orientation of a point. The gradient orientation of a point x on a curve in either X or Y is defined to be the angle of the tangent to the point. Let g be a Φ-descriptive mapping of X into Y. Then $g(a) = b$ in Fig. 2, provided $\Phi(a) = \Phi(b)$, i.e., provided points $a \in A$ and $b \in B$ have the same gradient orientation.

Theorem 2.1 Let (X,δ_Φ), (Y,δ_Φ) be descriptive EF-proximity spaces with $A, A' \subseteq X$, $B \subseteq Y$. If $g: A \to B$ is a Φ-descriptive mapping which is defined by $g(a) = b$ such that $\Phi(a) = \Phi(b)$ for each $a \in A$ and some $b \in B$, then $(A \cup A')\ \delta_\Phi\ g(A \cup A')$.

Proof. We can always find some $b \in B$ such that $g(a) = b$ and $\Phi(a) = \Phi(b)$. Consequently, $\mathcal{Q}(g(A)) \subseteq \mathcal{Q}(B)$ and we have $A \underset{\Phi}{\cap} B \neq \varnothing$. Therefore we get that $A\ \delta_\Phi\ g(A)$. Let $A \cup A' \subseteq X$ for $A, A' \subseteq X$ and so $\mathcal{Q}(g(A \cup A')) \subseteq \mathcal{Q}(B)$ since $\mathcal{Q}(g(A)) \subseteq \mathcal{Q}(B)$. Hence we obtain that $(A \cup A')\ \delta_\Phi\ g(A \cup A')$.

Corollary 2.1 Let (X,δ_Φ), (Y,δ_Φ) be descriptive EF-proximity spaces with $A, A' \subseteq X$, $B \subseteq Y$. If $g: A \to B$ is a Φ-descriptive mapping which defined by $g(a) = b$ such that $\Phi(a) = \Phi(b)$ for each $a \in A$ and some $b \in B$, then $A\ \delta_\Phi\ g(A \cup A')$ or $A'\ \delta_\Phi\ g(A \cup A')$.

Corollary 2.2 Let (X,δ_Φ), (Y,δ_Φ) be descriptive EF-proximity spaces with $A_1, \ldots, A_n \subseteq X$, $B \subseteq Y$. If $g: A \to B$ is a Φ-descriptive mapping which defined by $g(a) = b$ such that $\Phi(a) = \Phi(b)$ for each $a \in A$ and some $b \in B$, then $\bigcup_{i=1}^{n} A_i \delta_\Phi\ g\left(\bigcup_{i=1}^{n} A_i\right)$.

A binary operation on a set S is a mapping of $S \times S$ into S, where $S \times S$ is the set of all ordered pairs of elements of S. A groupoid is a system $S(*)$ consisting of a nonempty set S together with a binary operation "$*$" on S. A proximal groupoid is a groupoid in proximity space.

Let $S(*)$ and $S'(\cdot)$ be groupoids. A mapping h of S into S' is called a *homomorphism* if $h(a*b) = h(a) \cdot h(b)$ for all $a, b \in S$. A one-to-one homomorphism h of S onto S' is called an *isomorphism* of S to S' [12, §1.3].

Consider groupoids $\mathcal{Q}(A)(*_1)$, $\mathcal{Q}(B)(*_2)$, where $A \subseteq X$, $B \subseteq Y$. A mapping

$$h_\Phi : \mathcal{Q}(B) \to \mathcal{Q}(A)$$

is called a *descriptive homomorphism*, provided

$$h_\Phi\left(\Phi_B(b_1) *_2 \Phi_B(b_2)\right) = h_\Phi\left(\Phi_B(b_1)\right) *_1 h_\Phi\left(\Phi_B(b_2)\right)$$

for all $\Phi_B(b_1), \Phi_B(b_2) \in \mathcal{Q}(B)$ [13].

Let $A(\cdot_1)$ and $B(\cdot_2)$ be groupoids, $h: B \to A$ be a homomorphism and $\Phi_A : A \to \mathcal{Q}(A)$, $a \mapsto \Phi(a)$ be an object description. The object description Φ_A of A into $\mathcal{Q}(A)$ is an object description homomorphism if $\Phi_A(a_1 \cdot_1 a_2) = \Phi_A(a_1) *_1 \Phi_A(a_2)$ for all $a_1, a_2 \in A$.

Lemma 2.1 [13] $h_\Phi \circ \Phi_B = \Phi_A \circ h$.

3. Proximal Groupoids

Let X be a nonempty set endowed with an EF-proximity relation and $A, B \subseteq X$. Let us consider the groupoids $A_\Phi(*), B_\Phi(*)$ (denoted by A_Φ, B_Φ) such that A, B are subsets of EF-proximity space (X, δ). A_Φ, B_Φ are called proximal groupoids. Notice that proximal groupoids A_Φ and B_Φ are near proximal groupoids, provided A_Φ and B_Φ have an element in common. Thus the intersection of A_Φ and B_Φ is not empty. Notice, for disjoint sets X, Y with $A \subseteq X$ and $B \subseteq Y$, the proximal groupoids A_Φ, B_Φ are not near proximal groupoids, since X and Y have no elements in common. Proximal groupoids A_Φ and B_Φ are descriptively near proximal groupoids, provided $\mathcal{Q}(A_\Phi)$ and $\mathcal{Q}(B_\Phi)$ have an element in common, i.e., $\mathcal{Q}(A_\Phi) \cap \mathcal{Q}(B_\Phi) \neq \varnothing$.

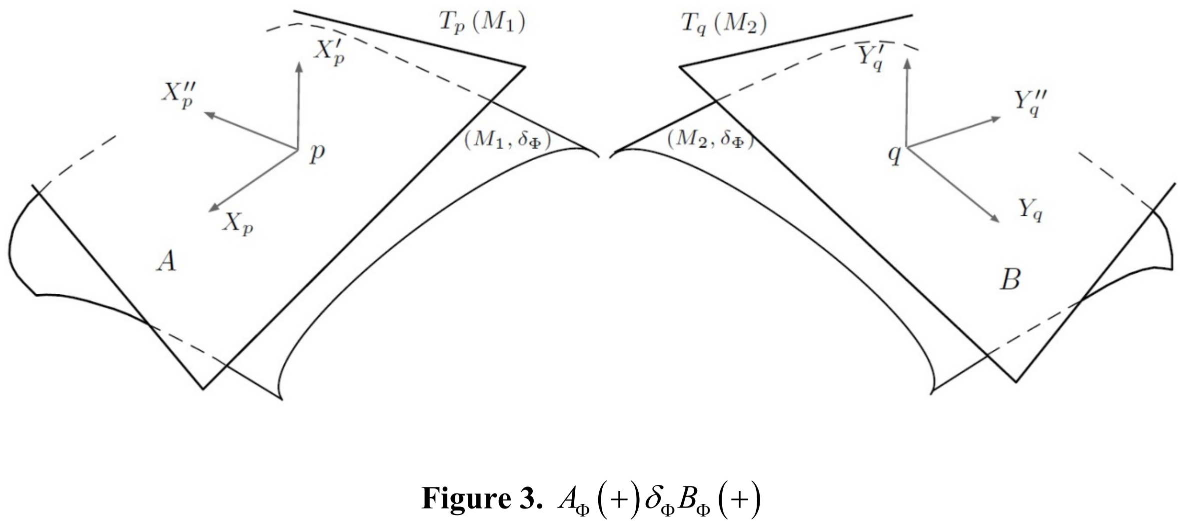

Figure 3. $A_\Phi(+)\delta_\Phi B_\Phi(+)$

Example 3.1 Let $<,>:\mathbb{R}_2^4 \times \mathbb{R}_2^4 \to \mathbb{R}$ be a semi-Euclidean metric and let M_1 and M_2 be differentiable manifolds endowed with descriptive proximity relation δ_Φ, where Φ contain a probe function that represents the norms of vectors in $M_1 = \mathbb{R}_2^4$ and $M_2 = \mathbb{R}_2^4$. Let $T_p(M_1)$, $T_q(M_2)$ be tangent spaces of M_1 and M_2, respectively (in Fig. 3). Assume that

$$A = Tp\{(1,0,0,0),(0,1,0,0)\}$$

and

$$B = Tp\{(0,0,1,0),(0,0,0,1)\}.$$

Let $A_\Phi(+)$ and $B_\Phi(+)$ be groupoids, where

$$+: A \times A \to A, \left(X'_p, X''_p\right) \mapsto X'_p + X''_p$$

and

$$+: B \times B \to B, \left(Y'_q, Y''_q\right) \mapsto Y'_q + Y''_q.$$

We can find $X_p \in A$ and $Y_q \in B$ such that norm of X_p matches the norm of Y_q, i.e., $\Phi(X_p) = \Phi(Y_q)$. Hence $A_\Phi(+)\,\delta_\Phi\,B_\Phi(+)$.

A descriptive proximal groupoid (denoted by $\mathcal{Q}(A)(*_\Phi)$ or shortly denoted by $A(*_\Phi)$) is defined relative to a binary operation $*_\Phi : \mathcal{Q}(A) \times \mathcal{Q}(A) \to \mathcal{Q}(A)$ on a set of objects S with descriptions, where A is a subset of proximity space X endowed with an EF-proximity relation. A descriptive proximal groupoid is obtained by considering a binary operation "$*_\Phi$" on $\mathcal{Q}(A)$ that maps each pair of descriptions of objects in $\mathcal{Q}(A) \times \mathcal{Q}(A)$ to a description in $\mathcal{Q}(A)$.

Let $A(*_\Phi)$, $B(*_\Phi)$ be a pair of descriptive proximal groupoids in X. For simplicity, we assume that each groupoid is defined in terms of the same binary operation. In general, this is not necessary.

To obtain a pair of proximal semigroups, assume "$*$" is associative and to obtain a pair of proximal monoids, assume A_Φ, B_Φ each has an identity element. To obtain a pair of proximal groups, assume A_Φ, B_Φ each has an identity element and assume that each member of A_Φ, B_Φ has an inverse. Similarly, we can obtain descriptive proximal semigroups, descriptive proximal monoids and descriptive proximal groups.

Example 3.2 From Example 3.1, if we consider the binary operations "$+_{\Phi_A}$" and "$+_{\Phi_B}$", where

$$+_{\Phi_A} : \mathcal{Q}(A)\times\mathcal{Q}(A)\to\mathcal{Q}(A), \left(\Phi\left(X_p^{'}\right),\Phi\left(X_p^{''}\right)\right)\mapsto \Phi\left(X_p^{'}\right)+_{\Phi_A}\Phi\left(X_p^{''}\right)$$

and

$$+_{\Phi_B} : \mathcal{Q}(B)\times\mathcal{Q}(B)\to\mathcal{Q}(B), \left(\Phi\left(Y_q^{'}\right),\Phi\left(Y_q^{''}\right)\right)\mapsto \Phi\left(Y_q^{'}\right)+_{\Phi_B}\Phi\left(Y_q^{''}\right),$$

then $A\left(+_{\Phi_A}\right)$ and $B\left(+_{\Phi_B}\right)$ are descriptive proximal groups on descriptive proximity differentiable manifolds (M_1, δ_Φ) and (M_2, δ_Φ), respectively.

4. Exactness of Descriptive Homomorphisms

Let $A_\Phi(*), B_\Phi(*), C_\Phi(*)$ be proximal monoids and $h: B_\Phi \to A_\Phi$, $h': C_\Phi \to B_\Phi$ be homomorphisms. A pair of homomorphisms

$$C_\Phi \overset{h'}{\to} B_\Phi \overset{h}{\to} A_\Phi$$

is said to be exact at B_Φ in case $Imh' = Kerh$. In general, a sequence of homomorphisms

$$\ldots \overset{h_{n-1}}{\to} (A_\Phi)_{n-1} \overset{h_n}{\to} (A_\Phi)_n \overset{h_{n+1}}{\to} (A_\Phi)_{n+1} \to \ldots$$

is exact in case each sequential pair h_n, h_{n+1} are exact at each $(A_\Phi)_n$ for $n \in \mathbb{N}$.

$$\begin{tikzcd}
C_\Phi \arrow[r, "h'"] \arrow[d, "\Phi_C"] & B_\Phi \arrow[r, "h"] \arrow[d, "\Phi_B"] & A_\Phi \arrow[d, "\Phi_A"] \\
\mathcal{Q}(C) \arrow[r, "h'_\Phi"'] & \mathcal{Q}(B) \arrow[r, "h_\Phi"'] & \mathcal{Q}(A)
\end{tikzcd}$$

Figure 4.

Lemma 4.1 Let $h: B_{\Phi} \rightarrow A_{\Phi}$ be a homomorphism, Φ_A, Φ_B be object descriptive homomorphisms and $h_{\Phi}: \mathcal{Q}(B) \rightarrow \mathcal{Q}(A)$ be a descriptive homomorphism represented in Fig. 4. If h is a monomorphism and Φ_A is object descriptive monomorphism, then Φ_B is object descriptive monomorphism.

Theorem 4.1 Let $A_{\Phi}(*)$, $B_{\Phi}(*)$, $C_{\Phi}(*)$ be proximal monoids, $A(*_{\Phi})$, $B(*_{\Phi})$, $C(*_{\Phi})$ be descriptive proximal monoids and $C_{\Phi} \xrightarrow{h'} B_{\Phi} \xrightarrow{h} A_{\Phi}$ be exact, as represented in Fig. 4. If Φ_A, Φ_B are object descriptive monomorphisms, then $\mathcal{Q}(C) \xrightarrow{h'_{\Phi}} \mathcal{Q}(B) \xrightarrow{h_{\Phi}} \mathcal{Q}(A)$ is exact.

Proof. Since Φ_A, Φ_B are object descriptive monomorphisms,

$$\begin{aligned}
Im\, h'_{\Phi} &= \left\{ \Phi_B(x) : \Phi_B(x) = h'_{\Phi}\left(\Phi_C(c)\right), \Phi_C(c) \in \mathcal{Q}(C) \right\} \\
&= \left\{ \Phi_B(x) : \Phi_B(x) = \Phi_B\left(h'(c)\right), c \in C \right\} \\
&= \left\{ \Phi_B(x) : x = h'(c), c \in C \right\} \\
&= \left\{ \Phi_B(x) : x \in Imh' \right\}
\end{aligned}$$

and

$$\begin{aligned}
Kerh_{\Phi} &= \left\{ \Phi_B(x) : h_{\Phi}\left(\Phi_B(x)\right) = e_{\mathcal{Q}(A)} \right\} \\
&= \left\{ \Phi_B(x) : \Phi_A\left(h(x)\right) = \Phi_A\left(e_A\right) \right\} \\
&= \left\{ \Phi_B(x) : h(x) = e_A \right\} \\
&= \left\{ \Phi_B(x) : x \in Kerh = Imh' \right\}.
\end{aligned}$$

Consequently, $Imh'_{\Phi} = Kerh_{\Phi}$. Hence $\mathcal{Q}(C) \xrightarrow{h'_{\Phi}} \mathcal{Q}(B) \xrightarrow{h_{\Phi}} \mathcal{Q}(A)$ is exact.

Theorem 4.2 In Fig. 4, let $A_{\Phi}(*)$, $B_{\Phi}(*)$, $C_{\Phi}(*)$ be proximal monoids and $A(*_{\Phi})$, $B(*_{\Phi})$, $C(*_{\Phi})$ be descriptive proximal monoids. Then

(i) If Φ_A, Φ_C are object descriptive monomorphisms, h'_{Φ} is a descriptive monomorphism and $C_{\Phi} \xrightarrow{h'} B_{\Phi} \xrightarrow{h} A_{\Phi}$ is exact, then Φ_B is an object descriptive monomorphism.

(ii) If Φ_B is an object descriptive epimorphism, Φ_A is an object descriptive monomorphism and h'_Φ is a descriptive monomorphism, then Φ_C is an object descriptive epimorphism.

Proof. (i) Let $b \in Ker\Phi_B$. Since the diagrams commute by Lemma 2.1,

$$\left(h_\Phi \circ \Phi_B\right)(b) = \left(\Phi_A \circ h\right)(b) = \Phi_A\left(h(b)\right)$$

for all $b \in B$. Then $\left(h_\Phi \circ \Phi_B\right)(b) = h_\Phi\left(\Phi_B(b)\right) = h_\Phi\left(e_{Q(B)}\right) = e_{Q(A)} = \Phi_A\left(e_A\right)$. Hence $\Phi_A\left(h(b)\right) = \Phi_A\left(e_A\right)$ and since Φ_A is an object descriptive monomorphism, $h(b) = e_A$. Thus $b \in Kerh = Imh'$, so there exists $c \in C$ such that $b = h'(c)$. Since the diagrams commute,

$$\left(h'_\Phi \circ \Phi_C\right)(c) = \left(\Phi_B \circ h'\right)(c)$$

for all $c \in C$. Then we obtain $\left(\Phi_B \circ h'\right)(c) = \Phi_B\left(h'(c)\right) = \Phi_B(b) = e_{Q(B)} = h'_\Phi\left(e_{Q(C)}\right)$. Hence $h'_\Phi\left(\Phi_C(c)\right) = h'_\Phi\left(e_{Q(C)}\right)$ and since h'_Φ is a descriptive monomorphism, $\Phi_C(c) = e_{Q(C)} = \Phi_C\left(e_C\right)$. Thus, since Φ_C is an object descriptive monomorphism, $c = e_C$. Consequently, $b = h'(c) = h'\left(e_C\right) = e_B$. Hence $Ker\Phi_B = \left\{e_B\right\}$.

(ii) Straightforward.

Corollary 4.1 In Fig. 4, let $A_\Phi(*)$, $B_\Phi(*)$, $C_\Phi(*)$ be proximal monoids and $A(*_\Phi)$, $B(*_\Phi)$, $C(*_\Phi)$ be descriptive proximal monoids. Then

(i) If Φ_A, Φ_C are object descriptive monomorphisms, h'_Φ is a descriptive monomorphism and $C_\Phi \xrightarrow{h'} B_\Phi \xrightarrow{h} A_\Phi$ is exact, then $Q(C) \xrightarrow{h'_\Phi} Q(B) \xrightarrow{h_\Phi} Q(A)$ is exact.

(ii) If Φ_A, Φ_C are object descriptive monomorphisms and $e \to C_\Phi \xrightarrow{h'} B_\Phi \xrightarrow{h} A_\Phi \to e$ is short exact sequence, then $e_\Phi \to Q(C) \xrightarrow{h'_\Phi} Q(B) \xrightarrow{h_\Phi} Q(A) \to e_\Phi$ is a short exact sequence.

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