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The solution of differential equation with Hulthen potential in curved space

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Abstract. The solution of the Schrödinger equation for a physical system in quantum mechanics is of great importance, because the knowledge of wave- function and energy spectrum contain all possible information about the physical properties of a system. In this paper, we have give solution of the Schrödinger equation in three dimensional curved space with Hulthen potential on the positive constant curvature. Then we achieve the wave-function and energy spectrum for the Hulthen potential. In order to solve the corresponding Schrödinger equation, we use of Nikiforov-Uvarov (N.U) method [1]. The N.U method is based on solving the second- order linear differential equations by reducing to a generalized equation of hypergeometric type.

Keywords: Hulthen potential, Schrödinger equation, (N.U) method

Eğik uzayda Hulthen potansiyelli diferansiyel denklem çözümü

Özet. Kuantum mekaniğinde fiziksel bir sistem için Schrödinger denkleminin çözümü büyük önem taşır çünkü dalga fonksiyonu ve enerji spektrumu bilgisi, bir sistemin fiziksel özellikleri hakkında mümkün olan tüm bilgileri içerir. Bu makalede, pozitif sabit eğrilik üzerinde Hulthen potansiyeli ile üç boyutlu kavisli uzayda Schrödinger denkleminin çözümünü veriyoruz. Daha sonra Hulthen potansiyeli için dalga fonksiyonu ve enerji spektrumu elde ediyoruz. Karşılık gelen Schrödinger denklemini çözmek için, Nikiforov-Uvarov (N.U) yöntemini kullanırız [1]. N.U yöntemi, hipergeometrik tipteki genelleştirilmiş bir denklemi indirgeyerek ikinci mertebeden lineer diferansiyel denklemlerin çözülmesine dayanmaktadır.

Anahtar Kelimeler: Hulthen potansiyeli, Schrödinger denklemi, (N.U) metodu

1. INTRODUCTION

One of the interesting problems of the nonrelativistic quantum mechanics is to find exact solutions to the Schrödinger equation for certain potentials of the physical interest. In recent years, considerable efforts have been done to obtain the analytical solution of non-central problems. The notion of the constant curvature and the accidental degeneracy first began with Schrödinger[2]. Essential advances of these systems with accidental degeneracy have been made by Nishino[4], Higgs[5] and Leemon[6]. At the same time, some papers on curved spherical spaces are concerned with some applications of physics such as linear and non-linear optics[7] and quantum dots[8, 9]. Furthermore, in[3], the authors studied Lie Algebraic Extensions of the Mie-type interactions with Positive Constant Curvature. This paper is organized as follows. Firstly, we take advantage from curvature space and make the Hulthen potential in spherical coordinates with spaces of constant curvature. Then by using the Laplace-Beltrami operator and N.U method, we solve the above pointed corresponding Schrödinger. In that case we achieve the wave-function and energy spectrum for the Hulthen potential.

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2. PRELIMINARIES

As we know the three-dimensional space of constant positive curvature can also be realized geometrically on the three-dimensional sphere S^3 of the radius R, embedded into the four-dimensional Euclidean space when the equation of S^3 has a form,

$$S^3 = \{(\lambda_0, \lambda_i) \in R^4 : \lambda_0^2 + \lambda_i \lambda_i = R^2\}, \tag{1}$$

where the tangent space X_i $(i = 1,2,3)$ are the coordinates and λ_i is,

$$\lambda_i = \frac{X_i}{\sqrt{1 + \frac{r^2}{R^2}}}, \tag{2}$$

and

$$\lambda_0 = \frac{R}{\sqrt{1 + \frac{r^2}{R^2}}}, \tag{3}$$

In order to write the Schrödinger like equation in curved space-time, we have to change the corresponding potential in flat space-time to curved space-time. So, in that case we try to write the general form of Hulthen potential in constant curvature space. We define $r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2$ and following potential,

$$V(r) = -V_0 \frac{e^{-\delta r}}{1 - e^{-\delta r}}, \tag{4}$$

where V_0 is the constant and δ is screening parameter. By inserting the above new coordinate r into λ, one can obtain the corresponding potential as,

$$V(\lambda) = -V_0 \frac{e^{-\delta \frac{\lambda}{\sqrt{1-\frac{\lambda^2}{R^2}}}}}{1 - e^{-\delta \frac{\lambda}{\sqrt{1-\frac{\lambda^2}{R^2}}}}}. \tag{5}$$

On the other hand, the spherical coordinates lead us to have following equations,

$$\lambda_1 = R \sin\psi \sin\theta \cos\phi, \tag{6}$$

$$\lambda_2 = R \sin\psi \sin\theta \sin\phi, \tag{7}$$

$$\lambda_3 = R\sin\psi\cos\theta, \tag{8}$$

$$\lambda_4 = R\cos\psi, \tag{9}$$

where $0 \le \psi \le \pi$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. Here also we will obtain the metric background for the above corresponding system. The form of metric help us to write the second order equation. By using the variation with respect to angles ψ , θ and ϕ (R is constant curvature) one can obtain the metric background in four dimension, which is given by,

$$ds^2 = R^2(d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)). \tag{10}$$

The above information help us to calculate the Hulthen potential in form of angle, so the $V(\psi)$ will be as,

$$V(\psi) = -V_0 \frac{e^{-\delta R\tan\psi}}{1-e^{-\delta R\tan\psi}}. \tag{11}$$

Now we are ready to arrange the general form of Schrödinger like equation for (11) on the constant curvature,

$$(-\frac{h^2}{2\mu}\Delta + V)\Psi = E\Psi, \tag{12}$$

where Δ is Laplace-Beltrami operator which is a restriction of the Laplace operator on the sphere. So, the Laplace-Beltrami operator will be following,

$$\Delta = \frac{1}{g}\sum_{i,k=1}^{3}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ik}\frac{\partial}{\partial x^k}). \tag{13}$$

As we see in Laplace-Beltrami operator correspond to the metric of space time, in flat space time we have just usual Laplace. We can define the general form of metric for the arbitrary space-time which is given by following expression,

$$ds^2 = g_{ik}dx^i dx^k, \tag{14}$$

where $g = det|g_{ik}|$ and by the chain rule $g^{ik} = (g_{ik})^{-1}$. Thus, using (6), (7), (8), (9) and (13), Schrödinger equation takes the form,

$$(\frac{1}{\sin^2\psi}\frac{\partial}{\partial\psi}\sin^2\psi\frac{\partial}{\partial\psi})\Psi + \frac{2\mu R^2}{h^2}[E - \frac{h^2}{2\mu R^2}\frac{m(m+1)}{\sin^2\psi} - V_0\frac{e^{-\delta R\tan\psi}}{1-e^{-\delta R\tan\psi}}]\Psi = 0. \tag{15}$$

Using a transformation of the wave-function in (15), The solution of differential equation

$$\Psi(\psi) = \frac{\phi(\psi)}{\sin(\psi)}, \quad (16)$$

$$C_1 = \frac{2\mu R^2}{h^2} E, \quad (17)$$

$$C_2 = -\frac{2\mu R^2}{h^2}(\frac{h^2 m(m+1)}{2\mu R^2}), \quad (18)$$

$$C_3 = -\frac{2\mu R^2}{h^2} V_0, \quad (19)$$

$$C_4 = -\delta R, \quad (20)$$

(15) turns into

$$\frac{d^2\phi}{d\psi^2} + (C_1 + C_2 \csc^2\psi + C_3 \frac{e^{C_4 \tan\psi}}{1-e^{C_4 \tan\psi}})\phi = 0. \quad (21)$$

3. THE SOLUTION WITH THE NIKIFOROV-UVAROV METHOD

The main equation which is closely associated whit the method is given in folloing form (Nikiforov-Uvarov , 1988)

$$\frac{d^2\phi}{ds^2} + \frac{\bar{\tau}(s)}{\sigma(s)}\frac{d\phi}{ds} + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\phi(s) = 0, \quad (22)$$

where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials at most second-degree and $\bar{\tau}(s)$ is a first-degree polynomial and $\psi(s)$ is a function of the hypergepmetric-type.

We turn (22) to

$$\frac{d^2\phi}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{d\phi}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{[s(1-\alpha_3 s)]^2}\phi = 0. \quad (23)$$

The above equation have the recursive equation as follows:

$$\phi(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{(-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3})} P_n^{(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s), \quad (24)$$

and eigenvalue for (23) is as:

$$\alpha_2 n - (2n+1)\alpha_5 + (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_5}) + n(n+1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2(\sqrt{\alpha_8\alpha_9}) = 0, \quad (25)$$

where α_i $(i = 1,2,3,...,13)$ are special functions and we define as follows:

$$\alpha_4 = \frac{1}{2}(1-\alpha_1), \tag{26}$$

$$\alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \tag{27}$$

$$\alpha_6 = (\alpha_5)^2 + \xi_1, \tag{28}$$

$$\alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \tag{29}$$

$$\alpha_8 = (\alpha_4)^2 + \xi_3, \tag{30}$$

$$\alpha_9 = \alpha_6 + \alpha_3\alpha_7 + (\alpha_3)^2\alpha_8, \tag{31}$$

$$\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \tag{32}$$

$$\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}), \tag{33}$$

$$\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \tag{34}$$

$$\alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}). \tag{35}$$

Now we assume,

$$C_4 = 1, \tag{36}$$

$$\tan\psi ; \sin\psi ; \psi, \tag{37}$$

(21) turn to,

$$\frac{d^2\phi}{d\psi^2} + (C_1 + \frac{C_2}{\psi^2} + C_3\frac{e^\psi}{1-e^\psi})\phi = 0. \tag{38}$$

With using of equivalence, we have,

$$e^\psi ; 1+\psi \to \psi ; e^\psi - 1 \to \frac{-1}{\psi} ; \frac{1}{1-e^\psi}, \tag{39}$$

and so

$$\frac{1}{\psi^2} ; \frac{1}{(1-e^\psi)^2}. \tag{40}$$

We put (40) in (38) as:

$$\frac{d^2\phi}{d\psi^2}+[C_1+C_2\frac{1}{(1-e^{\psi})^2}+C_3\frac{e^{\psi}}{1-e^{\psi}}]\phi=0. \tag{41}$$

Moreover let us substitute,

$$s=e^{\psi}, \tag{42}$$

hence we have,

$$\frac{d\phi}{d\psi}=e^{\psi}(\frac{d\phi}{ds}), \tag{43}$$

and also

$$\frac{d^2\phi}{d\psi^2}=e^{\psi}(\frac{d\phi}{ds})+e^{2\psi}(\frac{d^2\phi}{ds^2}). \tag{44}$$

Using (43),(44) and substituting into (41), we obtain,

$$e^{2\psi}(\frac{d^2\phi}{ds^2})+e^{\psi}(\frac{d\phi}{ds})+[C_1+C_2\frac{1}{(1-e^{\psi})^2}+C_3\frac{e^{\psi}}{1-e^{\psi}}]\phi=0, \tag{45}$$

and with substituting (42) into (45), we have,

$$\frac{d^2\phi}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{d\phi}{ds}+\frac{[s^2(C_3-C_1)+s(C_3-2C_1)+(C_1+C_2)]}{[s(1-s)]^2}\phi=0. \tag{46}$$

(46)is Similar to(23) where $\alpha_1=\alpha_2=\alpha_3=1$ and,

$$\xi_1=C_1-C_3, \tag{47}$$

$$\xi_2=C_3-2C_1, \tag{48}$$

$$\xi_3=C_1+C_2, \tag{49}$$

$$\alpha_4=0, \tag{50}$$

$$\alpha_5=-\frac{1}{2}, \tag{51}$$

$$\alpha_6=\frac{1}{4}+C_1-C_3, \tag{52}$$

$$\alpha_7=2C_1-C_3, \tag{53}$$

$$\alpha_8 = C_1 + C_2, \tag{54}$$

$$\alpha_9 = \frac{1}{4} + 4C_1 + C_2 - 2C_3, \tag{55}$$

$$\alpha_{10} = 1 + 2(\sqrt{C_1 + C_2}), \tag{56}$$

$$\alpha_{11} = 2(1 + \sqrt{\frac{1}{4} + 4C_1 + C_2 - 2C_3} + \sqrt{C_1 + C_2}), \tag{57}$$

$$\alpha_{12} = \sqrt{C_1 + C_2}, \tag{58}$$

$$\alpha_{13} = -\frac{1}{2} - (\sqrt{\frac{1}{4} + 4C_1 + C_2 - 2C_3} + \sqrt{C_1 + C_2}). \tag{59}$$

Therefore with using (N.U)method and (24), the solution for(46)is as:

$$\phi(s) = s^{\sqrt{C_1+C_2}}(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+4C_1+C_2-2C_3}} P_n^{(2[\sqrt{(C_1+C_2)}],2[1+\sqrt{\frac{1}{4}+4C_1+C_2-2C_3}])}(1-2s). \tag{60}$$

According to (42)we have,

$$s = e^{\psi} \rightarrow \psi = \ln s, \tag{61}$$

then (60) turn to,

$$\phi(\psi) = \psi(\sqrt{C_1 + C_2})(1-\psi)^{\frac{1}{2}+\sqrt{\frac{1}{4}+4C_1+C_2-2C_3}} P_n^{(2[\sqrt{(C_1+C_2)}],2[1+\sqrt{\frac{1}{4}+4C_1+C_2-2C_3}])}(1-2\psi). \tag{62}$$

Now according to (16), wave-function as follows:

$$\Psi(\psi) = \frac{\psi(\sqrt{C_1 + C_2})(1-\psi)^{\frac{1}{2}+\sqrt{\frac{1}{4}+4C_1+C_2-2C_3}} P_n^{(2[\sqrt{(C_1+C_2)}],2[1+\sqrt{\frac{1}{4}+4C_1+C_2-2C_3}])}(1-2\psi)}{\sin\psi}. \tag{63}$$

By using the above equation and (25), one can obtain the eigenvalue as,

$$n + \frac{1}{2}(2n+1) + (2n+1)(M) + n(n-1) + 4C_1 + 2C_2 - C_3 + 2C_1 + 2C_2 + 2N = 0, \tag{64}$$

where

$$N = \sqrt{(\frac{1}{4} + 4C_1 + C_2 - 2C_3)(C_1 + C_2)}, \tag{65}$$

and

$$M=\sqrt{\frac{1}{4}+4C_1+C_2-2C_3}+\sqrt{C_1+C_2}. \tag{66}$$

So by using equations(18),(19),(64), we obtain C_1 and finally the energy spectrum will be as:

$$E_{m,n}=[\frac{2m(m+1)-2N-(n+\frac{1}{2})-\frac{1}{4}}{R^2}]-\frac{2M}{h^2}V_0. \tag{67}$$

In case of flat space $R\to\infty$,one can achieve the energy spectrum as,

$$E=-\frac{2M}{h^2}V_0. \tag{68}$$

So, we see the energy in curved space time depent to n and m as a quantum number. In flat space time, the energy not depent to n and m . We note here, different potential in curved space time give very interesting wave-function and eigenvalues. So, in future we can do different potential in three and four dimension in curved space.

4. CONCLUSION

We know that the time-independent has the second-order differential equation in the Schrödinger picture as well. Therefore, in this paper we confined our attention to this equation and its approximate solutions for the Hulthen potential. We have studied the Hulthen potential in spherical curved spaces with constant positive curvature through N.U method. It is seen that, Hulthen potential is transformed into other potentials such as Harmonic Oscillator, Coulomb, Kratzer, Morse in spherical spaces. The solution meant that we have obtained the energy spectrum and the corresponding wave-function of a particle subject to one of these potentials.

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