On the Timelike Surface with Constant Angle in Hyperbolic Space H^3

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Abstract
In this paper , we study constant timelike angle surface whose unit normal vector field make constant timelike with a fixed spacelike axis in R_1^4 in Hyperbolic space H^3. Let $x: M \to H^3$ be a space-like immersion and let ξ be a unit normal vector field to M. If there exists spacelike direction U such that timelike angle $\theta(\xi, U)$ is constant on M, then M is called constant timelike angle surfaces with spacelike axis in H^3. Also, conditions being a constant angle surface in H^3 have been determined and invariants of these surfaces have been investigated.

Keywords — Constant angle surface, hyperbolic space, helix, timelike surface

Hiperbolik Uzayda Sabit Açılı Zamansal Yüzeyler Üzerine

Özet
Bu çalışmada, yüzeyin birim normal vektör alanı ile R_1^4 de sabit uzaysal bir doğrultu ile sabit bir zamansal açı yapan yüzeyler çalışılmıştır. $x: M \to H^3$ uzaysal bir immersiyon ve ξ, M yüzeyinin birim normal vektörü olsun. Eğer M yüzeyi üzerinde $\theta(\xi, U)$ zamansal açısı sabit olacak şekilde sabit bir uzaysal U doğrultusu varsa, M yüzeyine H^3 hiperbolik uzayında sabit uzaysal eksenli zamansal açılı yüzey denir. Ayrıca hiperbolik uzayda sabit açılı yüzey olma koşulları belirlenmiş ve bu yüzeylerin değişmezleri araştırılmıştır.
Anahtar Kelimeler — Sabit açılı yüzey, hiperbolik uzay, helis, timelike yüzey

1 Introduction

A constant angle curve whose tangents make constant angle with a fixed direction in ambient space is called a helix. A surface whose tangent planes make a constant angle with a fixed vector field of ambient space is called constant angle surface. Constant angle surfaces have been studied for arbitrary dimension in Euclidean space E^n [13,14], and recently in product spaces $S^2 \times R$ [15], $H^2 \times R$ [16] or different ambient spaces Nil_3 [17]. In [1] , Lopez and Munteanu studied constant hyperbolic angle surfaces whose unit normal timelike vector field makes a constant hyperbolic angle with a fixed timelike axis in Minkowski space R_1^4. In particular, they had shown that these surfaces are flat.

Hyperbolic space is a good model for pyhsical cases and most of the physical cases can be explained by this model. Surface types in different spaces are important since this kind of surfaces can guide the fields involved with our daily life such as architecture and geometrical design. It is possible to see this on the structures in the history of architecture. For example, these structures have been used by firsty in Euclidean curves, then spherical curves in middle ages and hyperbolic curves in the modern ages. Probably, architectural structures and geometrical designs that use de-Sitter curves enter in to our life in the future. In the literatüre constant timelike and spacelike angle surface have not been investigated both in hyperbolic space H^3 and de-Sitter space S_1^3 Constant angle spacelike surface in hyperbolic space H^3 and constant angle spacelike surface in de-Sitter space S_1^3 are developed in our paper [19] and [20]. In this paper, a special class of surfaces which is called the constant timelike angle surfaces is given in hyperbolic space H^3. A constant timelike angle surface in hyperbolic space H^3 is a surface whose tangent planes make a constant timelike angle with a fixed spacelike vector field on R_1^4. In Minkowski space R_1^4, due to the variety of causal character of a vector field, there is a natural concept of variable angle between two arbitrary vector fields. Since x spacelike immersion into H^3, ξ is unit spacelike normal vector field to M .

2 Preliminaries

Let $x : M \to R_1^4$ be an immersion of a surface M into R_1^4. We say that x is timelike (resp. spacelike, lightlike) if the induced metric on M via x is Lorentzian (resp.Riemannian, degenerated). If $\langle x, x\rangle = -1, x_0 > 1$, then x is an immersion of hyperbolic space H^3. Let $Sp\{x, y\}$ be the subspace spanned by the vectors x and y. Let U be unit spacelike vector field on H^3, and $W = Sp\{\xi_p, U_p\}$ be the subspace spanned by U_p and ξ_p.

If U is unit spacelike vector field on H^3, then the subspace W can be spacelike, timelike or lightlike.

If W is timelike subspace (seen Fig 1 and Fig 2) the arclength of the hyperbolic line segment QR is called the measure of angle between ξ_p and U_p. In this case, there is a unique positive real number $\theta(\xi_p, U_p)$ such that $\left|\langle \xi_p, U_p\rangle\right| = \cosh\theta(\xi_p, U_p)$ [11]. The real number $\theta(\xi_p, U_p)$ is called timelike angle between spacelike vectors U_p and ξ_p.

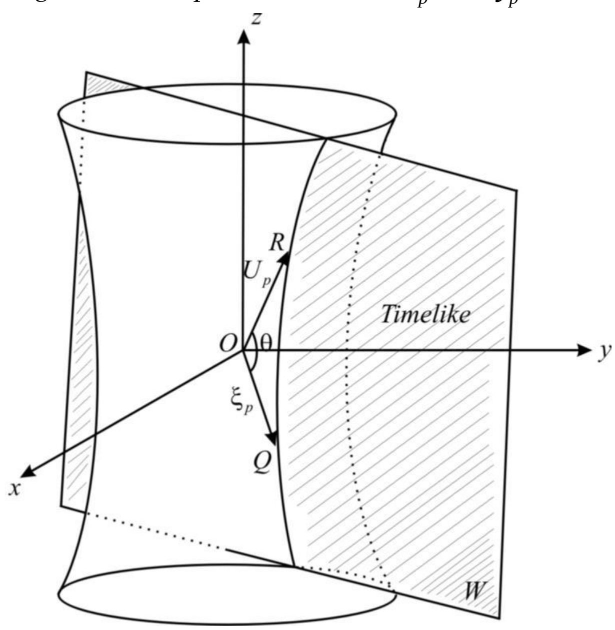

Figure 1. *Timelike angle between spacelike vectors U_p and ξ_p*

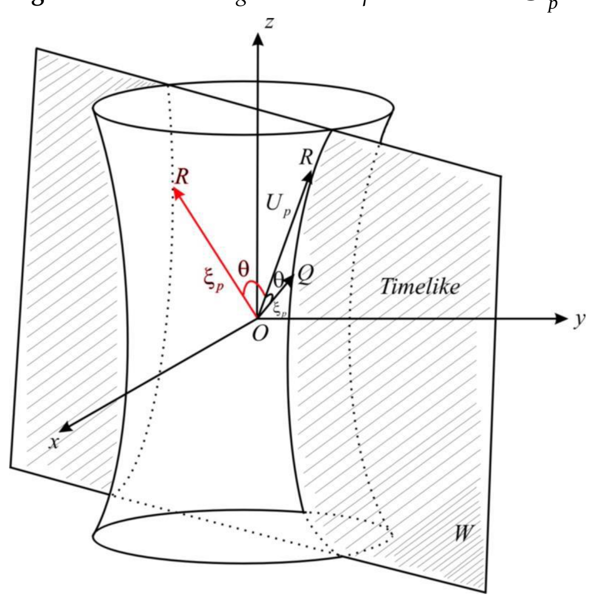

Figure 2. *Timelike angle between spacelike vectors U_p and ξ_p*

If W is spacelike subspace (see Fig 3) the arclenght of segment QR for each $p \in M$ is called the measure of angle between ξ_p and U_p. In this case, there is a unique real number $\theta(\xi_p, U_p) \in (0, \pi)$ such that $\langle \xi_p, U_p \rangle = \cos\theta(\xi_p, U_p)$.

The real number $\theta(\xi_p, U_p)$ is called spacelike angle between spacelike vectors ξ_p and U_p [11].

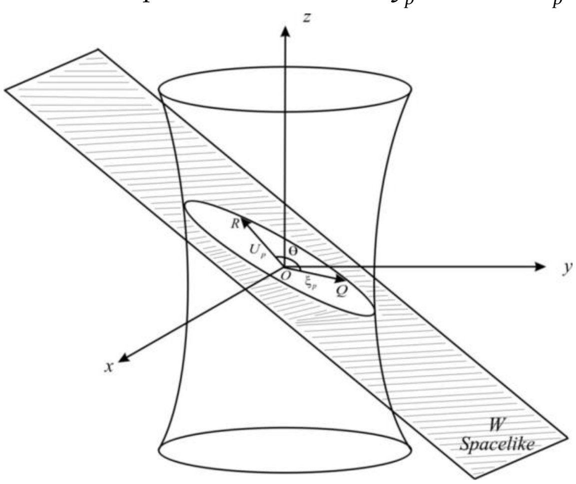

Figure 3. *Spacelike angle between spacelike vectors ξ_p and U_p*

Let $x : M \to H^3$ be a spacelike immersion and let ξ be a unit normal vector field to M. If there exists spacelike direction U such that timelike angle $\theta(\xi, U)$ is constant on M, then M is called constant timelike angle surfaces with spacelike axis.

Let R_1^4 be 4-dimensional vector space equipped with the scalar product $\langle , \rangle$ which is defined by

$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$.

Then R_1^4 is called Minkowskian or Lorentzian 4-space. From now on, the constant angle surface is proposed in Minkowskian ambient space R_1^4. The Lorentzian norm of x is defined to be

$\|x\| = |\langle x, y \rangle|^{\frac{1}{2}}$.

If $(x_0^i, x_1^i, x_2^i, x_3^i)$ is the coordinate of x_i with respect to canonical basis (e_0, e_1, e_2, e_3) of R_1^4, then the lorentzian cross product $x_1 \times x_2 \times x_3$ is defined by the symbolic determinant

$$x_1 \times x_2 \times x_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix}.$$

On can easly see that

$\langle x_1 \times x_2 \times x_3, x_4 \rangle = \det(x_1, x_2, x_3, x_4)$.

In [2],[3] and [5] Izumiya at all introduced and investigated differantial geometry of curves and surfaces Hyperbolic 3-space. The set

$\{x \in R_1^4, \langle x, x \rangle = -1, x_0 \geq 1\}$,

$\{x \in R_1^4, \langle x, x \rangle = 1\}$ and $\{x \in R_1^4, \langle x, x \rangle = 0, x_0 \geq 0\}$

is called Hyperbolic space H^3, de Sitter space S_1^3 and future lightcone at the origin LC^*. We can give the following background of context in [2].

Since H^3 is a Riemannian manifold and regular curve γ reparametrized by arclength, we may assume that $\gamma(s)$ is a unit speed curve. That is, there is a tangent vector $t(s) = \gamma'(s)$ with $\|t(s)\| = 1$. If $\langle t'(s), t'(s) \rangle \neq -1$, then there is a unit vector

$$n(s) = \frac{t'(s) - \gamma(s)}{\|t'(s) - \gamma(s)\|}$$

and also $e(s) = \gamma(s) \Lambda t(s) \Lambda n(s)$. Then we have a pseudo orthonormal frame $\{\gamma(s), t(s), n(s), e(s)\}$ of R_1^4 along γ.

Since $\langle t(s), t(s) \rangle \neq -1$, we have also the following Frenet-Serre type formulas is obtained

$$\begin{cases} \gamma' = t(s) \\ t'(s) = \kappa_h(s) n(s) + \gamma(s) \\ n'(s) = -\kappa_h(s) t(s) + \tau_h(s) e(s) \\ e'(s) = -\tau_h(s) n(s) \end{cases}$$

where

$$\kappa_h(s) = \|t'(s) - \gamma(s)\|$$

and

$$\tau_h(s) = -\frac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{[\kappa_h(s)]^2}.$$

Since $\langle t(s), t(s) \rangle \neq -1$, it is easily seen that

$\kappa_h(s) \neq 0$.

We can show that $\kappa_h(s) = 0$ if and only if there exists a lightlike vector c such that $\gamma(s) - c$ is a geodesic.

Let $U \subset R^2$ is an open subset and $x: U \to H^3$ is a regular surface. $M = x(U)$ is embedding of x. If

$$e(u) = \frac{x(u) \Lambda x_1(u) \Lambda x_2(u)}{\|x(u) \Lambda x_1(u) \Lambda x_2(u)\|},$$

Then $\langle e, x\rangle \equiv \langle e, x_i\rangle \equiv 0, \langle e, e\rangle = 1$ where $x_i = \frac{\partial x}{\partial u_i}$.

Thus there is de Sitter Gauss image of x which is defined by mapping $E: U \subset R^2 \to S_1^3, E(u) = e(u)$. The lightcone Gauss image of x is defined by map $L^{\pm}: U \subset R^2 \to LC^*, L^{\pm}(u) = x(u) \pm e(u)$.

Since $dx(u_0)$ and I_{TpM} is identify mapping on the tangent space TpM, the derivative $dx(u_0)$ can be identified with TpM relate to identification of U and M. That is $dL^{\pm}(u_0) = I_{TpM} \pm dE(u_0)$. The lineer transformation

$$S_p^{\pm} := -dL^{\pm}(u_0): TpM \to TpM$$

is called the hyperbolic shape operator of $M = x(u)$ at $p = x(u_0)$. Also the

$$A_p := -dE(u_0): TpM \to TpM$$

is called the de Sitter shape operator of $M = x(u)$ at $p = x(u_0)$. The eigenvalues of $S_p^{\pm}$ and A_p are denoted by $\overline{K_i^{\pm}}(p)$ and $K_i(p), i = 1,2$. The eigen values $\overline{K_i^{\pm}}(p)$ and $K_i(p)$ of $S_p^{\pm}$ and A_p is called the principal curvatures of M in H^3 and R_1^4. Since $S_p^{\pm} = -I_{TpM} \pm A_p$, $S_p^{\pm}$ and A_p have same eigenvectors and relations

$$\overline{K_i^{\pm}}(p) = -1 \pm K_i(p).$$

$\overline{K_i^{\pm}}(p)$, $(i = 1,2)$ are called hyperbolic principal curvatures and $K_i(p)$, $(i = 1,2)$ are called de Sitter principal curvature of $M = x(u)$ at $p = x(u_0)$.

Let $\gamma(s) = x(u_1(s), u_2(s))$ be a unit speed curve on $M = x(u)$, with $p = \gamma(s_0)$. We have the hyperbolic curvature vector $k(s) = t'(s) - \gamma(s)$ and the de Sitter normal curvature

$$K_n^{\pm}(s_0) = \langle k(s_0), L^{\pm}(u_1(s_0), u_2(s_0))\rangle$$
$$= \langle t'(s_0), L^{\pm}(u_1(s_0), u_2(s_0))\rangle + 1$$

of $\gamma(s)$ at $p = \gamma(s_0)$. The de Sitter normal curvature depends on the point p and the unit tangent vector of M at p analogous to the Euclidean case. Hyperbolic normal curvature of $\gamma(s)$ is given by

$$\overline{K_n^{\pm}}(s) = K_n^{\pm}(s) - 1.$$

The Hyperbolic Gauss curvature $\overline{K_h^{\pm}}(u_0)$ and the Hyperbolic mean curvature $\overline{H_h^{\pm}}(u_0)$ at $p = x(u_0)$, is given by

$$\overline{K_h^{\pm}}(u_0) = \det S_p^{\pm} = \overline{K_1^{\pm}}(p)\overline{K_2^{\pm}}(p),$$

$$H_h^{\pm}(u_0) = \frac{1}{2} Trace S_p^{\pm} = \frac{\overline{K_1^{\pm}}(p) + \overline{K_2^{\pm}}(p)}{2}.$$

The extrinsic (de Sitter) Gauss curvature $K_e(u_0)$ and the de Sitter mean curvature $H_d(u_0)$ at $p = x(u_0)$, is obtained

$$K_e = \det Ap = K_1(p) K_2(p),$$

$$H_d(u_0) = \frac{1}{2} Trace Ap = \frac{K_1(p) + K_2(p)}{2}.$$

3 Constant Timelike Angle Surfaces with Spacelike Axis

Let $\chi(M)$ be the tangent vector field space on M. Levi-Civita connections of IR_1^4, H^3 and M denote by $\overline{\overline{D}}, \overline{D}, D$. If the tangent and normal component of $\overline{\overline{D}}_X Y$ denoted by superscript T and $\perp$, we have $D_X Y = \left(\overline{\overline{D}}_X Y\right)^T$ and $\tilde{V}(X,Y) = \left(\overline{\overline{D}}_X Y\right)^{\perp}$.

By using this notation, we obtain

$$\begin{cases} \overline{\overline{D}}_X Y = \overline{D}_X Y - \langle X, Y\rangle x \\ \overline{\overline{D}}_X Y = D_X Y + \tilde{V}(X,Y) \end{cases} \tag{3.1}$$

for each $X, Y \in \chi(M)$.

The first and second equation of (3.1) is called the Gauss formula of H^3 and M in IR_1^4. If ξ is a normal vector field to M in H^3, then the Weingarten Endomorphism $S_{\xi}^{\pm}(X)$ and $A_x(X)$ is given by the tangent component of $\left(-\overline{\overline{D}}_X\xi\right)^T$ and $\left(-\overline{\overline{D}}_X x\right)^T$. Thus, the Weingarten equations of the vector field ξ and x is abtained

$$\begin{cases} S_\xi^{\pm}(X) = -\overline{\overline{D}}_X\xi + \left\langle \overline{\overline{D}}_X x, \xi \right\rangle x \\ A_x(X) = -\overline{\overline{D}}_X x + \left\langle \overline{\overline{D}}_X x, \xi \right\rangle \xi \end{cases} \tag{3.2}$$

It is obvious that $S_\xi^{\pm}$ and A_x is linear and self adjoint map for each $p \in M$. Moreover, if $X, Y \in \chi(M)$, we have

$$\begin{cases} \left\langle S^{\pm}(X), Y \right\rangle = \left\langle \tilde{V}(X,Y), \xi \right\rangle \\ \left\langle A(X), Y \right\rangle = \left\langle \tilde{V}(X,Y), x \right\rangle \end{cases},$$

and

$$\tilde{V}(X,Y) = \left\langle S^{\pm}(X), Y \right\rangle \xi - \left\langle A(X), Y \right\rangle x,$$

$$\overline{\overline{D}}_X Y = D_X Y + \left\langle S^{\pm}(X), Y \right\rangle \xi - \left\langle A(X), Y \right\rangle x$$

Since

$$\overline{D}_X Y = D_X Y - \left\langle S^{\pm}(X), Y \right\rangle \xi$$

and

$$\overline{\overline{D}}_X Y = \overline{D}_X Y + \left\langle X, Y \right\rangle x$$

we obtain

$$\overline{\overline{D}}_X Y = D_X Y - \left\langle S^{\pm}(X), Y \right\rangle \xi + \left\langle X, Y \right\rangle x \tag{3.3}$$

Let $\{v_1, v_2\}$ be a basis in the tangent plane TpM and let

$$\tilde{V}_{ij} = \left\langle \tilde{V}(v_i, v_j), \xi \right\rangle = \left\langle S^{\pm}(v_i), v_j \right\rangle,$$

$$\tilde{W}_{ij} = \left\langle \tilde{V}(v_i, v_j), x \right\rangle = \left\langle A(v_i), v_j \right\rangle.$$

Then we have

$$\overline{\overline{D}}_{v_i} v_j = D_{v_i} v_j - \tilde{V}_{ij}\xi + \left\langle v_i, v_j \right\rangle x \tag{3.4}$$

If this basis is orthonormal, by (3.1) and (3.2)

$$\overline{\overline{D}}_{v_i} v_j = D_{v_i} v_j - \tilde{V}_{ij}\xi \tag{3.5}$$

$$\overline{\overline{D}}_{v_i} \xi = -\tilde{v}_{i1} v_1 - \tilde{v}_{i2} v_2 \tag{3.6}$$

$$\overline{\overline{D}}_{v_i} x = -\tilde{w}_{i1} v_1 - \tilde{w}_{i2} v_2 \tag{3.7}$$

Let M be a constant timelike angle surface with spacelike axis. If timelike angle $\theta = 0$, then $\xi = U$. Throghout this section, without loss of generality we assume that θ. If U^T is the projection of U on the tangent plane of M, then we decompose U as

$U = U^T - (\cosh\theta)\xi + (\sinh\varphi)x$,

where φ is angle between x and U.

Let $e_1 = \dfrac{U^T}{\left\|U^T\right\|}$ and let consider e_2 be a unit vector field on M orthogonal to e_1. Then we have an oriented ortonormal basis $\{e_1, e_2, \xi, x\}$ for IR_1^4. The constant vector field U_h is given by

$$U_h = \sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}\, e_1 - (\cosh\theta)\xi \tag{3.8}$$

Since U_h is constant vector field on H^3 and $\overline{\overline{D}}_{e_2} U_h = \overline{D}_{e_2} U_h = 0$, we have

$$\begin{aligned} \overline{\overline{D}}_{e_2} U_h &= \sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}\, \overline{\overline{D}}_{e_2} e_1 - (\cosh\theta)\overline{\overline{D}}_{e_2}\xi \\ &= 0 \end{aligned} \tag{3.9}$$

if we take scalar product both side of (3.9) by ξ, we obtain

$$\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}\left\langle \overline{\overline{D}}_{e_2} e_1, \xi \right\rangle - \cosh\theta \left\langle \overline{\overline{D}}_{e_2}\xi, \xi \right\rangle = 0$$

or

$$\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}\, \tilde{v}_{21} = 0.$$

Since $\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|} \neq 0$,

we conclude $\tilde{v}_{21} = \tilde{v}_{12} = 0$. Using (3.6) in (3.9), it follows that

$$\overline{\overline{D}}_{e_2} e_1 = -\frac{\cosh\theta}{\sqrt{\sinh^2\varphi - \sinh^2\theta}}\, \tilde{v}_{22} e_2 \tag{3.10}$$

Since U_h is a constant vector field on H^3, then we have

$$\overline{D}_{e_1} U_h = 0$$

and

$$\overline{\overline{D}}_{e_1} U_h = \sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}\, x \tag{3.11}$$

By (3.8), we obtain

$$\begin{aligned} \overline{\overline{D}}_{e_1} U_h &= \sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}\, \overline{\overline{D}}_{e_1} e_1 \\ &\quad -(\cosh\theta)\overline{\overline{D}}_{e_1}\xi \end{aligned} \tag{3.12}$$

By (3.11) and (3.12), we conclude that

$$\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}\,\overline{\overline{D}}_{e_1}e_1-(\cosh\theta)\overline{\overline{D}}_{e_1}\xi$$
$$=\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}\,x \tag{3.13}$$

if we take salar product both side of (3.13), we obtain

$$\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}\left\langle\overline{\overline{D}}_{e_1}e_1,\xi\right\rangle=0,$$

or

$$\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}\,\tilde{v}_{11}=0$$

Since $\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}\neq 0$, we conclude $\tilde{v}_{11}=0$. Using (3.6) in (3.11), we obtain

$$\overline{\overline{D}}_{e_1}e_1=x \tag{3.14}$$

Now we have the following theorem.

Theorem 1 The Levi-Civita connection D for a constant timelike angle spacelike surface in H^3 is given by

$$D_{e_1}e_1=0 \qquad D_{e_2}e_1=\frac{-\cosh\theta}{\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}}\tilde{v}_{22}e_2$$

$$D_{e_1}e_2=0 \qquad D_{e_2}e_2=\frac{\cosh\theta}{\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}}\tilde{v}_{22}e_1.$$

Corollary 1 Given a constant angle spacelike surface M in H^3, there exist local coordinates u and v such that the metric on M writes as $\langle,\rangle:=du^2+\beta^2dv^2$, where $\beta=\beta(u,v)$ is a smooth function on M, i.e. the coefficients of the first fundamental form are $E=1,F=0,G=\beta^2$.

By the above parametrization $x(u,v)$ and Theorem 1 one can obtain the following corollary.

Corollary 2 There exist a system for constant timelike angle surface in H^3 which is

$$\begin{cases} x_{uu}=x \\ x_{uv}=\dfrac{\beta_u}{\beta}x_v \\ x_{vv}=-\beta\beta_u x_u+\dfrac{\beta_v}{\beta}x_v-\beta^2\tilde{v}_{22}\xi+\beta^2 x \end{cases} \tag{3.15}$$

Corollary 3 Let M be a constant angle surface with unit normal vector ξ. Then we have the following system

$$\begin{cases} \xi_u=\overline{\overline{D}}_{x_u}\xi=0 \\ \xi_v=\overline{\overline{D}}_{x_v}\xi=-\tilde{v}_{22}x_v \end{cases} \tag{3.16}$$

Since $\xi_{uv}=\xi_{vu}=0$, we have $\overline{\overline{D}}_{x_u}\left(-\tilde{v}_{22}x_v\right)=0$. Using $\tilde{v}_{12}=0$, $\overline{\overline{D}}_{x_u}x_v=\overline{\overline{D}}_{x_v}x_u$ and Theorem-1, we obtain

$$\left(\tilde{v}_{22}\right)_u x_v+\tilde{v}_{22}\left[\frac{\cosh\theta}{\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}}\right]\tilde{v}_{22}\frac{x_v}{\left\|x_v\right\|}=0.$$

Therefore, we have

$$\left(\tilde{v}_{22}\right)_u-\frac{\cosh\theta}{\sqrt{\left|\sinh^2\varphi-\sinh^2\theta\right|}}\left(\tilde{v}_{22}\right)^2=0 \tag{3.17}$$

The second equality of (3.15), we have

$$\left(\tilde{v}_{22}\right)_u+\frac{\beta_u}{\beta}\tilde{v}_{22}=0 \tag{3.18}$$

and so

$$\left(\beta\tilde{v}_{22}\right)_u=0 \tag{3.19}$$

By (3.19), we see that there exist a smooth function $\psi=\psi(v)$ depending on v such that

$$\beta\tilde{v}_{22}=\psi(v).$$

Proposition 1 Let $x=x(u,v)$ be parametrization of a constant angle spacelike surface in H^3. If $\tilde{v}_{22}=0$ on M, then the x describes an affine plane of H^3.

Proof Let ξ be unit normal vector of the constant angle surface M. By (3.16) we obtain

$$\begin{cases} \xi u=\overline{\overline{D}}_{x_u}\xi=0 \\ \xi v=\overline{\overline{D}}_{x_v}\xi=-\tilde{v}_{22}x_v. \end{cases}$$

If $\tilde{v}_{22}=0$ on M, then

$$\begin{cases} \xi u=0 \\ \xi v=0 \end{cases}$$

Consequently ξ is a constant vector field along M. This complates the proof.

From now on, we will assume that $\tilde{v}_{22}\neq 0$. By solving equation (3.15), we obtain a function $\alpha=\alpha(v)$ such that

$$\begin{cases} \tilde{v}_{22} = \dfrac{-\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}}{u\cosh\theta + \alpha(v)} \\ \alpha(v) = \sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}\,\bar{\alpha}(v) \end{cases}$$

We have

$$\beta(u,v) = \frac{-\psi(v)}{\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}}\left(u\cosh\theta + \alpha(v)\right)$$

In the spacial case of $\psi(v) = -v\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}$ and $\alpha(v) = \frac{1}{v}$,we obtain

$$\begin{cases} x_{uu} = x \\ x_{uv} = \dfrac{v\cosh\theta}{uv\cosh\theta + 1}x_v \\ x_{vv} = -v\cosh\theta(uv\cosh\theta+1)x_u + \dfrac{u\cosh\theta}{uv\cosh\theta+1}x_v - v\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}(uv\cosh\theta+1)\xi + (uv\cosh\theta+1)^2 x \end{cases} \tag{3.20}$$

Now we have the following Theorem by (3.20).

Theorem 2 If M is a constant timelike angle surface with spacelike immersion, then the parametrization of M is

$$\begin{cases} x_i(u,v) = \dfrac{-C_{1i}(v)}{2v\cosh\theta(uv\cosh\theta+1)^2} + C_{2i}(v) \\ i = 1,2,3,4 \end{cases} \tag{3.21}$$

One can calculate the hyperbolic principle curvatures, hyperbolic Gauss and mean curvatures of the constant timelike angle surfaces with spacelike axis in H^3 as follows

$\overline{K_1^{\pm}}(p) = 0$ ve $\overline{K_2^{\pm}}(p) = \tilde{v}_{22}$,

$\overline{K_h^{\pm}} = 0$,

$H_h^{\pm} = \frac{1}{2}\tilde{v}_{22}$,

where $\tilde{v}_{22}$ is

$$\tilde{v}_{22} = \frac{v\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}}{1 + uv\cosh\theta}.$$

Corollary 4 If a constant timelike angle surface M is minimal surface, then M is hyperbolic plane in H^3.

On the other hand, we shall denote eigenvalues of linear transformation A_p and $S_p^{\pm}$ by $K_i(p)$ and $\overline{K_i^{\pm}}(p), i = 1,2$ respectively. We know that A_p and $S_p^{\pm}$ have same eigenvectors and

$\overline{K_i^{\pm}}(p) = -1 \pm K_i(p)$

(see [2]). Therefore we get

$K_1(p) = \pm 1 \qquad K_2(p) = \pm(1 + \tilde{v}_{22})$.

Hence de Sitter Gauss and mean curvaturse of M at p are

$K_e = \pm(1 + \tilde{v}_{22})$,

$H_d = \dfrac{\pm(2 + \tilde{v}_{22})}{2}$.

Let $\gamma(s) = x(u_1(s), u_2(s))$be a curve with unit speed at $p = \gamma(s_0)$on surface M. Then de Sitter normal curvature of $\gamma(s)$ is zero. Since $\overline{K_n^{\pm}}(s_0) = K_n^{\pm}(s_0) - 1$ (see [2]) , we have

$\overline{K_n^{\pm}}(s) = -1$.

Corollary 5 In Hyperbolic-3 space , constant timelike angle surface with spacelike axis are flat.

Definition 1 $K_1(p) = K_2(p)$,then $p = x(u)$ is an umbilical point [2].

Since the eigenvectors of $S_p^{\pm}$ and Ap are the same, the above condition is equivalent to the condition $\overline{K_1^{\pm}}(p) = \overline{K_2^{\pm}}(p)$. We say that $M = x(u)$ is total umbilical if all points on M are umbilical.

Corollary 6 There is no any umbilic point for constant timelike angle surface with spacelike axis in H^3.

Definition 2 The total umbilical surface is called Horosphere in Hyperbolic space [2].

Corollary 7 The constant timelike angle surfaces with spacelike axis are not horosphere in H^3.

4 Constant Timelike Angle Tangent Surfaces

In this section we will study constant timelike angle tangent surfaces (See [2] and [6] for the Minkowski ambient space and Euclidean ambient space,respectively). Let $\alpha : I \to H^3 \subset IR_1^4$ be a regular curve given by arc-length. We define the tangent surface M generated by α as the surface parametrized by

$$\begin{cases} x(s,t) = (\cosh t)\alpha(s) + (\sinh t)\alpha'(s) \\ (s,t) \in I \times IR \end{cases} \tag{4.1}$$

The tangent plane at a point (s,t) of M is spanned by $\{x_s, x_t\}$, where

$$\begin{cases} x_s = (\cosh t)\alpha'(s) + (\sinh t)\alpha''(s) \\ x_t = (\sinh t)\alpha(s) + (\cosh t)\alpha'(s) \end{cases} \tag{4.2}$$

By computing the first fundamental form $\{E,F,G\}$ of M with respect to basis $\{x_s, x_t\}$, we obtain

$E = 1 + K_h^2 \sinh^2 t$, $F = 1$, $G = 1$.

Thus we have

$EG - F^2 = K_h^2 \sinh^2 t$.

Since $EG - F^2 > 0$, M is spacelike surface. From Frenet-Serre type formulae , we obtain

$$\begin{cases} x(s,t) = (\cosh t)\alpha(s) + (\sinh t)t(s) \\ x_s(s,t) = (\sinh t)\alpha(s) + (\cosh t)t(s) \\ +K_h(s)(\sinh t)n(s) \\ x_t(s,t) = (\sinh t)\alpha(s) + (\cosh t)t(s) \end{cases} \tag{4.3}$$

Now we calculate normal vector of M . We know that normal vector of M is

$$e = \frac{x \times x_s \times x_t}{\|x \times x_s \times x_t\|} = \mp \frac{\alpha \times \alpha' \times \alpha''}{|K_h|} \tag{4.4}$$

Since (3.8) and

$$e_1 = \frac{x_s}{\|x_s\|}, \|x_s\| = \sqrt{1 + (\sinh^2 t) K_h^2},$$

we obtain

$$U_h = (\sinh t \sqrt{\frac{|\sinh^2 \varphi - \sinh^2 \theta|}{1 + \sinh^2 t K_h^2}})\alpha(s)$$
$$+(\cosh t \sqrt{\frac{|\sinh^2 \varphi - \sinh^2 \theta|}{1 + \sinh^2 t K_h^2}})t(s)$$
$$+(K_h(s) \sinh t \sqrt{\frac{|\sinh^2 \varphi - \sinh^2 \theta|}{1 + \sinh^2 t K_h^2}})n(s)$$
$$-(\cosh \theta) e(s) \tag{4.5}$$

Theorem 3 Let $\alpha : I \subset IR \to H^3$ curve be different from hyperbolic line. If $x(s,t)$ tangent surface is constant timelike angle surface with spacelike axis then α curve lie hyperbolic plane.

Proof Suppose that $x(s,t)$ tangent surface is constant timelike angle surface with spacelike axis such that α is a curve different from hyperbolic line. Since

$$\xi = \frac{x \times x_s \times x_t}{\|x \times x_s \times x_t\|} = e(s) ,$$

there exsist a positive real number θ such that

$\langle \xi, U_h \rangle = \langle e(s), U_h \rangle = -\cosh \theta$.

If we calculate the derivative of the last equation in s , then we get that

$\langle e'(s), U_h \rangle = 0$.

Hence we get

$$\langle n(s), U_h \rangle = 0 \quad \text{veya} \quad \tau_h(s) = 0 \tag{4.6}$$

If in equation (4.6) $\langle n(s), U_h \rangle = 0$ then scalar product of (4.6) equation with $n(s)$ that we have $t = 0$. This is contradict with definition of tangent surface. Therefore using equation (4.7) $\tau_h(s) = 0$ is obvious. It means that α lie hyperbolic plane.

Remark 1 Since stereografik projection is conformal map, using stereografik projection, constant angle surface in Minkowskian model of hyperbolic space H^3 is visulized in Poincare ball model of hyperbolic space H^3.

By using that idea, we can give the following example.

Example 1 Let $\alpha : I \to H^3 \subset IR_1^4$ be a regular curve given by arc-length

$\alpha(s) = (\sqrt{1+s^2}, s\cos(\arcsin h(s))$
$, s\sin(\arcsin h(s)), 0)$

The tangent surface M generated by α as the surface parametrized by

$x(s,t) = (\cosh t)\alpha(s) + (\sinh t)\alpha'(s),\ (s,t) \in I \times IR$

. The pictures of the Stereografik projection of tangent surface appear in Figure 4.

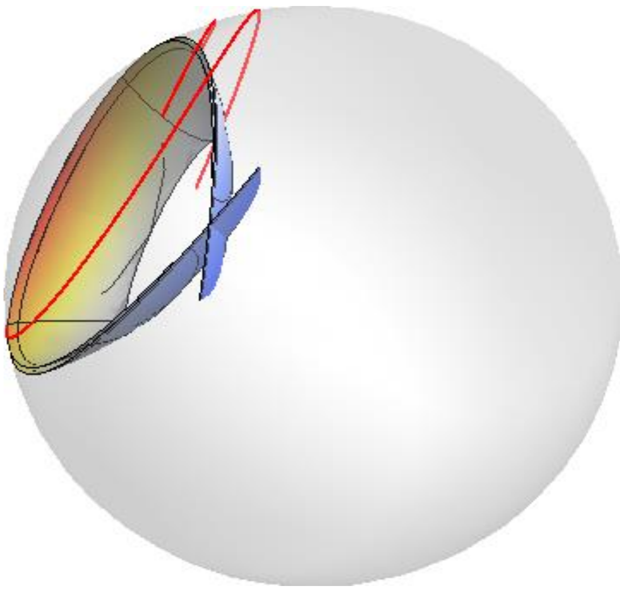

Figure 4 *Stereografik projection of tangent surface*

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