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SOLUTION OF A 1-D CONSERVATION LAWS WITHOUT CONVEXITY

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ABSTRACT

In this paper the exact solution for Cauchy problem of first order nonlinear partial equation with piece-wise initial condition described scalar conservation laws without convexity of the state function. In particular, the state functions having four and one point of inflection are considered. The structure of solutions is investigated.

Keywords: *Scalar conservation laws, state function without convexity, convex and concave hull, weak solution*

ÖZET

Bu makalede bükeyliği olmayan durum fonksiyonuna sahip birinci mertebeden nonlineer kısmi türevli diferansiyel denklem için yazılmış parçalı sürekli başlangıç koşullu Cauchy probleminin gerçek çözümleri elde edilmiştir. Özel olarak, sırasıyla dört ve bir dönüm noktalarına sahip durum fonksiyonları ele alınmış ve çözümün yapısı incelenmiştir.

Anahtar Kelimeler: *Scaler korunum kanunları, bükeyliği olmayan durum fonksiyonu, konveks ve konkav katman, zayıf çözüm.*

1. INTRODUCTION

In this study we construct the exact solutions of a scalar conservation law in one dimension as

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0, \quad x\in R^1,\ t>0. \tag{1}$$

We assume that the flux function is in $C^1(R^1)$ and has finite number of infection points. The equation (1) with the following initial condition

$$u(x,0)=u_0(x), \quad x\in R^1 \tag{2}$$

have been investigated in [2], [4]–[7], [11]–[13], when $u_0(x)\in L_\infty(R^1)$. In [3] has been construct fundamental solution of the equation (1) with initial condition

$$\lim_{t\to\infty} u(x,t)=M\delta(x), \quad x\in R^1, t>0,$$

where the initial value in distribution sense and $M\in R^1$ is the total mass, and $\delta(x)$ is the Dirac function. It should be noted the problem (1), (2) has been investigated in [1] from practical point of view. In [8]-[11] the method for obtaining the exact solution of this kind problem is proposed.

2. CONSTRUCTION OF THE EXACT SOLUTION

In this section we will construct two problems for different state functions with piecewise constant initial function. As the state function $f(u)$ we consider the functions $-\frac{\cos 2u}{2}$ and $\frac{u^3}{3}$, respectively.

2.1. The case of $f(u)=\frac{\cos 2u}{2}$

In order to find the exact solutions of these problems, according to [4], [12] we formulate the following definitions. We denote by $\aleph$

the set functions of $\tilde{f}$ defined on $[\alpha, \beta]$ which satisfy the inequality $\tilde{f} \geq f(u)$.

Definition 1. The function defined by the relation $\hat{f} = \inf_{\tilde{f} \in \aleph} \tilde{f}(u)$ is called a convex hull on $[\alpha, \beta]$ of a function $f(u)$.

Definition 2. The function defined by the relation $\check{f} = \inf_{\tilde{f} \in \aleph} \tilde{f}(u)$ is called a concave hull on $[\alpha, \beta]$ of a function $f(u)$.

1. At first we will consider the problem (1), (2) for the case $f(u) = -\frac{\cos 2u}{2}$ on the interval $\left[-\frac{5\pi}{4}, \frac{5\pi}{4}\right]$. As it is seen from the Figure 1a) the function $0.5\cos 2u$ has four inflection points on the interval $\left[-\frac{5\pi}{4}, \frac{5\pi}{4}\right]$, that is we consider the following equation

$$\frac{\partial u}{\partial t} + \sin 2u \frac{\partial u}{\partial x} = 0 \tag{3}$$

with the following initial distribution

$$u_0(x) = \begin{cases} \frac{5\pi}{4}, & x < 0, \\ -\frac{5\pi}{4}, & x > 0. \end{cases} \tag{4}$$

According to Definition 1, we construct the convex hull of the function $-\frac{\cos 2u}{2}$ on the interval $\left[-\frac{5\pi}{4}, \frac{5\pi}{4}\right]$. It is obvious that $u = -\frac{\pi}{2}$ and $u = \frac{\pi}{2}$ are the roots of the equation $\cos 2u = -1$ on the interval $\left[-\frac{5\pi}{4}, \frac{5\pi}{4}\right]$. In order to construct the convex hull of the function $-\frac{\cos 2u}{2}$, we draw the tangential lines from the points $\left(-\frac{5\pi}{4}, 0\right)$ and $\left(\frac{5\pi}{4}, 0\right)$ to the graph of the function $-\frac{\cos 2u}{2}$, respectively. We denote the abscissa of points of intersection of these tangential lines with graph of the $f(u) = -\dfrac{\cos 2u}{2}$ by v_1 and v_2.

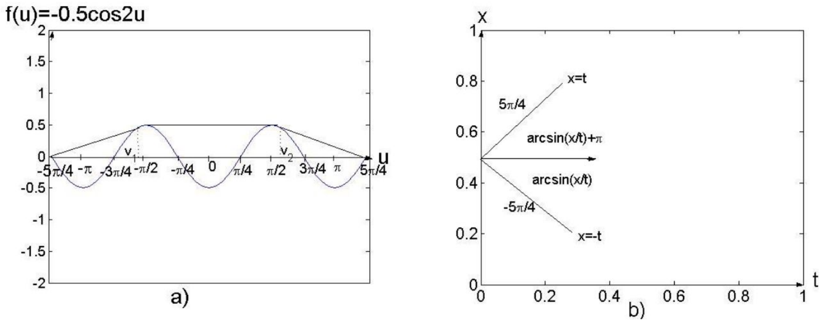


Figure 1: a) The convex hull of the function $f(u) = -0.5\cos 2u$; b) weak solution on (x,t) plane

Therefore, the convex hull of the function $-\dfrac{\cos 2u}{2}$ on the interval $\left[-\dfrac{5\pi}{4}, \dfrac{5\pi}{4}\right]$ consists of the following: tangential line from the point $\left(-\dfrac{5\pi}{4}, 0\right)$, the part of the graph lying $\left[v_1, -\dfrac{\pi}{2}\right]$, the line connecting the points $A = \left(-\dfrac{\pi}{2}, \dfrac{1}{2}\right)$ and $B = \left(\dfrac{\pi}{2}, \dfrac{1}{2}\right)$, the part of the graph lying on the interval $\left[\dfrac{\pi}{2}, v_2\right]$ and the tangential straight line from the point $\left(\dfrac{5\pi}{4}, 0\right)$. The graph of the convex hull of the function $-\dfrac{\cos 2u}{2}$ is shown in Fig.1a.

Since the tangential lines are symmetric, it is easily shown that the equations of the tangential line from the point $\left(-\dfrac{5\pi}{4}, 0\right)$ and $\left(\dfrac{5\pi}{4}, 0\right)$ are $x = -kt$ and $x = kt$, respectively. Here,

$-k = \frac{f(v_1)}{v_1} = \sin 2v_1$. We can not obtain the values of v_1, v_2 and k naturally, but we may say that v_1 is the least positive root of the equation $\cot 2v_1 = -2v_1$, and $v_1 + v_2 = \frac{5\pi}{2}$, $v_2 = \frac{5\pi}{2} - v_1$.

Since all solutions of the equation (1) are lines having slopes $f'(u)$ and passing through the origin of the (x,t) plane, we convert the function $\xi = \frac{x}{t} = f'(u) = \sin 2u$ on the interval $[-\pi, 0]$ and $[0, \pi]$. Solving this equation we find $u(x,t) = -\frac{1}{2}\arcsin\frac{x}{t}$ and $u(x,t) = \frac{1}{2}\arcsin\frac{x}{t}$, respectively.

The graph of the weak solution of the problem (3), (4) on the (x,t) plane is illustrated in Fig. 1b. Therefore, the exact solution of the problem (3), (4) is

$$u(x,t) = \begin{cases} \frac{5\pi}{4}, & x < kt, \\ -\frac{1}{2}\arcsin\frac{x}{t}, & -kt < x < 0, \\ \frac{1}{2}\arcsin\frac{x}{t}, & 0 < x < kt, \\ -\frac{5\pi}{4}, & x \geq kt. \end{cases} \qquad (5)$$

The time evaluation of the solutions of (5) at value $T = 1.0$ is shown in Fig. 2a.

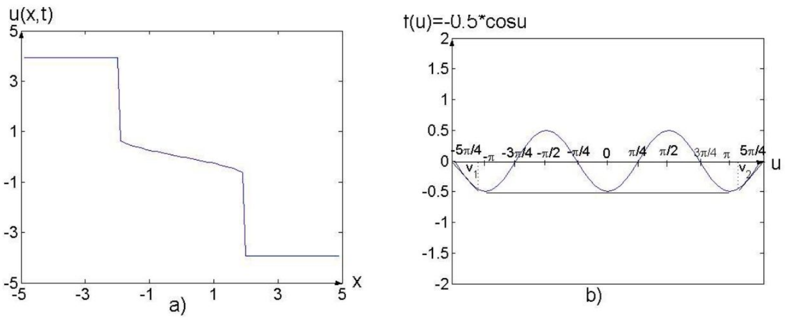


Figure 2: a) Time evaluation of the exact solution $u(x,t)$ a) $T = 1.0$; b) The concave hull of the function $f(u) = -0.5\cos 2u$

Now, we investigate the Eq. (3) with the following initial function

$$u_0(x) = \begin{cases} -\dfrac{5\pi}{4}, & x < 0, \\ \dfrac{5\pi}{4}, & x > 0. \end{cases} \quad (6)$$

In this case we construct the concave hull of the function $f(u) = -\dfrac{\cos 2u}{2}$ on the interval $\left[-\dfrac{5\pi}{4}, \dfrac{5\pi}{4}\right]$. For this aim, we first find the solutions of $\cos 2u = 1$, which are $u = -\pi$ and $u = \pi$. We denote by A and B the points $\left(-\pi, -\dfrac{1}{2}\right)$ and $\left(\pi, -\dfrac{1}{2}\right)$, respectively, and we connect the points A and B by a straight line. We draw the tangential lines from points $\left(-\dfrac{5\pi}{4}, 0\right)$ and $\left(\dfrac{5\pi}{4}, 0\right)$, respectively. We also denote the intersections of these tangential lines with the graph $f(u) = -\dfrac{\cos 2u}{2}$ by v_1 and v_2, respectively.

Therefore, the concave hull of the function $-\dfrac{\cos 2u}{2}$ on the interval $\left[-\dfrac{5\pi}{4}, \dfrac{5\pi}{4}\right]$ consists of the following: tangential line from the point $\left(-\frac{5\pi}{4},0\right)$, the part the graph lying $\left[v_1,-\frac{\pi}{2}\right]$, the straight line connected of the points $A=\left(-\pi,-\frac{1}{2}\right)$ and $B=\left(\pi,-\frac{1}{2}\right)$, the part of the graph lying on the interval $\left[v_2,\frac{\pi}{2}\right]$ and the tangential line from the point $\left(\frac{5\pi}{4},0\right)$, Fig. 2b.

As above, since the tangential lines are symmetric, their equations are $x=-k_1t$ and $x=k_1t$, respectively. The values of v_1, v_2 and k_1 can not be found, in general, but we may say that v_1 is the least positive root of the equation $-k=-\frac{\cos 2v_1}{2v_1}=\sin 2v_1$ and $v_1+v_2=\frac{5\pi}{2}$, $v_2=\frac{5\pi}{2}-v_1$.

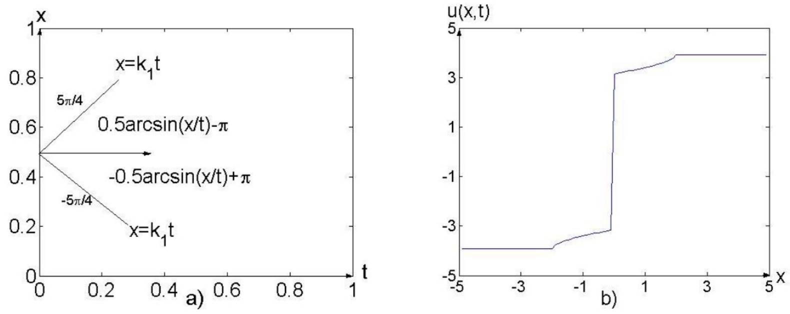


Figure 3: a) weak solution on (x,t) plane; b) Time evaluation of the exact solution $u(x,t)$ $T=1.0$

Now, we must find the inverse function of $f'(u)=\sin 2u$ on the interval $\left[-\frac{5\pi}{4},-\pi\right]$ and $\left[\pi,\frac{5\pi}{4}\right]$, respectively. It is clear that these inverse functions are $u(x,t)=-\frac{1}{2}\arcsin\frac{x}{t}-\pi$ and $u(x,t)=\frac{1}{2}\arcsin\frac{x}{t}+\pi$.

As a result, the exact solution of this problem is

$$u(x,t)=\begin{cases} -\dfrac{5\pi}{4}, & x<k_1t, \\ -\dfrac{1}{2}\arcsin\dfrac{x}{t}-\pi, & -k_1t<x<0, \\ \dfrac{1}{2}\arcsin\dfrac{x}{t}+\pi, & 0<x<k_1t, \\ \dfrac{5\pi}{4}, & x\geq k_1t. \end{cases} \tag{7}$$

The graph of the weak solution of this problem on the (x,t) plane is illustrated in Fig. 3a. The time evaluation of the solution (7) at $T=1.0$ is shown in Fig. 3b.

2.2. The case of $f(u)=\dfrac{u^3}{3}$

According to Definition 1, at first we will construct the convex hull of the function $f(u)=\dfrac{u^3}{3}$ on the interval $[-2,2]$. For this aim we draw the tangential line from point $\left(2,\dfrac{8}{3}\right)$ to graph of $f(u)$ and note by $\left(u_0,\dfrac{u_0^3}{3}\right)$ the point of intersection of this line with curve

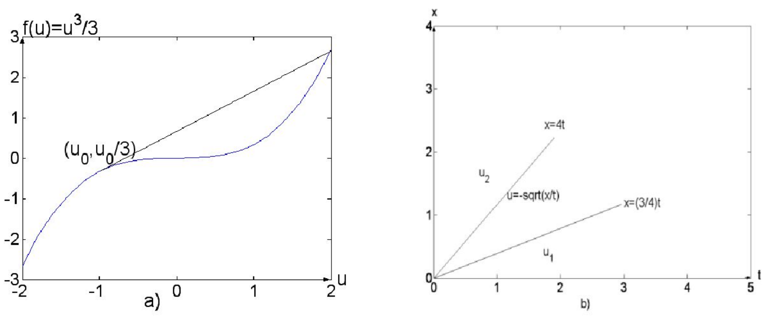


Figure 4. a) The graph of the convex hull of the function $f(u)=\dfrac{u^3}{3}$; b) The weak solution of $u(x,t)$ on (x,t) plane

$\frac{u^3}{3}$. It is clear that, the value of u_0 is found from the relation $m = f'(u_0) = \frac{f(u_1) - f(u_0)}{u_1 - u_0}$ as $u_0 = -1$.

Therefore the convex hull of the function $\frac{u^3}{3}$ on the interval $[-2,2]$ consists of the following: tangential line from the point $\left(2,\frac{8}{3}\right)$ and the part of the graph of $\frac{u^3}{3}$ lies between point of $\left(-1,-\frac{1}{3}\right)$ and $\left(-2,-\frac{1}{8}\right)$, Fig. 4a.

It is clear that the solution is exposed to jump on the line $x = \xi t,\ \ t > 0$ which is paralel to the tangential line. From Rankino-Hugoniot condition we have $\xi = \frac{f(u_2) - f(u_0)}{u_2 - u_0} = 1$. This jump lies between $u_0 = -1$ $(x > \xi t)$ and $\psi\left(\frac{x}{t}\right)$. Here, $\psi\left(\frac{x}{t}\right)$ is inverse function of the $\xi = f'(u) = u^2$ on the interval $[-2,-1]$. Hence, $u = \psi\left(\frac{x}{t}\right) = (f')^{-1} = -\sqrt{\frac{x}{t}},\ \ 1 \le \xi \le 4$.

It is clear that intersection of the function $\psi\left(\frac{x}{t}\right)$ with line $u_2 = -2$ take place on the ray $x = 4t$. But this ray is paralel to the tangential line leaving from point $(u_2, f(u_2)) = \left(u_2, \frac{u_2^3}{3}\right)$.

Therefore, the solution of the problem (1), (8) is

$$u(x,t)=\begin{cases} 2, & x<t \\ -\sqrt{\dfrac{x}{t}}, & t<x<4t \\ -2, & x\geq 4t. \end{cases} \qquad (8)$$

The weak solution of the problem (1), (8) is shown in Fig. 4b. The time evaluation of the solution (1), (8) is demonstrated in Fig. 6b.

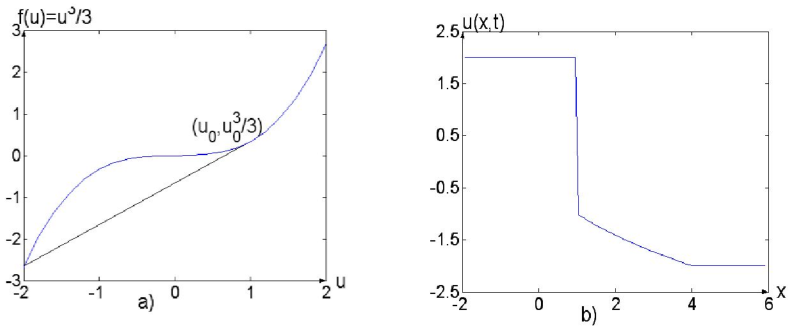
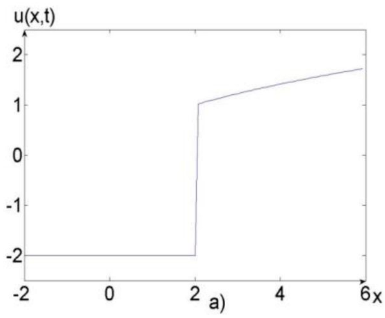


Figure 6. a) The exact solution of $u(x,t)$ at the value $T=1.0$; b) The concave hull of the function $f(u)=\dfrac{u^3}{3}$ on the interval $[-2,2]$

Now, we will investigate the case $u_1=-2$ and $u_2=2,\ u_1<u_2$. In this case we will construct the concave hull the function $f(u)=\dfrac{u^3}{3}$ on the $[u_1,u_2]$. Let us draw the tangential line from point $A(-2,f(u))=A\left(-2,-\dfrac{8}{3}\right)$ to graph of the function $f(u)$. It is clear that, this tangential line will be intersect the graph of $f(u)$ at the point $\left(u_0,\dfrac{u_0^3}{3}\right)=\left(1,\dfrac{1}{3}\right)$.



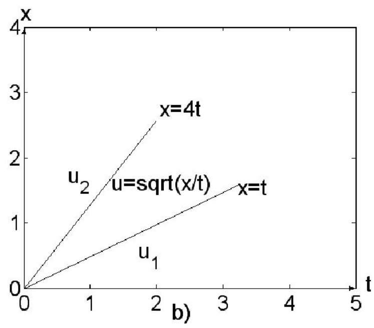


Figure 7. a) The graph of the function (9); b)The weak solution of the problem (1), (8) on the (x,t) plane

Therefore, the concave hull of the function $f(u)=\frac{u^3}{3}$ on the interval $[-2,2]$ consists the part of the graph of $f(u)$ lies between points of $\left(1,\frac{1}{3}\right)$ and $\left(-2,-\frac{8}{3}\right)$, and tangential line leaving from the point of $\left(-2,-\frac{8}{3}\right)$, Figure 6a. In this case the solution is exposed to jump on the ray $x=\xi t,\ t>0$ with is paralel to tangential line originated from point $\left(-2,-\frac{8}{3}\right)$. As above, from Rankino-Hugoniot condition we get $\xi=1$. This jump take place between $u_0=2$ (for $x>t$) and $\psi\left(\frac{x}{t}\right)$. Here $\psi\left(\frac{x}{t}\right)$ is inverse function of $\xi=f'(u)=u^2$ on the interval $[1,2]$. From here we have $u=\sqrt{\xi}$, $1\le\xi\le4$. Therefore the exact solution of the problem (1), (8) (in the case $u_1<u_2$) is

$$u(x,t)=\begin{cases}-2, & x<t \\ \sqrt{\dfrac{x}{t}}, & t<x<4t \\ 2, & x\geq 4t.\end{cases} \quad (9)$$

The graph of the function (9) is shown in Figure 7a. The weak solution of the problem (1), (8) is given Figure 7b.

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