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ON THE BEHAVIORS OF SOLUTIONS IN LINEAR NONHOMOGENEOUS DELAY DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract. This paper deals with the behaviors of solutions for linear nonhomogeneous delay differential equations. In this study, a periodic solution, an asymptotic result and a useful exponential estimate of the solutions are established. Our results are obtained by the use of real roots of the corresponding characteristic equation.

1. Introduction and Preliminaries

The delay differential equation is considered as:

$$x'(t) = a(t)x(t) + \sum_{i\in I} b_i(t)x(t-\tau_i) + f(t), \quad t \geq 0, \tag{1.1}$$

$$x(t) = \phi(t)\ , \quad -\tau \leq t \leq 0. \tag{1.2}$$

where I is the initial segment of natural numbers, a and b_i for $i \in I$ the continuous real-valued functions on the interval $[0,\infty)$, f is a continuous real-valued function on the interval $[0,\infty)$, and τ_i for $i \in I$ positive real numbers with $\tau_{i_1} \neq \tau_{i_2}$ for $i_1, i_2 \in I$ such that $i_1 \neq i_2$. Suppose that the functions b_i for $i \in I$ are not identically zero on $[0,\infty)$ and also the coefficients a and b_i for $i \in I$ are the periodic functions with a common period $T > 0$ where $\tau_i = m_iT$ for positive integers m_i for $i \in I$. τ is positive number such that

$$\tau = \max_{i\in I} \tau_i.$$

ϕ is continuous real-valued given *the initial function* on the interval $[-\tau, 0]$.

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In the case where the function f is identically zero on the interval $[0, \infty)$, the delay differential equation (1.1) reduces to

$$x'(t) = a(t)x(t) + \sum_{i \in I} b_i(t)x(t - \tau_i), \quad t \geq 0. \tag{1.3}$$

As far as the applications' point of view is concerned, our literature review comprehensively offers the behaviors based on the solutions of delay differential equations [1-6]. As it concers the applications point view, first order linear delay differential equations appear as models in various problems in science and tecnology. For example, in [7], first order linear delay differential equations have been used for description of different economic processes. For the basic theory of delay differential equations with periodic coefficients, the reader is referred to the books by Farkas [8].

Our aim in this article is to obtain periodic solutions of the given equation, and to present some new results on asymptotic behavior for linear delay differential equations with periodic coefficients. Our results are motivated by those in two excellent papers by Philos [9] and Farkas [11]. The very recent results given by Philos [9] (and also [10]) for periodic first order linear (homogeneous) delay differential equations can be obtained from the results of the present paper. Also, the results given here contain essentially ones obtained by Farkas [11] for the particular case of first order linear nonhomogeneous one constant delay differential equations. Our results are derived by the use of a real root (with an appropriate property) of the corresponding (in a sense) characteristic equation. A combination of several methods [6, 9-11] are referred for the used techniques.

The function $x(t)$ is described as a *solution* of the initial value problem (1.1)-(1.3) on $[-\tau, \infty)$. This paper uses the notation

$$A = \frac{1}{T} \int_0^T a(t)dt, \quad \text{and} \quad B_i = \frac{1}{T} \int_0^T b_i(t)dt \quad \text{for} \quad i \in I.$$

Furthermore, we associate the following equation with the differential equation (1.3)

$$\lambda = A + \sum_{i \in I} B_i e^{-\lambda \tau_i}, \tag{1.4}$$

specified as the *characteristic equation* of (1.3). There were given sufficient conditions to obtain a unique real root of characteristic equation (1.4) in Philos [9].

In what follows, the T-periodic extensions are denoted by $\tilde{a}$ and $\tilde{b}_i$ for $i \in I$ for the coefficients a and b_i for $i \in I$ respectively on the interval $[-\tau, \infty)$. In order to construct a suitable mapping for the asymptotic criterion of the solutions, we should reach a finding as follows. Suppose that λ_0 is a real root of (1.4). We can now write

$$h_{\lambda_0}(t) = \tilde{a}(t) + \sum_{i \in I} \tilde{b}_i(t) e^{-\lambda_0 \tau_i} \quad \text{for} \quad t \geq -\tau. \tag{1.5}$$

Next, we will establish some equalities needed below. For each index $i \in I$, we can use the assumption that the functions $\tilde{b}_i$ are T-periodic and that $\tau_i = m_i T$ to obtain for $t \geq 0$

$$\int_{t-\tau_i}^{t} \tilde{b}_i(u)du = \int_0^{\tau_i} b_i(u)du = \left[\frac{1}{\tau_i}\int_0^{\tau_i} b_i(u)du\right]\tau_i = \left[\frac{1}{T}\int_0^{T} b_i(u)du\right]\tau_i = B_i\tau_i. \tag{1.6}$$

In a similar manner, one can verify that

$$\int_{t-\tau_i}^{t} |\tilde{b}_i(u)|du = |B_i|\tau_i \quad \text{for every } t \geq 0 \quad \text{and all} \ \ i \in I. \tag{1.7}$$

Our aim in this paper is to study the periodic solutions of equation (1.1) when f is also T-periodic. We will show that, under certain conditions, equation (1.1) has periodic solutions. In the following discussion, without specific mention, we always assume that f is also T-periodic.

2. Periodic Solutions

In this section, we establish conditions under which equation (1.1) has a periodic solution. Consider, first, the homogeneous equation (1.3) and the equation without delay

$$x'(t) = a(t)x(t). \tag{2.1}$$

The general solution of (2.1) is

$$x(t) = c\ exp\left\{\int_0^t a(s)ds\right\}$$

where c is a constant. To find a solution of (1.3), we apply the variation of constants formula. Assume that

$$x(t) = C(t)\ exp\left\{\int_0^t \tilde{a}(s)ds\right\} \tag{2.2}$$

where

$$\tilde{a}(t) = \begin{cases} a(t)\ , & t \geq 0, \\ a(t+\tau)\ , & -\tau \leq t \leq 0, \end{cases}$$

is a solution of (1.3). Substituting this into (1.3) yields the condition

$$C'(t) = \sum_{i\in I} b_i(t)C(t-\tau_i)exp\left\{-\int_{-\tau_i}^0 \tilde{a}(s)ds\right\} \tag{2.3}$$

for all $t \geq 0$ on $C(t)$. We define

$$g(t) = \sum_{i\in I}\tilde{b}_i(t),$$

where

$$\tilde{b}_i(t) = \begin{cases} b_i(t)\ , & t \geq 0, \\ b_i(t+\tau)\ , & -\tau \leq t \leq 0. \end{cases}$$

Assume that (2.3) has a solution of the form

$$C(t) = exp\left\{\mu\int_0^t g(s)ds\right\}. \tag{2.4}$$

Then, by using (2.4) in (2.3) for $t \geq 0$ we obtain

$$\mu\sum_{i\in I} b_i(t) = \sum_{i\in I} b_i(t)exp\left\{-\mu\int_{t-\tau_i}^t g(s)ds\right\}exp\left\{-\int_{-\tau_i}^0 \tilde{a}(s)ds\right\}.$$

Since the functions $b_i(t)$ are T-periodic, from the last equation

$$\mu \sum_{i\in I} b_i(t) = \sum_{i\in I} b_i(t) exp\left\{-\mu \int_{-\tau_i}^{0} g(s)ds\right\} exp\left\{-\int_{-\tau_i}^{0} \tilde{a}(s)ds\right\}$$

or

$$\mu \sum_{i\in I} b_i(t) = \sum_{i\in I} b_i(t) exp\left\{-\int_{0}^{\tau_i} (a(s)+\mu g(s))\, ds\right\}. \tag{2.5}$$

Next, for each index $i \in I$, we can use the assumption that the functions a and b_i are T-periodic and that $\tau_i = m_i T$ to obtain for $t \geq 0$

$$\begin{aligned}\int_0^{\tau_i} (a(s)+\mu g(s))\, ds &= \left[\frac{1}{\tau_i}\int_0^{\tau_i} (a(s)+\mu g(s))\, ds\right]\tau_i = \left[\frac{1}{T}\int_0^{T} (a(s)+\mu g(s))\, ds\right]\tau_i \\ &= \left\{\left[\frac{1}{T}\int_0^{T} a(s)ds\right] + \mu \sum_{i\in I}\left[\frac{1}{T}\int_0^{T} b_i(s)\right]\right\}\tau_i \\ &= \left(A + \mu \sum_{i\in I} B_i\right)\tau_i.\end{aligned}$$

Thus, from (2.5) we get

$$\mu \sum_{i\in I} b_i(t) = \sum_{i\in I} b_i(t) exp\left\{-\left(A+\mu\sum_{i\in I} B_i\right)\tau_i\right\}. \tag{2.6}$$

If we assume that $\sum_{i\in I} b_i(t) \neq 0$ for $t \geq -\tau$ and $A + \mu\sum_{i\in I} B_i = 0$ hold with $\mu = 1$, (2.6) establishes

$$C(t) = exp\left\{\int_0^t \sum_{i\in I} \tilde{b}_i(s)ds\right\}$$

is a solution of (2.3). Hence, from (2.2)

$$x(t) = k\ exp\left\{\int_0^t \left(\tilde{a}(s) + \sum_{i\in I} \tilde{b}_i(s)\right) ds\right\}, \tag{2.7}$$

where k is a constant, is a solution of equation (1.3). Also, since $A + \sum_{i\in I} B_i = 0$, it is easy to see that

$$\int_0^{\sigma} \left(a(s) + \sum_{i\in I} b_i(s)\right) ds = 0,$$

where $\sigma = \min_{i\in I} \tau_i$. Then, (2.7) is a σ-periodic solution of equation (1.3).

Now, consider the original nonhomogeneous equation (1.1). The variation of constants formula is applied again. Assume that (1.1) has a solution of the form

$$x_p(t) = K(t)\ exp\left\{\int_0^t \left(\tilde{a}(s) + \sum_{i\in I} \tilde{b}_i(s)\right) ds\right\}. \tag{2.8}$$

Using $A + \sum_{i\in I} B_i = 0$, substituting this into (1.1) yields the condition

$$K'(t) + \sum_{i\in I} b_i(t)\left(K(t) - K(t-\tau_i)\right) = f(t) exp\left\{\int_0^t -\left(a(s) + \sum_{i\in I} b_i(s)\right) ds\right\}.$$

The equation (2.8) is a periodic solution of (1.1) if and only if $K(t)$ is periodic. But, this means that $K(t)-K(t-\tau_i)=0$, and so the differential equation for K is

$$K'(t)=f(t)exp\left\{\int_0^t -\left(a(s)+\sum_{i\in I} b_i(s)\right)ds\right\}.$$

It follows that

$$K(t)=\int_0^t f(u)exp\left\{\int_0^u -\left(a(s)+\sum_{i\in I} b_i(s)\right)ds\right\}du.$$

By noting that this function is the integral of a σ-periodic function, we see that it is a σ-periodic function if and only if

$$\int_0^\sigma f(u)exp\left\{\int_0^u -\left(a(s)+\sum_{i\in I} b_i(s)\right)ds\right\}du=0.$$

Substituting this into (2.8), we have the following result.

Theorem 2.1. *Assume that*

$$\sum_{i\in I} b_i(t)\neq 0 \quad \textit{for} \quad t\geq -\tau,$$

$$A+\sum_{i\in I} B_i=0$$

where $A=\frac{1}{T}\int_0^T a(t)dt, \quad B_i=\frac{1}{T}\int_0^T b_i(t)dt$*, and suppose that*

$$\int_0^\sigma f(u)exp\left\{\int_0^u -\left(a(s)+\sum_{i\in I} b_i(s)\right)ds\right\}du=0,$$

where $\sigma=\min_{i\in I}\tau_i$*. Then, for each* $c\in\mathbb{R}$*,*

$$x(t)=c\ exp\left\{\int_0^t \left[a(s+\tau)+\sum_{i\in I} b_i(s+\tau)\right]ds\right\}+x_p(t) \quad \textit{for} \quad t\geq -\tau,$$

where

$$\begin{aligned} x_p(t) =& exp\left\{\int_0^t \left[a(s+\tau)+\sum_{i\in I} b_i(s+\tau)\right]ds\right\} \\ &\times\left\{\int_0^t f(u)exp\left[\int_0^u -\left(a(s+\tau)+\sum_{i\in I} b_i(s+\tau)\right)ds\right]du\right\}\end{aligned}$$

is a σ*-periodic solution of equation* (1.1).

Example 2.2. Consider

$$x'(t)=-2x(t)+(1-sint)x(t-2\pi)+(1+cost)x(t-4\pi)+sint-cost, \quad t\geq 0. \quad (2.9)$$

Since $A=\frac{1}{2\pi}\int_0^{2\pi}(-2)dt=-2, \quad B_1=\frac{1}{2\pi}\int_0^{2\pi}(1-sint)dt=1$ and $B_2=\frac{1}{2\pi}\int_0^{2\pi}(1+cost)dt=1$, we have $A+B_1+B_2=0$. Also

$$\int_0^{2\pi}(sinu-cosu)exp\left\{\int_0^u (sins-coss)\,ds\right\}du=0.$$

Therefore, the conditions of Theorem 2.1 are satisfied. Then, for each $c \in \mathbb{R}$,

$$x(t) = c\, exp\left\{\int_0^t [coss - sins]\, ds\right\} + x_p(t) \quad \text{for} \quad t \geq -4\pi,$$

where

$$x_p(t) = exp\left\{\int_0^t [coss - sins]\, ds\right\}\left\{\int_0^t (sinu - cosu) exp\left[\int_0^u -(coss - sins)\, ds\right] du\right\}$$

or

$$x(t) = (c-1)\, exp\,\{sint + cost - 1\} + 1 \quad \text{for} \quad t \geq -4\pi$$

is 2π-periodic solution of equation (2.9).

3. An asymptotic result and estimation of solutions

We give a fundamental asymptotic criterion as a theorem to solve the problem (1.1)-(1.2).

Theorem 3.1. *Assume that λ_0 be a real root of the characteristic equation* (1.4) *and that the root λ_0 satisfies*

$$\mu(\lambda_0) = \sum_{i \in I} |B_i| \tau_i e^{-\lambda_0 \tau_i} + \int_0^\infty |f(u)| \exp\left[-\int_0^u h_{\lambda_0}(s) ds\right] du < 1, \tag{3.1}$$

where h_{λ_0} is defined as in (1.5). *Then, for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of* (1.1)-(1.2) *satisfies*

$$\lim_{t \to \infty}\left\{x(t) \exp\left[-\int_0^t h_{\lambda_0}(u) du\right]\right\} = \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \tag{3.2}$$

where

$$\begin{aligned} L(\lambda_0; \phi) =& \phi(0) + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{-\tau_i}^0 \tilde{b}_i(s) \phi(s) \exp\left[-\int_0^s h_{\lambda_0}(u) du\right] ds \\ & + \int_0^\infty f(u) \exp\left[-\int_0^u h_{\lambda_0}(s) ds\right] du \end{aligned} \tag{3.3}$$

and

$$\beta(\lambda_0) = \sum_{i \in I} B_i \tau_i e^{-\lambda_0 \tau_i}. \tag{3.4}$$

Note: It is guaranteed by the property (2.1) that $0 < 1 + \beta(\lambda_0) < 2$ and $\int_0^\infty f(u) \exp\left[-\int_0^u h_{\lambda_0}(s) ds\right]$ is finite.

Proof. By (3.1), we have $|\beta(\lambda_0)| \leq \mu(\lambda_0) < 1$. So, this yields that $0 < 1 + \beta(\lambda_0) < 2$ and
$-1 < \int_0^\infty f(u) \exp\left[-\int_0^u h_{\lambda_0}(s) ds\right] du < 1.$

Let us define

$$y(t) = x(t) \exp\left[-\int_0^t h_{\lambda_0}(u) du\right] \quad \text{for} \quad t \geq -\tau. \tag{3.5}$$

Then, we obtain for every $t \geq 0$

$$y'(t) = \big(a(t) - h_{\lambda_0}(t)\big)y(t) + \sum_{i\in I} b_i(t)e^{-\lambda_0\tau_i}y(t-\tau_i) + f(t)\exp\Big[-\int_0^t h_{\lambda_0}(u)du\Big].$$

Thus, using (1.5), the fact that x satisfies (1.1) for all $t \geq 0$ is equivalent to

$$y'(t) = -\sum_{i\in I} b_i(t)e^{-\lambda_0\tau_i}\big[y(t) - y(t-\tau_i)\big] + f(t)\exp\Big[-\int_0^t h_{\lambda_0}(u)du\Big]. \tag{3.6}$$

Furthermore, the initial condition (1.2) is equivalent to

$$y(t) = \phi(t)\exp\Big[-\int_0^t h_{\lambda_0}(u)du\Big], \quad t \in [-\tau, 0]. \tag{3.7}$$

When equation (3.6) is integrated from 0 to t, by taking into account the fact that the functions $\tilde{b}_i$ for each index $i \in I$ are T-periodic and that the delays τ_i, $i \in I$ are multiples of T, we can verify that (3.6) is equivalent to

$$y(t) = L(\lambda_0;\phi) - \sum_{i\in I} e^{-\lambda_0\tau_i}\int_{t-\tau_i}^t \tilde{b}_i(s)y(s)ds - \int_t^\infty f(u)\exp\Big[-\int_0^u h_{\lambda_0}(s)ds\Big]du. \tag{3.8}$$

Now, for $t \geq -\tau$ we define

$$z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}.$$

Hence, from the equation (3.8) it is reduced to the equation as below

$$z(t) = -\sum_{i\in I} e^{-\lambda_0\tau_i}\int_{t-\tau_i}^t \tilde{b}_i(s)z(s)ds - \int_t^\infty f(u)\exp\Big[-\int_0^u h_{\lambda_0}(s)ds\Big]du \quad \text{for } t \geq 0. \tag{3.9}$$

Moreover, the initial condition (3.7) can be equivalently

$$z(t) = \phi(t)\exp\Big[-\int_0^t h_{\lambda_0}(u)du\Big] - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}. \tag{3.10}$$

Using y and z, we should prove the equality (3.2), i.e.

$$\lim_{t\to\infty} z(t) = 0. \tag{3.11}$$

Put

$$W(\lambda_0;\phi) = \max\Big\{1\ ,\ \max_{t\in[-\tau,0]}\Big|\phi(t)\exp\Big[-\int_0^t h_{\lambda_0}(u)du\Big] - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\Big|\Big\}.$$

Thus, by (3.10) we obtain

$$|z(t)| \leq W(\lambda_0;\phi) \quad \text{for } -\tau \leq t \leq 0. \tag{3.12}$$

Now, the following inequality will be proved

$$|z(t)| \leq W(\lambda_0;\phi) \quad \text{for } t \geq -\tau. \tag{3.13}$$

To this end, let us consider an arbitrary number $\epsilon > 0$. We claim that

$$|z(t)| < W(\lambda_0;\phi) + \epsilon \quad \text{for } t \geq -\tau. \tag{3.14}$$

Otherwise, because of (3.13), there exists a point $t^* > 0$ such that

$$|z(t)| < W(\lambda_0;\phi) + \epsilon \quad \text{for } t \in [-\tau, t^*) \quad \text{and} \quad |z(t^*)| = W(\lambda_0;\phi) + \epsilon.$$

Then, by using (3.1) and (1.7), from (3.9) we obtain

$$\begin{aligned}
W(\lambda_0;\phi) + \epsilon &= |z(t^*)| \\
&= \left| -\sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t^*-\tau_i}^{t^*} \tilde{b}_i(s)z(s)ds - \int_t^\infty f(u)\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \right| \\
&\le \sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t^*-\tau_i}^{t^*} |\tilde{b}_i(s)||z(s)|ds + \int_t^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \\
&\le \left\{ \sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t^*-\tau_i}^{t^*} |\tilde{b}_i(s)|ds + \int_0^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \right\} \\
&\quad \times W(\lambda_0;\phi) + \epsilon \\
&\le \mu(\lambda_0)(W(\lambda_0;\phi) + \epsilon) < W(\lambda_0;\phi) + \epsilon.
\end{aligned}$$

This is a contradiction and so (3.14) holds true. Since (3.14) is satisfied for all $\epsilon > 0$, (3.13) is always fulfilled. Next, in view of (1.7), (3.1) and (3.13), from (3.9) we get for every $t \ge 0$

$$\begin{aligned}
|z(t)| &= \left| -\sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t-\tau_i}^{t} \tilde{b}_i(s)z(s)ds - \int_t^\infty f(u)\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \right| \\
&\le \sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t-\tau_i}^{t} |\tilde{b}_i(s)||z(s)|ds + \int_t^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \\
&\le \left\{ \sum_{i\in I} e^{-\lambda_0\tau_i}|B_i|\tau_i + \int_0^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \right\} W(\lambda_0;\phi) \\
&\le \mu(\lambda_0)W(\lambda_0;\phi).
\end{aligned}$$

In other words, we have

$$|z(t)| \le \mu(\lambda_0)W(\lambda_0;\phi) \quad \text{for } t \ge 0. \tag{3.15}$$

By (3.1), (3.13) and (3.15), using an easy induction, that z satisfies

$$|z(t)| \le [\mu(\lambda_0)]^n W(\lambda_0;\phi) \quad \text{for } t \ge n\tau - \tau \ (n = 0, 1, \cdots). \tag{3.16}$$

Due to (2.1) , we get $\lim_{n\to\infty} [\mu(\lambda_0)]^n = 0$. Thus, from (3.16) we get

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \left\{ x(t)\exp\left[-\int_0^t h_{\lambda_0}(u)du\right] - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \right\} = 0$$

i.e. (3.2) satisfies. Theorem 3.1 has been already proven. □

Corollary 3.2. *Assume that*

$$a(t) + \sum_{i\in I} b_i(t) = 0 \quad \textit{for} \quad t \in [0, \infty) \tag{3.17}$$

and

$$\sum_{i\in I} |B_i|\tau_i + \int_0^\infty |f(u)|du < 1. \tag{3.18}$$

Thus, the solution x of (1.1)-(1.2) *satisfies for any* $\phi \in ([-\tau, 0], \mathbb{R})$,

$$\lim_{t\to\infty} x(t) = \frac{\phi(0) + \sum_{i\in I} \int_{-\tau_i}^{0} \tilde{b}_i(s)\phi(s)ds + \int_0^\infty f(u)du}{1 + \sum_{i\in I} B_i\tau_i}.$$

Note: It is guaranteed by (3.18) that $2 > 1 + \sum_{i\in I} B_i\tau_i > 0$.

Proof. It immediately follows from (3.17) that $A + \sum_{i\in I} B_i = 0$ and hence $\lambda_0 = 0$ is a real root of (1.4). By using again (3.18), we see that, for $\lambda_0 = 0$, we have $h_{\lambda_0} = 0$ on the interval $[-\tau, \infty)$. Moreover, (3.18) facilitates the verification of which the root $\lambda_0 = 0$ of (1.4) has the property (2.1). Therefore this can be applied Theorem 3.1. □

Theorem 3.3. *Let λ_0 be a real root of the characteristic equation* (1.4) *with the property* (3.1)*, and let $h_{\lambda_0}(t)$ and $\beta(\lambda_0)$ are specified by* (1.5) *and* (3.4)*, respectively. Set*

$$N(\lambda_0) = \frac{(1 + \mu(\lambda_0))^2}{1 + \beta(\lambda_0)} + \mu(\lambda_0). \tag{3.19}$$

Then, for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of (1.1)-(1.2) *satisfies*

$$|x(t)| \le N(\lambda_0)R(\lambda_0;\phi) \exp\left[\int_0^t h_{\lambda_0}(u)du\right], \quad \textit{for all} \quad t \ge 0, \tag{3.20}$$

where

$$R(\lambda_0;\phi) = \max\left\{1\ ,\ \max_{-\tau\le t\le 0} |\phi(t)|\ ,\ \max_{-\tau\le t\le 0} \left[|\phi(t)| \exp\left[-\int_0^t h_{\lambda_0}(u)du\right]\right]\right\}. \tag{3.21}$$

Note: It is guaranteed by the property (2.1) *that* $0 < 1 + \beta(\lambda_0) < 2$.

Proof. Suppose that x is the solution of (1.1)-(1.2) and y, z are defined as above, i.e. for $t \ge -\tau$

$$y(t) = x(t) \exp\left[-\int_0^t h_{\lambda_0}(u)du\right] \quad \text{and} \quad z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1 + \beta(\lambda_0)},$$

where $L(\lambda_0;\phi)$ is defined as in (3.3). Therefore, we specify $W(\lambda_0;\phi)$ as in the proof of Theorem 3.1. Hence, as in Theorem 3.1, it can be also proved that z satisfies inequality (3.15), and thus for $t \ge 0$ we get

$$|y(t)| \le \mu(\lambda_0)W(\lambda_0;\phi) + \frac{|L(\lambda_0;\phi)|}{1 + \beta(\lambda_0)}. \tag{3.22}$$

Using (3.1) and (3.21), from (3.3) we obtain

$$
\begin{aligned}
|L(\lambda_0;\phi)| \leq |\phi(0)| &+ \sum_{i\in I} e^{-\lambda_0\tau_i}\int_{-\tau_i}^{0} |\tilde{b}_i(s)||\phi(s)|\exp\left[-\int_0^s h_{\lambda_0}(u)du\right]ds \\
&+ \int_0^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \\
&\leq \left(1+\sum_{i\in I} e^{-\lambda_0\tau_i}\int_{-\tau_i}^{0}|\tilde{b}_i(s)|ds + \int_0^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du\right) \\
&\times R(\lambda_0;\phi) \\
&= \left(1+\sum_{i\in I}|B(i)|\tau_i e^{-\lambda_0\tau_i} + \int_0^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du\right) \\
&\times R(\lambda_0;\phi) \\
" &= (1+\mu(\lambda_0))R(\lambda_0;\phi).
\end{aligned}
$$

Furthermore, using the definition of $W(\lambda_0;\phi)$ we have

$$
\begin{aligned}
W(\lambda_0;\phi) &\leq \max\left\{1\ ,\ R(\lambda_0;\phi)+\frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)}\right\} = R(\lambda_0;\phi)+\frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} \\
&\leq R(\lambda_0;\phi)+\frac{(1+\mu(\lambda_0))R(\lambda_0;\phi)}{1+\beta(\lambda_0)} = \left(1+\frac{(1+\mu(\lambda_0))}{1+\beta(\lambda_0)}\right)R(\lambda_0;\phi).
\end{aligned}
$$

So, using (3.19) and (3.21), by (3.22) we reach for $t \geq 0$

$$
\begin{aligned}
|y(t)| &\leq \mu(\lambda_0)\left(1+\frac{(1+\mu(\lambda_0))}{1+\beta(\lambda_0)}\right)R(\lambda_0;\phi)+\frac{(1+\mu(\lambda_0))R(\lambda_0;\phi)}{1+\beta(\lambda_0)} \\
&= \left\{\mu(\lambda_0)\left(1+\frac{(1+\mu(\lambda_0))}{1+\beta(\lambda_0)}\right)+\frac{(1+\mu(\lambda_0))}{1+\beta(\lambda_0)}\right\}R(\lambda_0;\phi) \\
&= N(\lambda_0)R(\lambda_0;\phi).
\end{aligned}
$$

Last of all, using the definition of y we get

$$|x(t)| \leq N(\lambda_0)R(\lambda_0;\phi)\exp\left[\int_0^t h_{\lambda_0}(u)du\right], \quad \text{for all} \quad t \geq 0.$$

Therefore, this completes the proof of the theorem.

□

Example 3.4. In the following example, we will apply Theorem 3.1 and Theorem 3.3. For simplicity of example we consider the problem as follows:

$$x'(t) = \left(\frac{1}{3}+\sin 2\pi t\right)x(t) - \left(\frac{1}{3}+\sin 2\pi t\right)x(t-1) - \frac{e^{-t}}{3}, \quad t \geq 0, \tag{3.23}$$

$$x(t) = 1\ , \quad -1 \leq t \leq 0 \tag{3.24}$$

where $\frac{1}{3}+\sin 2\pi t$ and $-\frac{1}{3}-\sin 2\pi t$ with period $T = 1$. The characteristic equation of the homogeneous equation of (3.23) is from (1.4)

$$\lambda = \frac{1}{3} - \frac{1}{3}e^{-\lambda}. \tag{3.25}$$

We have $\lambda_1 \approx -1.9$ and $\lambda_2 = 0$ are real roots of characteristic equation (3.25). Let $\lambda_0 \approx -1.9$. Then, the first term in (3.1) $\frac{e^{1.9}}{3} \approx 2.23$. Therefore, Theorem 3.1 and

Theorem 3.3 cannot be applied to equation (3.23). But, let $\lambda_0 = 0$. We check the condition for Theorem 3.1 as follows: Since $h_{\lambda_0}(t) = 0$, from (3.1) we obtained easily

$$\mu(\lambda_0) = \mu(0) = \frac{1}{3} + \int_0^\infty \frac{e^{-u}}{3} du = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1.$$

Therefore, (3.1) is satisfied. Then, from (3.2) and (3.20), the solution x of (3.23) and (3.24) satisfies

$$\lim_{t\to\infty} x(t) = \frac{1 - \int_{-1}^{0} \left(\frac{1}{3} + \sin 2\pi s\right) ds - \int_0^\infty \frac{e^{-t}}{3} du}{1 - \frac{1}{3}} = \frac{3}{2}$$

and

$$|x(t)| \leq \left(\frac{(1 + 2/3)^2}{1 - \frac{1}{3}} + \frac{2}{3} \right) = \frac{29}{6}, \quad \text{for all} \quad t \geq 0.$$

4. The Special Case of Linear Nonhomogeneous Delay Differential Equations with Constant Coefficients

In this section, we will consider the special case of first order linear nonhomogeneous delay differential equations with constant coefficients and constant delays. The linear autonomous delay differential equation is a special version of the delay differential equation (1.1)

$$x'(t) = ax(t) + \sum_{i\in I} b_i x(t - \tau_i) + f(t), \quad t \geq 0, \tag{4.1}$$

where a, b_i for $i \in I$ are the real constants, and τ_i for $i \in I$ the positive real numbers with $\tau_{i_1} \neq \tau_{i_2}$ for i_1, i_2 with $i_1 \neq i_2$ and f is a continuous real-valued function on the interval $[0, \infty)$. Let τ be defined by $\tau = \max_{i\in I} \tau_i$. and the initial function be given as in (1.2). The characteristic equation of the homogeneous equation of (4.1) is

$$\lambda = a + \sum_{i\in I} b_i e^{-\lambda\tau_i}. \tag{4.2}$$

There were given sufficient conditions to obtain a unique real root of characteristic equation (4.2) in Philos [2, Chapter 5]. The constant coefficients a and b_i of (4.1) can be considered as T -periodic functions, for each real number $T > 0$. Moreover, as it concerns the autonomous delay differential equation (4.1), the hypothesis that there exists positive integers m_i for $i \in I$ such that $\tau_i = m_i T$ holds by itself. After these observations, it is not difficult to apply the main results of this paper, i.e., Theorem 3.1, Corollary 3.2 and Theorem 3.3, to the special case of the autonomous linear nonhomogeneous delay differential equation (4.1). Because of equation (4.1) is a constant coefficient equation, we needn't to prove below Theorem 4.1 and Theorem 4.3.

Theorem 4.1. *Suppose that λ_0 be a real root of* (4.2) *with*

$$\mu(\lambda_0) = \sum_{i\in I} |b_i| \tau_i e^{-\lambda_0\tau_i} + \int_0^\infty |f(u)|\, e^{-\lambda_0 u} du < 1. \tag{4.3}$$

Thus the solution x of the system (4.1) *and* (1.2) *satisfies*

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right]=\frac{L(\lambda_0;\phi)}{1+\sum_{i\in I}b_i\tau_i e^{-\lambda_0\tau_i}},$$

where

$$L(\lambda_0;\phi)=\phi(0)+\sum_{i\in I}b_i e^{-\lambda_0\tau_i}\int_{-\tau_i}^{0}\phi(s)e^{-\lambda_0 s}ds+\int_0^\infty f(u)e^{-\lambda_0 u}du.$$

Note: It is guaranteed by the property (4.3) that $0<1+\sum_{i\in I}b_i\tau_i e^{-\lambda_0\tau_i}<2$.

Application of the Theorem 4.1 with $\lambda_0=0$ leads to the following corollary.

Corollary 4.2. *Assume that*

$$a+\sum_{i\in I}b_i=0 \quad \textit{and} \quad \sum_{i\in I}|b_i|\tau_i+\int_0^\infty f(u)du<1. \tag{4.4}$$

The solution x of the system (4.1) *and* (1.3) *satisfies*

$$\lim_{t\to\infty}x(t)=\frac{\phi(0)+\sum_{i\in I}b_i\int_{-\tau_i}^{0}\phi(s)ds+\int_0^\infty f(u)du}{1+\sum_{i\in I}b_i\tau_i}.$$

Theorem 4.3. *Assume that Theorem 4.1 is satisfied and Let λ_0 be a real root of* (4.2) *satisfying* (4.3) *and set*

$$R(\lambda_0;\phi)=\max\left\{1\,,\ \max_{-\tau\le t\le 0}|\phi(t)|\,,\ \max_{-\tau\le t\le 0}\left[e^{-\lambda_0 t}|\phi(t)|\right]\right\}.$$

Thus the solution x of the system (4.1) *and* (1.3) *satisfies*

$$|x(t)|\le N(\lambda_0)R(\lambda_0;\phi)e^{\lambda_0 t} \quad \textit{for} \quad t\ge 0,$$

where

$$N(\lambda_0)=\frac{(1+\mu(\lambda_0))^2}{1+\sum_{i\in I}b_i\tau_i e^{-\lambda_0\tau_i}}+\mu(\lambda_0).$$

5. Conclusions

In this study, firstly, we have obtained sufficient conditions for (1.1) to have periodic solutions. Later, we have proved that there is a basic asymptotic criterion for the solutions of the initial value problem (1.1)-(1.2). Finally, using this asymptotic criterion, we obtained a useful exponential boundary for solutions of (1.1)-(1.2). These results were obtained using a suitable real root for the characteristic equation. Namely that, this real root played an important role in establishing the results of the article. We have also presented the application in the special case of constant coefficients of the results obtained. We also gave two examples.

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