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ON THE $\lambda_h^\alpha-$STATISTICAL CONVERGENCE OF THE FUNCTIONS DEFINED ON THE TIME SCALE

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ABSTRACT. In this paper, we have introduced the concepts λ_h^α-density of a subset of the time scale $\mathbb{T}$ and λ_h^α-statistical convergence of order α $(0 < \alpha \leq 1)$ of $\Delta-$ measurable function f defined on the time scale $\mathbb{T}$ with the help of modulus function h and $\lambda = (\lambda_n)$ sequences. Later, we have discussed the connection between classical convergence, λ-statistical convergence and λ_h^α-statistical convergence. In addition, we have seen that f is strongly λ_h^α-Cesaro summable on T then f is λ_h^α-statistical convergent of order α.

1. INTRODUCTION

The concept of statistical convergence which is a generalization of classical convergence was first given by Zygmund [21] and later were introduced independently by Steinhaus [18] and Fast [4]. This concept is discussed under different names in spaces such as topological space, cone metric space, Banach space, time scale (see [10],[11],[12],[13],[15],[16],[17],[18],[19],[20],[26],[24],[25],[34],[41],[43]). Mursaleen [27] introduced the notion of λ-statistical convergence by using the sequence $\lambda = (\lambda_n)$ and then λ-statistical convergence on the time scales was introduced by Yılmaz et al[33] . The order of statistical convergence of a sequence of positive linear operators was introduced by Gadjiev and Orhan [36]. Later, Çolak [37] introduced and investigated the statistical convergence of order α $(0 < \alpha \leq 1)$ and strong p-Cesaro summability of order α of number sequences.

The time scale calculus was first introduced by Hilger in his Ph.D. thesis in 1988 (see [8],[9],[22]). In later years, the integral theory on time scales was given by Guseinov [7], and further studies were developed by Cabada-Vivero [3] and Rzezuchowski [16]. Recently, Seyyidoğlu and Tan [17] defined the density of the subset of the time scale. By using this definition, they gave $\Delta-$convergence and $\Delta-$Cauchy concepts for a real valued function defined on time scale. On the other side, the modulus function was first introduced by Nakano [14]. Aizpuru et al.[1] defined a new density concept with the help of a modulus function and obtained

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a new convergence concept between ordinary convergence and statistical convergence. Gürdal and Özgür [6] introduced ideal h-statistical convergence and ideal h-statistical Cauchy concepts in normed space using the modulus function h and ideals.

In this paper, we have aimed to define λ_h^α-statistical convergence of $\Delta-$ measurable functions of order α $(0<\alpha\leq 1)$ defined on the time scale by using modulus function h and $\lambda=(\lambda_n)$ sequences in light of works of Seyyidoğlu and Tan [17] and others [7], [2].

2. Prelimineries

The statistical convergence concept is based on the asymptotic (natural) density of a subset B in $\mathbb{N}$ (the set of positive integers) which is defined as

$$\delta(B)=\lim_{n\to\infty}\frac{|\{k\leq n : k\in B\}|}{n}, \tag{2.1}$$

where $|B|$ denotes the number of elements in B (see [29],[4],[5]). It has been generalized to α-density of a subset $B\subset\mathbb{N}$ and given the definition of $\alpha-$statistically convergence ($\alpha\in(0,1]$) by Colak [37]. The notion of λ-statistical convergence was introduced by Mursaleen [27] using the sequence $\lambda=(\lambda_n)$ which is a non-decreasing sequence of positive numbers tending to ∞ as $n\to\infty$ such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$, and $I_n=[n-\lambda_n+1,n]$. Lets denote by Λ the set of $\lambda=(\lambda_n)$ sequences. The λ- density of $B\subset\mathbb{N}$ is defined by

$$\delta_\lambda(B)=\lim_{n\to\infty}\frac{|\{k\in I_n : k\in B\}|}{\lambda_n} \tag{2.2}$$

and $\delta_\lambda(B)$ reduces to the natural density $\delta(B)$ in case of $\lambda_n=n$ for all $n\in\mathbb{N}$ (see [33]). A sequence $x=(x_n)$ is said to be λ- statistically convergent to L of order α ($\alpha\in(0,1]$) if for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{|\{k\in I_n : |x_k-L|\geq\epsilon\}|}{(\lambda_n)^\alpha}=0. \tag{2.3}$$

In this case, we write $s_{\lambda^\alpha}-limx=L$ (see [33],[27],[38],[28],[45],[46],[44]) and we denote by S_{λ^α} the set of λ^α- statistically convergent sequences of order α . If λ_n $=n$, S_{λ^α} reduces to S^α the set of statistically convergent number sequences of order α.

On the other hand, we recall that $h:[0,\infty)\to[0,\infty)$ is called modulus function, or simply modulus, if it is satisfies:

1. $h(s)=0$ *if and only if* $s=0$,
2. $h(s+p)\leq h(s)+h(p)$ *for every* $s,p\in[0,\infty)$,
3. h *is increasing*,
4. h *is continuous from the right at* 0.

A modulus may be bounded or unbounded . For instance, $h(x)=x^p$, where $0<p\leq 1$, is unbounded, but $h(x)=\frac{x}{1+x}$ is bounded (see [39], [23]).

Let h be an unbounded modulus function. The $\lambda_h^\alpha-$density of order α $(0<\alpha\leq 1)$ of a set $B\subseteq\mathbb{N}$ is defined by

$$\delta^{\lambda_h^\alpha}(B)=\lim_{n\to\infty}\frac{h(|\{n-\lambda_n+1\leq k\leq n : k\in B\}|)}{h((\lambda_n)^\alpha)} \tag{2.4}$$

whenever this limit exists.

In this study, we shall give a notion of $\lambda_h^\alpha-$statistical convergence on any time scales and its properties. Throughout this paper, we consider the time scales which are unbounded from above and have a minimum point. Lets remember some concepts.

A nonempty closed subset of $\mathbb{R}$ is called a time scale and is denoted by $\mathbb{T}$. We suppose that a time scale has the topology inherited from $\mathbb{R}$ with the standart topology. For $t \in \mathbb{T}$, we consider the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. In this definition, we take $\inf \emptyset = sup\mathbb{T}$. For $t \in \mathbb{T}$ with $a \leq b$, it is defined the interval $[a, b]$ in $\mathbb{T}$ by $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Let $\mathbb{T}$ be a time scale. Denote by $\mathcal{F}$ the family of all left-closed and right-open intervals of $\mathbb{T}$ of the form $[a, b) = \{t \in \mathbb{T} : a \leq t < b\}$ with $a, b \in \mathbb{T}$ and $a \leq b$. It is clear that the interval $[a, a)$ is an empty set, $\mathcal{F}$ is semiring of subsets of $\mathbb{T}$. Let $m : \mathcal{F} \to [0, \infty)$ be the set function on $\mathcal{F}$ that assings to each interval $[a, b)$ its lenght $b - a, m([a, b)) = b - a$. Then m is a countably additive measure on $\mathcal{F}$. We denote by μ_Δ the Caratheodory extension of the set function m associated with family $\mathcal{F}$ (for the Caratheodory extension see [17]) and is denoted by μ_Δ, the Lebesgue Δ-measure on $\mathbb{T}$, and that is a countably additive measure . In this case, it is known that if $a \in \mathbb{T} - \{max\mathbb{T}\}$, then the single point set $\{a\}$ is Δ-measurable and $\mu_\Delta(a) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$ then $\mu_\Delta(a, b)_\mathbb{T} = b - \sigma(a)$. If $a, b \in \mathbb{T} - \{max\mathbb{T}\}$, $a \leq b$; $\mu_\Delta(a, b]_\mathbb{T} = \sigma(b) - \sigma(a)$ and $\mu_\Delta[a, b]_\mathbb{T} = \sigma(b) - a$. It can be easily seen that the measure of a subset of $\mathbb{N}$ is equal to its cardinality (see [17],[32]).

Turan and Duman [30] introduced the concept of statistical convergence of Δ-measurable real-valued functions defined on time scales as follows. Suppose that Ω be a Δ-measurable subset of $\mathbb{T}$. Then, the set $\Omega(t)$ is defined by $\Omega(t) =: \{s \in [t_0, t]_\mathbb{T} : s \in \Omega\}$ for $t \in \mathbb{T}$. In this case, the density of Ω on $\mathbb{T}$ can be defined as

$$\delta_\mathbb{T}(\Omega) = \lim_{t\to\infty} \frac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0, t]_\mathbb{T})} \tag{2.5}$$

provided that the limit exists. In case of $\mathbb{T} = \mathbb{N}$, this reduces to the classical concept of asymptotic density. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta-$ measurable function. Then, f is statistically convergent to a real number L on $\mathbb{T}$ if for every $\epsilon > 0$, $\delta_\mathbb{T}(\{t \in \mathbb{T} : \ |f(t) - L| \geq \epsilon\}) = 0$. In this case, it can be written $s_\mathbb{T} - \lim\limits_{t\to\infty} f(t) = L$.

Later, the λ-statistical convergence on time scale was introduced by Yılmaz et al [33], [31]. It is said that f is $\lambda-$statistically convergent on $\mathbb{T}$ to a real number L if

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}(\{s \in [t - \lambda_t + t_0, t]_\mathbb{T} : \ |f(s) - L| \geq \epsilon\})}{\mu_{\Delta_\lambda}([t - \lambda_t + t_0, t]_\mathbb{T})} = 0 \tag{2.6}$$

for every $\epsilon > 0$. In this case, we can writes $s_\mathbb{T}^\lambda - \lim\limits_{t\to\infty} f(t) = L$. The set of all $\lambda-$statistically convergence functions on $\mathbb{T}$ will be denoted by $S_\mathbb{T}^\lambda$. Here and afterwards Δ_λ shows that Δ depends on λ.

3. Main Results

Definition 3.1. *Let Ω be a Δ_λ-measurable subset of $\mathbb{T}$, h be an unbounded modulus function and α be any real number $(0 < \alpha \leq 1)$. Then, one defines the set $\Omega(t, \lambda)$*

by $\Omega(t,\lambda) =: \{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : s \in \Omega\}$ *for* $t \in \mathbb{T}$*. In this case, the* $\lambda_h^\alpha-$*density of* Ω *on* $\mathbb{T}$ *of order* α *can be defined as*

$$\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = \lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)} \tag{3.1}$$

provided that the limit exists.

We can easily get $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = \delta_{\mathbb{T}}^{h^\alpha}(\Omega)$ if $\lambda_t = t$ and $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = \delta_{\mathbb{T}}^{\lambda^\alpha}(\Omega)$ if we take $h(x) = x$ on $\mathbb{T}$.

Definition 3.2. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a* Δ_λ*- measurable function. Then, one says that* f *is* $\lambda_h^\alpha-$ *statistically convergent to a real number* L *of order* α $(0 < \alpha \leq 1)$ *on* $\mathbb{T}$ *if*

$$\lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : \ |f(s)-L| \geq \epsilon\}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)} = 0 \tag{3.2}$$

for every $\epsilon > 0$.

In this case, one writes $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty} f(t) = L$. The set of all $\lambda_h^\alpha-$ statistically convergence functions on $\mathbb{T}$ will be denoted by $S_{\mathbb{T}}^{\lambda_h^\alpha}$.

If we take $\lambda_t = t$ in (8), we get classical statistically convergent on $\mathbb{T}$ to a real number L, for the function f which is defined by [17],[30] in (7). This shows that our results are generalizations of classical conclusions.

As will be noted that, when $\alpha = 1$, $\lambda_h^\alpha-$density of Ω on $\mathbb{T}$ of order α returns to λ_h-density. In case $h(x) = x$, $\lambda_h^\alpha-$density becomes $\lambda^\alpha-$density. If $\alpha = 1$ and $h(x) = x$, then $\lambda_h^\alpha-$density reduces to $\lambda-$density of Ω on $\mathbb{T}$.

The equality $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) + \delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\diagdown\Omega) = 1$ does not hold for α $(0 < \alpha \leq 1)$ and an unbounded modulus h, in general. For instance, if we take $h(x) = x^p$, $0 < p \leq 1$, $0 < \alpha < 1$ and $\Omega = \{2n : n \in \mathbb{N}\}$, then $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = \delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\diagdown\Omega) = \infty$. Also, finite sets have zero $\lambda_h^\alpha-$density for any unbounded modulus h and α $(0 < \alpha \leq 1)$ (see [30], [38]).

Lemma 3.1. *Let* α $(0 < \alpha \leq 1)$ *be any real number,* Ω *be a* Δ_λ*-measurable subset of* $\mathbb{T}$ *and* h *be an unbounded modulus function. If* $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = 0$*, then* $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\diagdown\Omega) \neq 0$.

Proof. Let α $(0 < \alpha \leq 1)$ be any given real number and the equality $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega) = 0$ be valid for any unbounded modulus h. Suppose that $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\diagdown\Omega) = 0$. Let us say $\Omega(t,\lambda)_{\mathbb{T}} =: \{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : s \in \Omega(t)\}$ for $t \in \mathbb{T}$ and $\mathbb{T}\diagdown\Omega(t,\lambda)_{\mathbb{T}} =: \{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : s \in \mathbb{T} \diagdown (\Omega)(t)\}$ for $t \in \mathbb{T}$. Since $\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}) = \mu_{\Delta_\lambda}(\Omega(t,\lambda)_{\mathbb{T}}) + \mu_{\Delta_\lambda}(\mathbb{T}\diagdown\Omega(t,\lambda)_{\mathbb{T}})$ for $t \in \mathbb{T}$ and h is subadditive, we have

$$h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})) \leq h(\ \mu_{\Delta_\lambda}\Omega(t,\lambda)_{\mathbb{T}}) + h(\ \mu_{\Delta_\lambda}(\mathbb{T}\diagdown\Omega(t,\lambda)_{\mathbb{T}})) \tag{3.3}$$

Hence we may write

$$\begin{aligned} & \lim_{t\to\infty} \frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)} \\ \leq \quad & \lim_{t\to\infty} \frac{h(\ \mu_{\Delta_\lambda}\Omega(t,\lambda)_{\mathbb{T}})}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)} + \lim_{t\to\infty} \frac{h(\ \mu_{\Delta_\lambda}(\mathbb{T}\diagdown\Omega(t,\lambda)_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha)}. \end{aligned} \tag{3.4}$$

Since $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega)=0$ and $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\mathbb{T}\diagdown\Omega)=0$, the right side of the inequality is zero and thus

$$\lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}=0.$$

This is a contradiction. Because $\frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}\geq 1$ for α $(0<\alpha\leq 1)$ and therefore

$$\lim_{t\to\infty}\frac{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}\geq 1. \tag{3.5}$$

□

For any unbounded modulus h and $0<\alpha\leq 1$, if $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega)=0$ then $\delta_{\mathbb{T}}^{\lambda^\alpha}(\Omega)=0$, but the inverse of this does not need to be true ([40]). Namely, a set having zero α-density for some α $(0<\alpha\leq 1)$ might have non-zero $\lambda_h^\alpha-$density for some unbounded modulus h, with the same α. Similarly a set having zero $\lambda-$ density might have non-zero $\lambda_h^\alpha-$density for some unbounded modulus h and $0<\alpha\leq 1$. For example, let $h(x)=log(x+1)$ and $\Omega=\{1,4,9,...\}$. Then $\delta^{\lambda}(\Omega)=0$ and $\delta_{\mathbb{T}}^{\lambda^\alpha}(\Omega)=0$ for $1/2<\alpha\leq 1$, but $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega)\geq\delta_{\mathbb{T}}^{\lambda_h}(\Omega)=1/2$ and therefore $\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Omega)\neq 0$.

If $\Phi\subseteq\mathbb{T}$ has zero $\lambda_h^\alpha-$density for some unbounded modulus h and for some α $(0<\alpha\leq 1)$, then it has zero $\lambda^\alpha-$density and hence zero $\lambda-$density (see [3]).

Lemma 3.2. [40] *Let h be an unbounded modulus and $\Phi\subseteq\mathbb{T}$. If $0<\alpha\leq\beta\leq 1$, then $\delta_{\mathbb{T}}^{\lambda_h^\beta}(\Phi)\leq\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Phi)$.*

Thus, for any unbounded modulus h and $0<\alpha\leq\beta\leq 1$, if Φ has zero $\lambda_h^\alpha-$density in that case, it has zero $\lambda_h^\beta-$density. Specially, a set having zero $\lambda_h^\alpha-$density for some α $(0<\alpha\leq 1)$ has zero λ_h-density. But, the inverse is not correct. For instance, let $h(x)=x^p$ for $0<p\leq 1$ and $\Phi=\{1,4,9,...\}$. Then

$$\begin{aligned}\delta_{\mathbb{T}}^{\lambda_h}(\Phi) &= \lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}\Phi(t,\lambda)_{\mathbb{T}})}{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))} && (3.6)\\ &\leq \lim_{t\to\infty}\frac{h(\lceil\sqrt{\Phi(t,\lambda)}\rceil)}{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))} && (3.7)\\ &= \lim_{t\to\infty}\frac{(\lceil\sqrt{\Phi(t,\lambda)}\rceil)^p}{(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^p}=0\end{aligned}$$

but, if we get $0<\alpha\leq 1/2$,

$$\begin{aligned}\delta_{\mathbb{T}}^{\lambda_h^\alpha}(\Phi) &= \lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}\Phi(t,\lambda)_{\mathbb{T}})}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)} && (3.8)\\ &= \lim_{t\to\infty}\frac{(\lceil\sqrt{\Phi(t,\lambda)}\rceil)^p}{((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)^p}=\infty\end{aligned}$$

where $\lceil r\rceil$ denotes the integer part of number r.

Proposition 3.3. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be a Δ_λ- measurable functions such that $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim_{t\to\infty}f(t)=L_1$ and $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim_{t\to\infty}g(t)=L_2$. Then the following statements hold:*

i) $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty}(f(t)+g(t)) = L_1 + L_2$,

ii) $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty}(cf(t)) = cL_1$.

Proof. It is easy to prove and we omit it. □

Theorem 3.4. $S_{\alpha\mathbb{T}}^{h} \subseteq S_{\mathbb{T}}^{\lambda_h^\alpha}$ *if and only if*

$$\liminf_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))}{h((\mu_{\Delta_\lambda}([t_0,t]_{\mathbb{T}})^\alpha)} > 0 \tag{3.9}$$

Proof. For given $\epsilon > 0$, we have

$$h(\mu_\Delta(\{s\in[t_0,t]_{\mathbb{T}}: \ |f(s)-L|\geq\epsilon\})) \supset h(\mu_\Delta(\{s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}: \ |f(s)-L|\geq\epsilon\})).$$

Then

$$\begin{aligned}
&\frac{h(\mu_{\Delta_\lambda}(\{s\in[t_0,t]_{\mathbb{T}}: \ |f(s)-L|\geq\epsilon\}))}{h((\mu_{\Delta_\lambda}([t_0,t]_{\mathbb{T}})^\alpha)} \\
\geq \quad &\frac{h(\mu_{\Delta_\lambda}(\{s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}: \ |f(s)-L|\geq\epsilon\}))}{h((\mu_{\Delta_\lambda}([t_0,t]_{\mathbb{T}})^\alpha)} \\
= \quad &\frac{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}\ ))}{h((\mu_{\Delta_\lambda}([t_0,t]_{\mathbb{T}})^\alpha)}\frac{1}{h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}\ ))} \\
&h(\mu_{\Delta_\lambda}(\{s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}: \ |f(s)-L|\geq\epsilon\}))
\end{aligned}$$

Hence by using (3.9) and taking the limit as $t\to\infty$, we get $s_{\mathbb{T}}^{\alpha}{}_{h} - \lim_{t\to\infty} f(s) \to L$ implies $s_{\mathbb{T}}^{\lambda_h^\alpha} - \lim_{t\to\infty} f(s) = L$. □

The definition of $p-Cesaro$ summability on time scales was given by Turan and Duman [30] as follows.

Definition 3.3. [30] *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function and $0<p<\infty$. Then, f is strongly $p-Cesaro$ summable on $\mathbb{T}$ if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{t\to\infty}\frac{1}{(\mu_\Delta([t_0,t]_{\mathbb{T}}))}\int\limits_{[t_0,t]_{\mathbb{T}}} |f(s)-L|^p\Delta s = 0. \tag{3.10}$$

The set of all $p-Cesaro$ summable functions on $\mathbb{T}$ is denoted by $[W_p]_{\mathbb{T}}$.

We need to emphasize that measure theory on time scales was first constructed by Guseinov [7] and *Lebesque* $\Delta - integral$ on time scales introduced by Cabada and Vivero [35].

Definition 3.4. *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ_λ- measurable function, $\lambda\in\Lambda$ and $0<p<\infty$. We say that f is strongly $\lambda_h^\alpha - Cesaro$ summable on $\mathbb{T}$ if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{t\to\infty}\frac{1}{(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))^\alpha}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\,\Delta s=0. \tag{3.11}$$

In this case we write $[W,\lambda_h^\alpha]_{\mathbb{T}}-lim\ f(s)=L$. The set of all strongly $\lambda_h^\alpha-$Cesaro summable functions on $\mathbb{T}$ will be denoted by $[W,\lambda_h^\alpha]_{\mathbb{T}}$. If we take $h(x)=x^p$ and $\alpha=1$ then we get $[W,\lambda_p]_{\mathbb{T}}$ the set of all strongly $\lambda_p-Cesaro$ summable functions on $\mathbb{T}$ (see [33]).

Lemma 3.5. *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ_λ- measurable function and $\Omega(t,\lambda)=\{\ s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}:h(|f(s)-L|)\geq\epsilon\ \}$ for $\epsilon>0$. In this case, we have*

$$\begin{aligned} h(\mu_{\Delta_\lambda}(\Omega(t,\lambda))) &\leq \frac{1}{\epsilon}\int\limits_{\Omega(t,\lambda)} h(|f(s)-L|)\,\Delta s &(3.12)\\ &\leq \frac{1}{\epsilon}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \ \Delta s &(3.13)\end{aligned}$$

Proof. It can be proved by using similar method with [30]. □

Theorem 3.6. *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ_λ- measurable function, $\lambda\in\Lambda$, $L\in\mathbb{R}$ and $0<p<\infty$. Then we get:*

i) $[W,\lambda_h^\alpha]_{\mathbb{T}}.\subset s_{\mathbb{T}}^{\lambda_h^\alpha}$.

ii) If f is strongly $\lambda_h^\alpha-Cesaro$ summable to L, then $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim\limits_{t\to\infty}f(t)=L$.

iii) If $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim\limits_{t\to\infty}f(t)=L$ and f is a bounded function, then f is strongly $\lambda_h^\alpha-Cesaro$ summable to L.

Proof. i) Let $\epsilon>0$ and $[W,\lambda_h^\alpha]_{\mathbb{T}}-lim\ f(s)=L$. We can write

$$\begin{aligned}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\,\Delta s &\geq \int\limits_{\Omega(t,\lambda)} h(|f(s)-L|)\ \ \Delta s &(3.14)\\ &\geq \epsilon\ \ h(\mu_{\Delta_\lambda}(\Omega(t,\lambda))). &(3.15)\end{aligned}$$

Therefore, $[W,\lambda_h^\alpha]_{\mathbb{T}}-lim\ f(s)=L$ implies $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim\limits_{t\to\infty}f(s)=L$.

ii) Let f is strongly $\lambda_h^\alpha-Cesaro$ summable to L. For given $\epsilon>0$, let $\Omega(t,\lambda)$ $=\{\ s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}:h(|f(s)-L|)\geq\epsilon\ \}$ on time scale $\mathbb{T}$. Then, it follows from lemma 9

$$\epsilon\ h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))\leq\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\,\Delta s.$$

Dividing both sides of the last equality by $h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}}))$ and taking limit as $t\to\infty$, we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)} &(3.16)\\ \leq\ &\frac{1}{\varepsilon}\lim_{t\to\infty}\frac{1}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\,\Delta s=0\end{aligned}$$

□

which yields that $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim_{t\to\infty} f(t)=L$.

iii) Let f be bounded and λ_h^α−statistically convergent to L on $\mathbb{T}$. Then, there exists a positive number M such that $|f(s)|\leq M$ for all $s\in\mathbb{T}$, and also

$$\lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}=0$$

where $\Omega(t,\lambda)=\{\ s\in[t-\lambda_t+t_0,t]_{\mathbb{T}}:h(|f(s)-L|)\geq\epsilon\ \}$ as stated before. Since

$$\begin{aligned}&\int_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \Delta s\\ =&\int_{\Omega(t,\lambda)} h(|f(s)-L|)\ \ \Delta s+\int_{[t-\lambda_t+t_0,t]_{\mathbb{T}}/\Omega(t,\lambda)} h(|f(s)-L|)\ \ \Delta s \qquad (3.17)\\ \leq&\ (h(M)+h(|L|))\int_{\Omega(t,\lambda)}\Delta s+\epsilon\int_{[t-\lambda_t+t_0,t]_{\mathbb{T}}/\Omega(t,\lambda)}\Delta s\\ =&\ (h(M)+h(|L|))\ h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))+\epsilon\ \ h(\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})),\end{aligned}$$

we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}\int_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} h(|f(s)-L|)\ \ \Delta s \qquad (3.18)\\ \leq&\ (h(M)+h(|L|))\lim_{t\to\infty}\frac{h(\mu_{\Delta_\lambda}(\Omega(t,\lambda)))}{h((\mu_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})^\alpha)}+\epsilon\end{aligned}$$

Since $\epsilon>0$ is arbitrary, the proof follows from (3.16) and (3.18).

Theorem 3.7. *Let f be a Δ_λ-measurable function. Then, $s_{\mathbb{T}}^{\lambda_h^\alpha}-\lim_{t\to\infty} f(t)=L$ if and only if there exists a Δ_λ-measurable set $\Omega\subseteq\mathbb{T}$ such that $\delta^{\lambda_h^\alpha}(\Omega)=1$ and $\lim_{t\to\infty}$ $h(|f(t)-L|)=0,\ \ (t\in\Omega(t,\lambda))$.*

Proof. It can be easily proved by using similar way in Theorem 3.9 of Turan and Duman (see, [30]). □

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