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Research Article

Some Inequalities on Half Lightlike Submanifolds of a Lorentzian Manifold with Semi-Symmetric Metric Connection

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ABSTRACT: In this paper, we introduce some inequalities for screen homothetic half lightlike submanifolds of a real space form $\tilde{M}(c)$ of constant sectional curvature c, endowed with the semi-symmetric metric connection. Using these inequalities, we derive some characterizations for such half lightlike submanifolds. Finally, Chen-Ricci inequalities are calculated. Moreover, the equality cases are considered and we get some results.

Keywords: Chen inequalities, Half lightlike submanifold, Lorentzian manifold, Semi-symmetric metric connection.

1. INTRODUCTION

Lightlike geometry has applications in mathematical physics and because of this, it is an important research field in differential geometry. Kupeli initiated the geometry of lightlike submanifolds in [1]. Then Duggal and Bejancu developed it [2]. Moreover, many authors studied it [3,4].

A semi-symmetric linear connection on a differentiable manifold is presented in [5], the geometers studied it in [6] and [7]. Then, in [8] and [9], the geometry of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection was investigated by Imai. The studies on Riemannian, semi-Riemannian, and Lorentzian manifolds with semi-symmetric metric connections belong to Nakao [10], Duggal-Sharma [11], and Konar-Biswas [12], respectively. Lightlike hypersurfaces of a semi-Riemannian manifold with a semi-symmetric metric connection were introduced in [13]. Later, Akyol, Vanlı, and Fernandez investigated the geometry of such connection on S manifolds in [14].

In addition to these, Chen introduced Chen inequalities and defined new types of curvature invariants in [15], and then, many authors worked on this topic [16-23]. Chen inequalities on submanifolds with constant and quasi-constant curvature with a semi-symmetric metric connection were studied in [24] and [25], respectively.

Firstly Chen inequalities on lightlike geometry were worked by Gülbahar, Kılıç, and Keleş in

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[26] and [27]. Then Poyraz, Doğan, and Yaşar studied Chen inequalities on the lightlike hypersurface of a Lorentzian manifold with a semi-symmetric metric connection in [28]. Some inequalities for screen conformal half lightlike submanifolds were established by Gülbahar and Kılıç in [29].

In this study, we introduce some inequalities for screen homothetic half lightlike submanifolds of a real space form $\tilde{M}(c)$ endowed with the semi-symmetric metric connection. Using these inequalities, we derive some characterizations for such half lightlike submanifolds. Finally, Chen-Ricci inequalities are calculated. Moreover, the equality cases are considered and we get some results.

2. PRELIMINARIES

Let $(\tilde{M},\tilde{g})$ be a semi-Riemannian manifold. A connection $\tilde{\nabla}$ $\tilde{M}$ is called a semi-symmetric metric connection if it is metric, i.e., $\tilde{\nabla}\tilde{g}=0$ and its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(\tilde{X},\tilde{Y})=\tilde{\pi}(\tilde{Y})\tilde{X}-\tilde{\pi}(\tilde{X})\tilde{Y} \tag{2.1}$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ of $\tilde{M}$, where $\tilde{\pi}$ is a 1-form defined by

$$\tilde{g}(\tilde{P},\tilde{X})=\tilde{\pi}(\tilde{X})$$

and $\tilde{P}$ is a vector field on $\tilde{M}$, which is called the torsion vector field.

Let $\tilde{M}$ be a semi-Riemannian manifold admits a semi-symmetric metric connection which is given by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y}=\mathring{\tilde{\nabla}}_{\tilde{X}}\tilde{Y}+\tilde{\pi}(\tilde{Y})\tilde{X}-\tilde{g}(\tilde{X},\tilde{Y})\tilde{P} \tag{2.2}$$

for any $\tilde{X},\tilde{Y}\in\Gamma(T\tilde{M})$, where $\mathring{\tilde{\nabla}}$ is the Levi-Civita connection concerning the semi-Riemannian metric $\tilde{g}$ [7].

Let $(\tilde{M},\tilde{g})$ be a $(n+3)-$dimensional semi-Riemannian manifold of the index $q\geq 1$ and M be a lightlike submanifold of codimension 2 of $\tilde{M}$. Then the radical distribution $Rad(TM)=TM\cap TM^{\perp}$ M is a vector subbundle of TM and the $TM^{\perp}$ of rank 1 or 2. If $rank(Rad(TM))=1$, then M is called half lightlike submanifold of $\tilde{M}$. Then there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, which are named the screen and the screen transversal distribution on M, respectively. Hence we derive

$$TM=Rad(TM)\perp S(TM),\ TM^{\perp}=Rad(TM)\perp S(TM^{\perp}). \tag{2.3}$$

Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\tilde{M}$. Then ξ and L belong to $\Gamma(S(TM)^{\perp})$. Hence we obtain

$$S(TM)^{\perp}=S(TM^{\perp})\perp S(TM^{\perp})^{\perp}, \tag{2.4}$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section $\xi\in Rad(TM)$ on a coordinate neighborhood $U\subset M$, there exists a uniquely determined null vector field $N\in\Gamma(ltr(TM))$ holding

$$\tilde{g}(N,\xi)=1,\quad \tilde{g}(N,N)=\tilde{g}(N,X)=\tilde{g}(N,L)=0,\ \forall X\in\Gamma(TM). \tag{2.5}$$

We say $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \perp ltr(TM)$ the lightlike transversal vector bundle and transversal vector bundle M for $S(TM)$, respectively. Hence we have

$$\begin{aligned} T\tilde{M} &= TM \oplus tr(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \end{aligned} \tag{2.6}$$

Using (2.6) we define the projection morphism $Q : \Gamma(TM) \to \Gamma(S(TM))$. Hence we have

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \tag{2.7}$$

$$\tilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.8}$$

$$\tilde{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.9}$$

$$\tilde{\nabla}_X L = -A_L X + \psi(X)N, \tag{2.10}$$

$$\nabla_X QY = \nabla^*_X QY + C(X,QY)\xi, \tag{2.11}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{2.12}$$

for any $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$, $U \in \Gamma(tr(TM))$, $N \in \Gamma(ltr(TM))$ and $L \in \Gamma(S(TM^{\perp}))$.

Then ∇ and ∇^* are named induced linear connections on TM and $S(TM)$ respectively, B and D are named the local second fundamental forms of M, C is named the local second fundamental form on $S(TM)$. A_N, A^*_{ξ} and A_L are named linear operators on TM. Also τ, ρ and ψ are named $1-$forms on TM.

This ∇ is not a metric connection and holds

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.13}$$

for any $X,Y,Z \in \Gamma(TM)$, where η is a $1-$form defined by

$$\eta(X) = \tilde{g}(X,N), \ \forall X \in \Gamma(TM). \tag{2.14}$$

But ∇^* is a metric connection. Using (2.1) and (2.13), we see that

$$T(X,Y) = \pi(Y)X - \pi(X)Y \tag{2.15}$$

and B and D are symmetric, where T is the torsion tensor for ∇. From (2.13) and (2.15), we see that ∇ is a semi-symmetric non-metric connection of M. Moreover, B and D are independent of the choice of $S(TM)$ and hold

$$B(X,\xi) = 0, \ D(X,\xi) = -\varepsilon\psi(X), \ \forall X \in \Gamma(TM). \tag{2.16}$$

Therefore one obtains

$$B(X,Y) = g(A^*_{\xi}X,Y), \quad g(A^*_{\xi}X,N) = 0, \tag{2.17}$$

$$C(X,QY) = g(A_N X,QY), \quad g(A_N X,N) = 0, \tag{2.18}$$

$$D(X,QY) = g(A_L X,QY), \quad g(A_L X,N) = \rho(X), \tag{2.19}$$

$$D(X,Y) = g(A_L X,Y) - \psi(X)\eta(Y), \quad \forall X,Y \in \Gamma(TM). \tag{2.20}$$

By (2.17) and (2.18), A^*_{ξ} and A_N are $\Gamma(S(TM))-$valued shape operators related to B and E, respectively and $A^*_{\xi}\xi = 0.$

Using (2.7), (2.12), and (2.16), one derives

$$\tilde{\nabla}_X \xi = -A^*_{\xi} X - \tau(X)\xi - \varepsilon\psi(X)L, \tag{2.21}$$

for any $X \in \Gamma(TM)$.

Definition 1. *A half lightlike submanifold* (M, g) *of a semi-Riemannian manifold* $(\tilde{M}, \tilde{g})$ *is named irrotational [1] if* $\tilde{\nabla}_X \xi \in \Gamma(TM)$ *for any* $X \in \Gamma(TM)$*. From (2.16) and (2.21), the definition of irrotational is equivalent to the condition* $\psi(X) = 0$*, that is,* $D(X, \xi) = 0$ *for any* $X \in \Gamma(TM)$.

Definition 2. *A half lightlike submanifold* (M, g) *of a semi-Riemannian manifold* $(\tilde{M}, \tilde{g})$ *is called umbilical in* $\tilde{M}$ *if there is a smooth vector field* $H \in \Gamma(tr(TM))$ *on any coordinate neighborhood* U *such that*

$$h(X,Y) = Hg(X,Y) \tag{2.22}$$

for any $X, Y \in \Gamma(TM)$*, where*

$$h(X,Y) = B(X,Y)N + D(X,Y)L \tag{2.23}$$

is the global second fundamental form tensor of M *. In the case of* $h = 0$ *on* U *, we call that* M *is totally geodesic [30].*

Besides, M is totally umbilical iff, on each coordinate neighborhood U, there exist smooth vector functions λ and δ such that

$$B(X,Y) = \lambda g(X,Y), D_2(X,Y) = \delta g(X,Y), \tag{2.24}$$

for any $X, Y \in \Gamma(TM)$.

Definition 3. *[30] The screen distribution* $S(TM)$ *of* M *is named totally umbilical if there is a smooth function* γ *on any coordinate neighborhood* $U \subset M$ *such that*

$$E(X, QY) = \gamma g(X,Y), \tag{2.25}$$

for any $X, Y \in \Gamma(TM)$*. If* $\gamma = 0$ *on* U*, then we say that* $S(TM)$ *is totally geodesic in* M *.*

Furthermore, $(M, g, S(TM))$ is named minimal if $\psi(X) = 0$ and

$$trace_{S(TM)} h = 0, \tag{2.26}$$

where $trace_{S(TM)}$ denotes the trace restricted to $S(TM)$ concerning the degenerate metric g [31].

Let $(M, g, S(TM))$ be a $(n+1)-$dimensional half-lightlike submanifold and $\{e_1, ..., e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. Let us consider

$$\mu_1 = \frac{1}{n}\sum_{j=1}^{n} B(e_j, e_j),\ \mu_2 = \frac{1}{n}\sum_{j=1}^{n} D(e_j, e_j). \tag{2.27}$$

Then it is clear from (2.26) and (2.27) that M is minimal iff $\mu_1 = \mu_2 = 0$.

A lightlike hypersurface (M, g) of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is called *screen locally conformal* if the shape operators A_N and A^*_ξ of M and $S(TM)$, respectively, are related by

$$A_N = \phi A^*_\xi, \tag{2.28}$$

i.e.,

$$C(X, PY) = \phi B(X,Y), \forall X, Y \in \Gamma(TM), \tag{2.29}$$

where ϕ is a non-vanishing smooth function on a neighborhood U in M . If ϕ is a non-zero constant, M is named screen homothetic [32].

We denote $\tilde{R}$, R and R^* the curvature tensors of the semi-symmetric metric connection of $\tilde{\nabla}$, ∇ and ∇^* , respectively.

Using (2.7)-(2.12) for M and $S(TM)$, we derive:

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,QW) &= g(R(X,Y)Z,QW)\\ &+B(X,Z)C(Y,QW)-B(Y,Z)C(X,QW)\\ &+D(X,Z)D(Y,QW)-D(Y,Z)D(X,QW),\end{aligned} \tag{2.30}$$

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,\xi) &= (\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z)\\ &+[\tau(X)-\pi(X)]B(Y,Z)-[\tau(Y)-\pi(Y)]B(X,Z)\\ &+\psi(X)D(Y,Z)-\psi(Y)D(X,Z),\end{aligned} \tag{2.31}$$

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,N) &= g(R(X,Y)Z,N)\\ &+\rho(Y)D(X,Z)-\rho(X)D(Y,Z),\end{aligned} \tag{2.32}$$

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)\xi,N) &= g(A_\xi^* X, A_N Y)-g(A_\xi^* Y, A_N X)\\ &-2d\tau(X,Y)+\rho(X)\psi(Y)-\rho(Y)\psi(X),\end{aligned} \tag{2.33}$$

$$\begin{aligned}g(R(X,Y)QZ,QW) &= g(R^*(X,Y)Z,QW)+B(Y,QW)C(X,QZ)\\ &-B(X,QW)C(Y,QZ),\end{aligned} \tag{2.34}$$

$$\begin{aligned}\tilde{g}(R(X,Y)QZ,N) &= (\nabla_X C)(Y,QZ)-(\nabla_Y C)(X,QZ)\\ &+[\tau(Y)+\pi(Y)]C(X,QZ)-[\tau(X)+\pi(X)]C(Y,QZ),\end{aligned} \tag{2.35}$$

for any $X,Y,Z\in\Gamma(TM)$ [33].

Now let us choose a $2-$dimensional non-degenerate plane section

$$\Pi = Span\{X,Y\}, \tag{2.36}$$

in T_pM, $p\in M$. Then the sectional curvature at p is expressed by [34]

$$K(\Pi)=\frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y)-g(X,Y)^2}. \tag{2.37}$$

Let $p\in M$ and ξ be the null vector of T_pM. A plane Π of T_pM is said to be null plane if it contains ξ and e_i such that $g(\xi,e_i)=0$ and $g(e_i,e_i)=\varepsilon_i=\pm1$. The null sectional curvature of Π is defined by

$$K_i^{null}=\frac{g(R_p(e_i,\xi)\xi,e_i)}{g_p(e_i,e_i)}.$$

The Ricci tensor $\tilde{R}ic$ of $\tilde{M}$ and the induced Ricci type tensor $R^{(0,2)}$ of M are given by

$$\begin{aligned}\tilde{R}ic(X,Y) &= trace\{Z\to\tilde{R}(Z,X)Y\},\ \ \forall X,Y\in\Gamma(T\tilde{M}),\\ R^{(0,2)}(X,Y) &= trace\{Z\to R(Z,X)Y\},\ \ \forall X,Y\in\Gamma(TM),\end{aligned} \tag{2.38}$$

where

$$R^{(0,2)}(X,Y)=\sum_{i=1}^{n}\varepsilon_i g(R(e_i,X)Y,e_i)+\tilde{g}(R(\xi,X)Y,N) \tag{2.39}$$

for the quasi-orthonormal frame $\{e_1,...,e_n,\xi\}$ of T_pM. From the equations (2.30)-(2.33), it can be shown that the Ricci type tensor doesn't need to be symmetric as the sectional curvature map. This tensor is called Ricci tensor if it is symmetric.

One defines scalar curvature τ by

$$\tau(p)=\sum_{i,j=1}^{n}K_{ij}+\sum_{i=1}^{n}K_i^{null}+K_{iN}, \tag{2.40}$$

where $K_{iN} = \tilde{g}(R(\xi,e_i)e_i,N)$ for $i \in \{1,...,n\}$.

3. CHEN RICCI INEQUALITIES

Let M be a $(n+1)-$dimensional half lightlike submanifold of a $(n+3)-$dimensional of a Lorentzian manifold $\tilde{M}$ with a semi-symmetric metric connection and $\{e_1,....,e_n,\xi\}$ be a basis of $\Gamma(TM)$ where $\{e_1,....,e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. For $k \leq n$, we establish $\pi_{k,\xi} = sp\{e_1,....,e_k,\xi\}$ is a $(k+1)-$dimensional degenerate plane section and $\pi_k = sp\{e_1,....,e_k\}$ is $k-$dimensional non-degenerate plane section. The $k-$degenerate Ricci curvature and the $k-$ Ricci curvature are defined by

$$Ric_{\pi_{k,\xi}}(X) = R^{(0,2)}(X,X) = \sum_{j=1}^{k} g(R(e_j,X)X,e_j) + \widehat{g}(R(\xi,X)X,N), \tag{3.1}$$

$$Ric_{\pi_k}(X) = R^{(0,2)}(X,X) = \sum_{j=1}^{k} g(R(e_j,X)X,e_j), \tag{3.2}$$

respectively for a unit vector $X \in \Gamma(TM)$. Also, $k-$degenerate scalar curvature and $k-$scalar curvature at $p \in M$ are given by

$$\tau_{\pi_{k,\xi}}(p) = \sum_{i,j=1}^{k} K_{ij} + \sum_{i=1}^{k} K_i^{null} + K_{iN}, \tag{3.3}$$

$$\tau_{\pi_k}(p) = \sum_{i,j=1}^{k} K_{ij}, \tag{3.4}$$

respectively. For $k = n, \quad \pi_n = sp\{e_1,....,e_n\} = \Gamma(S(TM)),$ we have the screen Ricci curvature and the screen scalar curvature given by

$$Ric_{S(TM)}(e_1) = Ric_{\pi_n}(e_1) = \sum_{j=1}^{n} K_{1j} = K_{12} + ... + K_{1n}, \tag{3.5}$$

and

$$\tau_{S(TM)} = \sum_{i,j=1}^{n} K_{ij}, \tag{3.6}$$

respectively.

Let $\tilde{M}(c)$ be a real space form of constant sectional curvature c endowed with a semi-symmetric metric connection $\tilde{\nabla}$. The curvature tensor $\mathring{\tilde{R}}$ to the Levi-Civita connection $\mathring{\tilde{\nabla}}$ on $\tilde{M}(c)$ is defined by

$$\tilde{g}(\mathring{\tilde{R}}(X,Y)Z,QW) = c\{g(X,W)g(Y,Z) - g(Y,W)g(X,Z)\}. \tag{3.7}$$

Using (2.2), we derive

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,QW) = \tilde{g}(\mathring{\tilde{R}}(X,Y)Z,QW) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) \\ -\alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z),\end{aligned} \tag{3.8}$$

for any $X,Y,Z,W \in \Gamma(TM)$, where α is a $(0,2)$ tensor field defined by

$$\alpha(X,Y) = (\mathring{\tilde{\nabla}}_X \pi)Y - \pi(X)\pi(Y) + \frac{1}{2}\pi(Q)g(X,Y) \tag{3.9}$$

[8].

From (2.30), (3.6), (3.7), and (3.8), we can write

$$\tau_{S(TM)}(p) = n(n-1)c - 2(n-1)\lambda + \sum_{i,j=1}^{n} B_{ii}C_{jj} - B_{ij}C_{ji} + \sum_{i,j=1}^{n} D_{ii}D_{jj} - D_{ij}D_{ji}, \tag{3.10}$$

where λ is the trace of α and $B_{ij} = B(e_i, e_j)$, $C_{ij} = C(e_i, e_j)$, $D_{ij} = D(e_i, e_j)$ for $i, j \in \{1, ..., n\}$.

Let M be a screen homothetic half lightlike submanifold of a $(n+3)-$dimensional Lorentzian space form $\tilde{M}(c)$. Using (2.29) and (3.10) we get

$$\tau_{S(TM)}(p) = n(n-1)c - 2(n-1)\lambda + \phi n^2 \mu_1^2 + n^2 \mu_2^2 - \sum_{i,j=1}^{n} [\phi(B_{ij})^2 + (D_{ij})^2]. \tag{3.11}$$

Lemma 4. *[35] Let $a_1, ..., a_n$ be $n-$ real numbers and define $A = \sum_{i<j} (a_i - a_j)^2$. Then*

(1) $A \geq \frac{n}{2}(a_1 - a_2)^2$ and equality holds iff

$$\frac{1}{2}(a_1 + a_2) = a_3 = ... = a_n.$$

(2) Let k, ℓ be integers such that $1 \leq k < \ell \leq n$ and $(k, \ell) \neq (1,2)$. If $A = \frac{n}{2}(a_1 - a_2)^2 = \frac{n}{2}(a_k - a_1)^2$ then $a_1 = a_2 = ... = a_n$.

Theorem 5. *Let M be a $(n+1)-$dimensional screen homothetic half lightlike submanifold with $\phi > 0$ of a $(n+3)-$dimensional Lorentzian space form $\tilde{M}(c)$ endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then we obtain*

$$\begin{aligned} \tau_{S(TM)}(p) \leq\, & n(n-1)c - 2(n-1)\lambda + \frac{n^3}{n+1}\phi\mu_1^2 \\ & - \frac{\phi n}{2(n+1)} \sum_{i,j=1}^{n} (B_{11} - B_{22})^2 + \frac{n^3}{n+1}\mu_2^2 \\ & - \frac{n}{2(n+1)} \sum_{i,j=1}^{n} (D_{11} - D_{22})^2. \end{aligned} \tag{3.12}$$

The equality case of (3.12) satisfies at $p \in M$ iff

$$\mu_1 = \frac{n}{2}(B_{11} + B_{22})^2, \mu_2 = \frac{n}{2}(D_{11} + D_{22})^2 \tag{3.13}$$

and

$$B_{ij} = D_{ij} = 0, \ \text{for } i \neq j \in \{1, ..., n\}. \tag{3.14}$$

Proof. From the Binomial theorem, we can write

$$\begin{aligned} & (B_{11} - B_{22})^2 + ... + (B_{11} - B_{nn})^2 + (B_{22} - B_{33})^2 + ... + (B_{22} - B_{nn})^2 \\ & \quad + ... + (B_{n-1n-1} - B_{nn})^2 = n\sum_{i=1}^{n} (B_{ii})^2 - 2\sum_{i \neq j}^{n} B_{ii}B_{jj}. \end{aligned} \tag{3.15}$$

By Lemma 4 and (3.15) we obtain

$$\sum_{i=1}^{n} (B_{ii})^2 \geq \frac{1}{n}\sum_{i \neq j} B_{ii}B_{jj} + \frac{1}{2}(B_{11} - B_{22})^2. \tag{3.16}$$

We also can derive

$$\frac{2}{n}\sum_{i \neq j} B_{ii}B_{jj} = n\mu_1^2 - \frac{1}{n}\sum_{i=1}^{n} (B_{ii})^2. \tag{3.17}$$

Using (3.16) and (3.17) we get

$$\sum_{i=1}^{n}(B_{ii})^2 \geq \frac{n^2}{n+1}\mu_1^2 + \frac{n}{2(n+1)}(B_{11}-B_{22})^2. \tag{3.18}$$

Similarly, we obtain

$$\sum_{i=1}^{n}(D_{ii})^2 \geq \frac{n^2}{n+1}\mu_2^2 + \frac{n}{2(n+1)}(D_{11}-D_{22})^2. \tag{3.19}$$

Finally, using (3.18) and (3.19) in (3.11), we obtain (3.12).

Taking consideration of case (1) of Lemma 4, the equality of (3.12) holds at $p \in M$ iff (3.13) and (3.14) hold. The converse part of the theorem is straightforward.

Considering Theorem 5, we derive Corollary 6 and Corollary 7.

Corollary 6. *Let* M *be a* $(n+1)-$*dimensional screen homothetic half lightlike submanifold with* $\phi > 0$ *of a* $(n+3)-$*dimensional Lorentzian space form* $\tilde{M}(c)$ *endowed with a semi-symmetric metric connection* $\tilde{\nabla}$. *Then we have*

$$\begin{aligned}\tau_{S(TM)}(p) \leq{}& n(n-1)c - 2(n-1)\lambda + \frac{n^3}{n+1}\phi\mu_1^2 - \frac{\phi n}{2(n+1)}\sum_{i,j=1}^{n}(B_{ii}-B_{jj})^2 \\ &+ \frac{n^3}{n+1}\mu_2^2 - \frac{n}{2(n+1)}\sum_{i,j=1}^{n}(D_{ii}-D_{jj})^2.\end{aligned} \tag{3.20}$$

The equality case of (3.20) satisfies for all $p \in M$ *iff* $S(TM)$ *is a totally umbilical in* M.

Corollary 7. *Let* M *be a* $(n+1)-$*dimensional irrotational screen homothetic half lightlike submanifold with* $\phi > 0$ *of a* $(n+3)-$*dimensional Lorentzian space form* $\tilde{M}(c)$ *endowed with a semi-symmetric metric connection* $\tilde{\nabla}$. *Then we have*

$$\begin{aligned}\tau_{S(TM)}(p) \leq{}& n(n-1)c - 2(n-1)\lambda + \frac{n^3}{n+1}\phi\mu_1^2 \\ &- \frac{\phi n}{2(n+1)}\sum_{i,j=1}^{n}(B_{ii}-B_{jj})^2 + \frac{n^3}{n+1}\mu_2^2 - \frac{n}{2(n+1)}\sum_{i,j=1}^{n}(D_{ii}-D_{jj})^2.\end{aligned} \tag{3.21}$$

The equality case of (3.21) holds for all $p \in M$ *iff* M *is totally umbilical.*

Lemma 8. *[15] If* $n \geq 2$ *and* $a_1, ..., a_n \in \mathbb{R}$ *are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + a\right),$$

then

$$2a_1 a_2 \geq a$$

with equality holding iff

$$a_1 + a_2 = a_3 = ... = a_n.$$

Theorem 9. *Let* M *be a* $(n+1)-$*dimensional screen homothetic half lightlike submanifold with* $\phi > 0$ *of a* $(n+3)-$*dimensional Lorentzian space form* $\tilde{M}(c)$ *endowed with a semi-symmetric metric connection* $\tilde{\nabla}$ *such that the vector field* P *is tangent to* M *and* $\Pi = Span\{e_1, e_2\}$ *be a* $2-$*dimensional non-degenerate plane section of* T_pM, $p \in M$. *Then we have*

$$\tau_{S(TM)}(p)-\tau(\Pi)\leq(n-2)\left(\frac{\phi n^2}{n-1}\mu_1^2+(n+1)c-2\lambda\right)$$
$$-2trace(\lambda|_{\Pi^\perp})+B(\Pi^\perp)^2+^2 n^2\mu_2^2(\Pi^\perp) \quad (3.22)$$
$$+\sum_{j>2}^{n}D_{11}D_{jj}+D_{22}D_{jj}$$

where

$$B(\Pi^\perp)^2=\sum_{i=3}^{n}(B_{ii})^2 \text{ and } n^2\mu_2^2(\Pi^\perp)=\sum_{i,j=3}^{n}D_{ii}D_{jj}. \quad (3.23)$$

The equality case of (3.22) holds for all $p\in M$ *if* $\mu_1=0$, $\mu_2=D_{11}+D_{22}$ *and the shape operator* A_ξ^* *of* M

$$A_\xi^*=\begin{pmatrix} B_{11} & B_{12} & \ldots & 0\\ B_{21} & B_{22} & \ldots & 0\\ & & & \\ 0 & 0 & \ldots & 0\end{pmatrix}. \quad (3.24)$$

Proof. If

$$\delta=\tau_{S(TM)}(p)-n(n-1)c+2(n-1)\lambda-\phi n^2\frac{(n-2)}{n-1}\mu_1^2$$
$$-n^2\mu_2^2+\sum_{i,j=1}^{n}(D_{ij})^2 \quad (3.25)$$

is said, we derive

$$\delta=\phi\frac{n^2}{n-1}\mu_1^2-\phi\sum_{i,j=1}^{n}(B_{ij})^2. \quad (3.26)$$

Thus, we can derive

$$\left(\sum_{i=1}^{n}B_{ii}\right)^2=(n-1)\left(\frac{\delta}{\phi}+\sum_{i=1}^{n}(B_{ii})^2+\sum_{i\neq j=1}^{n}(B_{ij})^2\right). \quad (3.27)$$

From Lemma 8, we obtain

$$2B_{11}B_{22}\geq\frac{\delta}{\phi}+\sum_{i\neq j=1}^{n}(B_{ij})^2. \quad (3.28)$$

Let choose $\Pi=Sp\{e_1,e_2\}$. Then, we get

$$\tau(\Pi)=2c-2(\alpha_{11}+\alpha_{22})+\phi\sum_{i,j=1}^{2}B_{ii}B_{jj}-(B_{ij})^2+\sum_{i,j=1}^{2}D_{ii}D_{jj}-(D_{ij})^2$$
$$\geq 2c-2(\alpha_{11}+\alpha_{22})+\delta+\phi\sum_{i\neq j=1}^{n}(B_{ij})^2-\phi\sum_{i\neq j=1}^{2}(B_{ij})^2$$
$$+\sum_{i,j=1}^{2}D_{ii}D_{jj}-(D_{ij}{}^2)$$
$$=2c-2(\alpha_{11}+\alpha_{22})+\delta+\phi\sum_{i,j=1}^{n}(B_{ij})^2-\phi\sum_{i=1}^{n}(B_{ii})^2 \quad (3.29)$$
$$-\phi\sum_{i,j=1}^{2}(B_{ij})^2-\phi\sum_{i=1}^{2}(B_{ii})^2+\sum_{i,j=1}^{2}D_{ii}D_{jj}-(D_{ij}{}^2)$$
$$\geq 2c-2(\alpha_{11}+\alpha_{22})+\delta-\phi\sum_{i=3}^{n}(B_{ii})^2+2D_{11}D_{22}-\sum_{i\neq j}^{2}(D_{ij})^2$$

We remark that

$$\alpha(e_1,e_1)+\alpha(e_2,e_2)=\lambda-trace(\lambda|_{\Pi^\perp}). \tag{3.30}$$

Using (3.26), (3.29), and (3.30) we derive

$$\begin{aligned}\tau(\Pi)\geq&-(n-2)(n+1)c+2(n-1)\lambda-2(\lambda-trace(\lambda|_{\Pi^\perp}))+\tau_{S(TM)}(p)\\&-\phi n^2\frac{(n-2)}{(n-1)}\mu_1^2-\sum_{i,j=3}^{n}D_{ii}D_{jj}-\sum_{j>2}^{n}(D_{11}D_{jj}+D_{22}D_{jj})\\&+\sum_{i,j=3}^{n}(D_{ij})^2-\phi\sum_{i=3}^{n}(B_{ii})^2.\end{aligned} \tag{3.31}$$

Thus we obtain (3.22).

The equality case of (3.22) holds for all $p\in M$ iff for all $i,j\in\{3,...,n\}$

$B_{ij}=D_{ij}=0,$

$B_{11}+B_{22}=B_{33}=0.$

Thus the proof is completed.

For the sectional curvature of screen conformal half lightlike submanifold is symmetric, the screen scalar curvature $r_{S(TM)}$ can be denoted by

$$r_{S(TM)}(p)=\sum_{1\leq i<j\leq n}K_{ij}=\frac{1}{2}\sum_{i,j=1}^{n}K_{ij}=\frac{1}{2}\tau_{S(TM)}(p). \tag{3.32}$$

By using (3.32), the equality (3.11) can be rewritten as follows:

$$\begin{aligned}2r_{S(TM)}(p)=&\,n(n-1)c-2(n-1)\lambda+\phi n^2\mu_1^2+n^2\mu_2^2\\&-\sum_{i,j=1}^{n}[\phi(B_{ij})^2+(D_{ij})^2].\end{aligned} \tag{3.33}$$

Lemma 10. *[36] Let* $a_1,a_2,....,a_n$, *be* n*-real number* $(n>1)$, *then*

$$\frac{1}{n}(\sum_{i=1}^{n}a_i)^2\leq\sum_{i=1}^{n}a_i^2$$

with equality if $a_1=a_2=....=a_n$.

Theorem 11. *Let* M *be a* $(n+1)-$*dimensional screen conformal half lightlike submanifold with* $\phi>0$ *of a* $(n+3)-$*dimensional Lorentzian space form* $\tilde{M}(c)$ *endowed with a semi-symmetric metric connection* $\tilde{\nabla}$. *Then we get*

$$2r_{S(TM)}(p)\leq n(n-1)(c+\phi\mu_1^2+\mu_2^2)-2(n-1)\lambda. \tag{3.34}$$

The equality of (3.34) satisfies for all $p\in M$ *iff* $S(TM)$ *is a totally umbilical in* M.

Proof. From (3.33), we have

$$\begin{aligned}2r_{S(TM)}(p)=&\,n(n-1)c-2(n-1)\lambda+\phi n^2\mu_1^2-\phi\sum_{i=1}^{n}(B_{ii})^2\\&-\phi\sum_{i\neq j=1}^{n}(B_{ij})^2+n^2\mu_2^2+\sum_{i=1}^{n}(D_{ii})^2+\sum_{i\neq j=1}^{n}(D_{ij})^2.\end{aligned} \tag{3.35}$$

Using Lemma 10 in (3.35) we find (3.34).

The equality of (3.34) holds for all $p\in M$ iff

$B_{11}=...=B_{nn},\ B_{ij}=0,$

$D_{11}=...=D_{nn},\ D_{ij}=0,$ for $i\neq j\in\{1,...,n\}$.

Thus $S(TM)$ is a totally umbilical in M.

The following corollary is obtained from the previous theorem.

Corollary 12. *Let M be a $(n+1)-$dimensional irrational screen conformal half lightlike submanifold with $\phi>0$ of a $(n+3)-$dimensional Lorentzian space form $\tilde{M}(c)$ endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then we have*

$$2r_{S(TM)}(p)\leq n(n-1)(c+\phi\mu_1^2+\mu_2^2)-2(n-1)\lambda. \tag{3.36}$$

The equality of (3.36) holds for all $p\in M$ iff M is totally umbilical.

One obtains the following equation from the Binomial theorem:

$$\begin{aligned}\sum_{i,j=1}^{n}(B_{ij})^2 &= \frac{1}{2}n^2\mu_1^2+\frac{1}{2}(B_{11}-B_{22}-...-B_{nn})^2\\ &+\sum_{j=2}^{n}(B_{1j})^2-\sum_{2\leq i<j\leq n}^{m}(B_{ii}B_{jj}-(B_{ij})^2).\end{aligned} \tag{3.37}$$

Theorem 13. *Let M be a $(n+1)-$dimensional screen homothetic half lightlike submanifold with $\phi>0$ of a $(n+3)-$dimensional Lorentzian space form $\tilde{M}(c)$ endowed with a semi-symmetric metric connection $\tilde{\nabla}$ such that the vector field P is tangent to M. Then, the followings are true.*

(i) *For* $X\in S^1(TM)=\{X\in S(TM):\ \langle X,X\rangle=1\}$

$$Ric_{S(TM)}(X)\leq\frac{1}{4}(\phi n^2\mu_1^2+n^2\mu_2^2)+(n-1)c-\lambda+(n-2)\alpha(X,X). \tag{3.38}$$

(ii) *The equality case of (3.38) is held by $X\in S^1(TM)$ iff*

$$\begin{aligned}&B(X,Y)=D(X,Y)=0,\ \text{ for all } Y\in T_p(M) \text{ orthogonal to } X,\\ &B(X,X)=\frac{n}{2}\mu_1,\ D(X,X)=\frac{n}{2}\mu_2\end{aligned} \tag{3.39}$$

(iii) *The equality case of (3.38) holds for all $X\in S^1(TM)$ if either $S(TM)$ is totally geodesic in M or $n=2$ and $S(TM)$ is totally umbilical in M.*

Proof. From (3.33) and (3.37), we get

$$\begin{aligned}\frac{1}{4}(\phi n^2\mu_1^2+n^2\mu_2^2) &= r_{S(TM)}(p)-\frac{n(n-1)}{2}c+(n-1)\lambda\\ &+\frac{1}{4}\phi(B_{11}-B_{22}-...-B_{nn})^2+\phi\sum_{j=2}^{n}(B_{1j})^2\\ &-\phi\sum_{2\leq i<j\leq n}B_{ii}B_{jj}-(B_{ij})^2+\frac{1}{4}(D_{11}-D_{22}-...-D_{nn})^2\\ &+\sum_{j=2}^{n}(D_{1j})^2-\sum_{2\leq i<j\leq n}D_{ii}D_{jj}-(D_{ij})^2.\end{aligned} \tag{3.40}$$

Moreover,

$$\begin{aligned}&\phi\sum_{2\leq i<j\leq n}B_{ii}B_{jj}-(B_{ij})^2+\sum_{2\leq i<j\leq n}D_{ii}D_{jj}-(D_{ij})^2\\ &=\sum_{2\leq i<j\leq n}K_{ij}-\frac{(n-2)(n-1)}{2}c+(n-2)\big(\lambda-\alpha(e_1,e_1)\big)\end{aligned} \tag{3.41}$$

is obtained. Using the two last equations, we have

$$Ric_{S(TM)}(e_1) = \frac{1}{4}(\phi n^2 \mu_1^2 + n^2 \mu_2^2) + (n-1)c - \lambda - (n-2)\alpha(e_1, e_1)$$
$$-\frac{1}{4}\phi(B_{11} - B_{22} - ... - B_{nn})^2 - \phi\sum_{j=2}^{n}(B_{1j})^2 \quad (3.42)$$
$$-\frac{1}{4}(D_{11} - D_{22} - ... - D_{nn})^2 - \sum_{j=2}^{n}(D_{1j})^2.$$

Choosing $e_1 = X$ as any vector of $T_p^1(M)$ in (3.42) we get (3.38).

Equality holds in (3.38) for $X \in T_p^1(M)$ iff

$$B_{12} = B_{13} = ... = B_{1n} = 0 \text{ and } B_{11} = B_{22} + ... + B_{nn}. \quad (3.43)$$

From (3.43) we obtain

$$n\mu_1 = B_{11} + ... + B_{nn} = 2B_{11}. \quad (3.44)$$

Similarly, we get

$$n\mu_2 = D_{11} + ... + D_{nn} = 2D_{11}. \quad (3.45)$$

From (3.44) and (3.45), we obtain (3.39).

Supposing the equality case of (3.38) for all $X \in T_p^1(M)$, considering (3.43), we derive

$$B_{ij} = 0, \quad i \neq j. \quad (3.46)$$

$$2B_{ii} = B_{11} + B_{22} + ... + B_{nn}, \quad i \in \{1,...,n\}. \quad (3.47)$$

From (3.47), we have $2B_{11} = 2B_{22} = ... = 2B_{nn} = B_{11} + B_{22} + ... + B_{nn}$, which implies that

$$(n-2)\sum_{i=1}^{n} B_{ii} = 0.$$

Similarly, we get

$$(n-2)\sum_{i=1}^{n} D_{ii} = 0.$$

Thus, either $\sum_{i=1}^{n} B_{ii} = \sum_{i=1}^{n} D_{ii} = 0$ or $n = 2$. If $\sum_{i=1}^{n} B_{ii} = \sum_{i=1}^{n} D_{ii} = 0$, then from (3.47) we get

$$B_{ii} = D_{ii} = 0, \forall\, i \in \{1,...,n\}. \quad (3.48)$$

From (3.46) and (3.48), we derive $B_{ij} = D_{ij} = 0$, $\forall i, j \in \{1,...,n\}$. Hence, $S(TM)$ *is totally geodesic in* M. If $n = 2$, then from (3.47),

$$2B_{11} = 2B_{22} = B_{11} + B_{22},$$

$$2D_{11} = 2D_{22} = D_{11} + D_{22},$$

that is, $S(TM)$ *is totally umbilical in* M. The converse of proof is trivial.

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