Araştırma Makalesi | ***BAUN Fen Bil. Enst. Dergisi, 22(1), 381-393, (2020)***

DOI:10.25092/baunfbed.685159 | ***J. BAUN Inst. Sci. Technol., 22(1), 381-393, (2020)***

A study on nonsymmetric cone normed spaces

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Geliş Tarihi (Received Date): 24.09.2019
Kabul Tarihi (Accepted Date): 30.01.2020

Abstract

In the realms of theoretical computer science and approximation theory, asymetric normed spaces play an important role. In this paper, by combining asymmetric norm and cone norm, it is defined asymmetric cone normed spaces. Also, it is introduced and studied some topological concepts in asymmetric cone normed spaces.

Keywords: *Asymmetric norm, cone norm, convergence, completeness.*

Simetrik olmayan konik normlu uzaylar üzerine bir çalışma

Öz

Teorik bilgisayar bilimi ve yaklaşım teorisi alanlarında asimetrik normlu uzaylar önemli bir rol oynamaktadır. Bu çalışmada asimetrik norm ve konik norm birleştirilerek asimetrik konik normlu uzaylar tanımlanmaktadır. Ayrıca asimetrik konik normlu uzaylarda bazı topolojik kavramlar tanıtılmakta ve çalışılmaktadır.

Anahtar kelimeler: *Asimetrik norm, konik norm, yakınsaklık, tamlık.*

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1. Introduction and preliminaries

In [1], the authors introduced the cone metric spaces by means of a partial ordering on a Banach space with a cone. By defining convergence and completeness in these spaces, they prove some fixed point theorems to generalize the corresponding ones in metric spaces. Later, the authors of [2] gave some generalized topological concepts and definitions in cone metric spaces and proved that every cone metric space is a topological space.

As a generalization of a norm, a cone norm is defined in [3,4] by replacing the set of real numbers with an ordered real Banach space. A real vector space with a cone norm is called a cone normed space. Cone normed spaces play an important role in fixed point theory, computer science and some other research areas of functional analysis.

Before giving the formal definition of a cone norm, we give some basic notions and results related to the topic.

Definition 1. [5] Let P be a subset of a real Banach space E. Then P is called a cone if the following conditions hold:
P is closed, $P \neq \{\theta_E\}$ and $P \neq \emptyset$,
$\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$ and $x, y \in P \Rightarrow \alpha x + \beta y \in P$,
$x \in P$ and $-x \in P \Rightarrow x = \theta_E$, where θ_E denotes the zero vector of the real vector space E.

Example 1. [5]
1. Let $E = \mathbb{R}^n$. Then $P = \{(x_1, \ldots, x_n) \in E : x_i \geq 0 \text{ for all } i = 1, \ldots, n\}$ is a cone on E.
2. Let $E = C[a, b]$. Then $P = \{f \in E : f(x) \geq 0 \text{ for all } x \in [a, b]\}$ is a cone on E.
3. Let $E = \ell_p$ $(1 \leq p < \infty)$. Then $P = \{(x_n) \in E : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$ is a cone on E.

Let P be a cone on a real Banach space E. For any $x, y \in E$, we mean
1. $x \preceq y \Leftrightarrow y - x \in P$,
2. $x \prec y \Leftrightarrow y - x \in P, x \neq y$,
3. $x \prec\prec y \Leftrightarrow y - x \in \operatorname{Int}P$, where $\operatorname{Int}P$ denotes the interior of P.

It is easy to see that $\preceq$ defines a partial ordering on E with respect to P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \preceq x \preceq y \Rightarrow \| x \| \leq K \| y \|$, where K is called the normal constant of P. P is called minihedral cone if for any $x, y \in E$, $\sup\{x, y\}$ exists, or equivalently $\inf\{x, y\}$ exists.

Lemma 1. [2] Let P be a cone on a real Banach space E. For every $c \in E$ with $\theta_E \prec\prec c$, there exists $\delta > 0$ such that $u \prec\prec c$ whenever $u \in E$ with $\| u \| < \delta$.

Lemma 2. [2] Let P be a cone on a real Banach space E. For every $c_1, c_2 \in E$ with $\theta_E \prec\prec c_1, c_2$, there exists $c \in E$ with $\theta_E \prec\prec c$ such that $c \prec\prec c_1, c_2$.

Lemma 3. [6] Let P be a cone on a real Banach space E. If $\theta_E \preceq u \prec\prec c$ for every $c \in$

E with $\theta_E \prec\prec c$, then $u = \theta_E$.

In our proofs, we use the following two facts:

$$\lambda \in \mathbb{R}, \lambda \geq 0, x \preceq y \Rightarrow \lambda x \preceq \lambda y. \quad (1)$$

$$\lambda \in \mathbb{R}, \lambda > 0 \Rightarrow \lambda \mathrm{Int}P \subset \mathrm{Int}P. \quad (2)$$

Throughout the study, $(E, \|.\|)$ is a real Banach space and P is a cone on E with $\mathrm{Int}P \neq \emptyset$.

Definition 2. [3,4] A cone norm on a real vector space X is a mapping $\|.\|_c: X \to E$ such that

$\theta_E \preceq \| x \|_c$

$\| x \|_c = \theta_E \Leftrightarrow x = \theta_X$

$\| \alpha x \|_c = |\alpha| \, \| x \|_c$

$\| x + y \|_c \preceq \| x \|_c + \| y \|_c$

hold for all $x, y \in X$ and $\alpha \in \mathbb{R}$, where θ_X denotes the zero vector of X. The ordered pair $(X, \|.\|_c)$ is called a cone normed space.

The study of asymmetric metrics goes back to Wilson [7] and then it became a subject of intensive research in the context of topology and theoretical computer science. Following his terminology, asymmetric metric is often called quasi metric. In the realms of pure and applied mathematics and materials science, there are many applications of asymmetric metric spaces.

An asymmetric norm is a positive definite sublinear functional on a real vector space. The definition is as follows:

Definition 3. [8] An asymmetric norm on real vector space X is a mapping $p: X \to \mathbb{R}$ such that

$p(x) \geq 0$

$p(x) = p(-x) = 0 \Leftrightarrow x = \theta_X$

$p(\alpha x) = \alpha p(x)$

$p(x + y) \leq p(x) + p(y)$

hold for all $x, y \in X$ and $\alpha \geq 0$. The ordered pair (X, p) is called an asymmetric normed space.

On the contrary to a norm, since the scalar multiplication is not continuous, an asymmetric norm does not induce a vector topology. An asymmetric norm defines an asymmetric metric which does not satisfy the symmetry condition of a metric. Hence one can obtain a topology induced by the asymmetric norm which is not necessarily Hausdorff. This innocent modification changes the whole theory, mainly related to completeness, compactness and totally boundedness. For instance, sequentially compactness and compactness are not the same notions contrary to the case of a normed space. Many authors have investigated the topological properties of asymmetric metric and related structures. We refer to [9-15] and references therein.

In this study, we give the definition of an asymmetric cone normed space as a generalization of the asymmetric normed space. Also, we introduce some topological concepts with basic results on asymmetric cone normed spaces.

2. Main results

In this section, we define asymmetric cone normed spaces and give some new results related to these spaces.

An asymmetric cone norm on a real vector space X is a mapping $p_c: X \to E$ satisfying the following conditions:

$\theta_E \preceq p_c(x)$

$p_c(x) = p_c(-x) = \theta_E \Leftrightarrow x = \theta_X$

$p_c(\alpha x) = \alpha p_c(x)$

$p_c(x+y) \preceq p_c(x) + p_c(y)$

for all $x, y \in X$ and $\alpha \geq 0$. The ordered pair (X, p_c) is called an asymmetric cone normed space.

The mapping $\bar{p}_c: X \to E$ defined by $\bar{p}_c(x) = p_c(-x)$ for all $x \in X$ is an asymmetric cone norm on X and called as the conjugate of asymmetric cone norm p_c.

Lemma 4. If P is a minihedral cone on E, then the mapping $p_c^s: X \to E$ defined by $p_c^s(x) = \sup\{p_c(x), \bar{p}_c(x)\}$ is a cone norm on X.

Proof. Clearly, $\theta_E \preceq p_c(x), \bar{p}_c(x) \preceq p_c^s(x)$ holds for every $x \in X$.

It can be easily seen that $p_c^s(x) = \theta_E \Leftrightarrow p_c(x) = p_c(-x) = \theta_E \Leftrightarrow x = \theta_X$.

Firstly, let $\alpha \geq 0$. By (1), we obtain $\alpha p_c(x), \alpha \bar{p}_c(x) \preceq \alpha p_c^s(x)$ and so $p_c(\alpha x), \bar{p}_c(\alpha x) \preceq \alpha p_c^s(x)$. This means that

$$p_c^s(\alpha x) \preceq \alpha p_c^s(x). \qquad (3)$$

Now, let $\alpha < 0$. It is clear that $p_c^s(-\alpha x) = \sup\{p_c(-\alpha x), \bar{p}_c(-\alpha x)\} = p_c^s(\alpha x)$. From (3), we obtain $p_c^s(-\alpha x) \preceq -\alpha p_c^s(x) = |\alpha| p_c^s(x)$ or equivalently, $p_c^s(\alpha x) \preceq |\alpha| p_c^s(x)$.

Hence, for $\alpha \neq 0$, we have $|\alpha| p_c^s(x) = |\alpha| p_c^s(\frac{1}{\alpha}\alpha x) \preceq |\alpha| |\frac{1}{\alpha}| p_c^s(\alpha x) = p_c^s(\alpha x)$. Consequently, it follows that $p_c^s(\alpha x) = |\alpha| p_c^s(x)$ for all $\alpha \in \mathbb{R}$.

We have for all $x, y \in X$, $p_c(x+y) \preceq p_c(x) + p_c(y) \preceq p_c^s(x) + p_c^s(y)$ and $\bar{p}_c(x+y) \preceq \bar{p}_c(x) + \bar{p}_c(y) \preceq p_c^s(x) + p_c^s(y)$ which imply together that $p_c^s(x+y) \preceq p_c^s(x) + p_c^s(y)$.

For $x \in X$ and $c \in E$ with $\theta_E \prec\prec c$, we define the open and closed balls by:

$$B_{p_c}(x, c) = \{y \in X : p_c(y - x) \prec\prec c\}$$

and

$$B_{p_c}[x, c] = \{y \in X : p_c(y - x) \preceq c\},$$

respectively.

For an asymmetric cone norm p_c, one can define an asymmetric cone metric q_{p_c} by the formula $q_{p_c}(x, y) = p_c(y - x)$ $(x, y \in X)$. Hence, a topology τ_{p_c} on X generated by the collection

$$\{B_{p_c}(x, c) : x \in X, c \in E \text{ with } \theta_E \prec\prec c\}$$

can be defined. That is, a set U in an asymmetric cone normed space X is open with respect to the topology τ_{p_c} if for all $x \in U$ there exists $c_x \in E$ with $\theta_E \prec\prec c_x$ such that $B_{p_c}(x, c_x) \subset U$.

In the same way, another topology $\tau_{\bar{p}_c}$ on X generated by the collection

$$\{B_{\bar{p}_c}(x, c) : x \in X, c \in E \text{ with } \theta_E \prec\prec c\}$$

can be defined. U is open with respect to $\tau_{\bar{p}_c}$ if for all $x \in U$ there exists $c_x \in E$ with $\theta_E \prec\prec c_x$ such that $B_{\bar{p}_c}(x, c_x) \subset U$.

Remark 1. The topology τ_{p_c} is not necessarily T_1.

Example 2. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$ and $p_c : X \to E$ defined by

$$p_c(x) = \begin{cases} (x,x), & \text{if} x > 0 \\ (0,0), & \text{if} x \leq 0. \end{cases}$$

Then, the topology τ_{p_c} is not T_1. In fact, given any $(c_1, c_2) \in E$ with $c_1, c_2 > 0$, we have $1 \in B_{p_c}(2, (c_1, c_2)) = (-\infty, c)$, where $c = \min\{c_1 + 2, c_2 + 2\}$. That is, there exists no neighbourhood of 2 with respect to τ_{p_c} which does not contain 1. Hence, τ_{p_c} cannot be T_1.

Theorem 1. Let (X, p_c) be an asymmetric cone normed space. The topology τ_{p_c} is T_1 if and only if $\theta_E \prec\prec p_c(x)$ for every $x \in X,\ x \neq \theta_X$.

Proof. $\Rightarrow$ Let τ_{p_c} be T_1. Then, given any $x \in X$ with $x \neq \theta_X$, there exist $c_x, c \in E$ with $\theta_E \prec\prec c_x, c$ such that $\theta_X \notin B_{p_c}(x, c_x)$ and $x \notin B_{p_c}(\theta_X, c)$; that is, $\theta_E \prec\prec c_x \preceq p_c(-x)$ and $\theta_E \prec\prec c \preceq p_c(x)$.

$\Leftarrow$ By hypothesis, we have $\theta_E \prec\prec p_c(y - x)$ and $\theta_E \prec\prec p_c(x - y)$ for every $x, y \in X$ with $x \neq y$. Then, we obtain that $y \notin B_{p_c}(x, c_x)$ and $x \notin B_{p_c}(y, c_y)$, where $c_x = p_c(y - x)$ and $c_y = p_c(x - y)$, respectively. Hence, we conclude that τ_{p_c} is T_1.

Lemma 5. $B_{p_c}(x, c)$ is open with respect to τ_{p_c}.

Proof. Let $y \in B_{p_c}(x, c)$. Then, we have $p_c(y - x) \prec\prec c$. Put $c_y = c - p_c(y - x)$. If $z \in B_{p_c}(y, c_y)$, we obtain

$$p_c(z - x) \preceq p_c(z - y) + p_c(y - x) \prec\prec c_y + p_c(y - x) = c$$

which means $z \in B_{p_c}(x, c)$. Hence, we conclude that $B_{p_c}(x, c)$ is open with respect to τ_{p_c} since given any $y \in B_{p_c}(x, c)$ the inclusion $B_{p_c}(y, c_y) \subset B_{p_c}(x, c)$ holds for some $c_y \in E$ with $\theta_E \prec\prec c_y$.

Lemma 6. $B_{p_c}[x, c]$ is closed with respect to $\tau_{\bar{p}_c}$.

Proof. Let $y \in X \backslash B_{p_c}[x, c]$. Then, we have $c \prec\prec p_c(y - x)$. Put $c_y = p_c(y - x) - c$. If $z \in B_{\bar{p}_c}(y, c_y)$, we obtain

$$p_c(y - x) \preceq p_c(y - z) + p_c(z - x) \prec\prec c_y + p_c(z - x)$$

which implies $c = p_c(y - x) - c_y \prec\prec p_c(z - x)$ and so $z \in X \backslash B_{p_c}[x, c]$. Hence, $X \backslash B_{p_c}[x, c]$ is open with respect to $\tau_{\bar{p}_c}$; that is $B_{p_c}[x, c]$ is closed with respect to $\tau_{\bar{p}_c}$.

Lemma 7. *In an asymmetric cone normed space* (X, p_c), $B_{p_c}(x, c) = x + \| c \| B_{p_c}(\theta_E, e)$ and $B_{p_c}[x, c] = x + \| c \| B_{p_c}[\theta_E, e]$ hold for every $x \in X$ and $c \in E$ with $\theta_E \prec\prec c$, where

$e = \frac{c}{\|c\|}$.

Proof. Let $y \in B_{p_c}(x,c)$. Then, we have $p_c(y-x) \prec\prec c$. From (2), we obtain $p_c(\frac{1}{\|c\|}(y-x)) \prec\prec \frac{c}{\|c\|}$ which means that $\frac{1}{\|c\|}(y-x) \in B_{p_c}(\theta_E, e)$ and so $y \in x + \| c \| B_{p_c}(\theta_E, e)$, where $e = \frac{c}{\|c\|}$. Hence, we conclude that $B_{p_c}(x,c) \subset x + \| c \| B_{p_c}(\theta_E, e)$.

For the reverse inclusion, let $y \in x + \| c \| B_{p_c}(\theta_E, e)$ $(e = \frac{c}{\|c\|})$. Then, we write $y = x + \| c \| z$ for $z \in E$ with $p(z) \prec\prec \frac{c}{\|c\|}$. It follows that $p(y-x) = \| c \| p(z) \prec\prec c$ and so $y \in B_{p_c}(x,c)$. Consequently, the inclusion $x + \| c \| B_{p_c}(\theta_E, e) \subset B_{p_c}(x,c)$ holds.

Theorem 2. The mapping p_c is upper semi continuous and lower semi continuous with respect to the topologies τ_{p_c} and $\tau_{\bar{p}_c}$, respectively.

Proof. Let $A = \{x \in X : p_c(x) \prec\prec u\}$ for any $u \in E$. Choose $x \in A$ and put $c = u - p_c(x)$ which satisfies $\theta_E \prec\prec c$. For $z \in B_{p_c}(x,c)$, we have

$$p_c(z) \preceq p_c(z-x) + p_c(x) \prec\prec c + p_c(x) = u$$

which means $z \in A$. Hence, the inclusion $B_{p_c}(x,c) \subset A$ holds. We conclude that A is τ_{p_c}-open and so p_c is upper semi continuous with respect to τ_{p_c}.

Let $B = \{x \in X : u \prec\prec p_c(x)\}$ for any $u \in E$. Choose $x \in B$ and put $c' = p_c(x) - u$ which satisfies $\theta_E \prec\prec c'$. For $z \in B_{\bar{p}_c}(x,c')$, we have

$$p_c(x) \preceq p_c(x-z) + p_c(z) = \bar{p}_c(z-x) + p_c(z) \prec\prec c' + p_c(z)$$

and therefore

$$u = p_c(x) - c' \prec\prec p_c(z)$$

which means $z \in B$. Hence, the inclusion $B_{\bar{p}_c}(x,c') \subset B$ holds. We conclude that B is $\tau_{\bar{p}_c}$-open and so p_c is lower semi continuous with respect to $\tau_{\bar{p}_c}$.

Definition 4. A sequence (x_n) in an asymmetric cone normed space (X, p_c) is said to be left (right) p_c-convergent to $x \in X$ if for every $c \in E$ with $\theta_E \prec\prec c$, there exists $n_c \in \mathbb{N}$ such that $p_c(x_n - x) \prec\prec c$ $(p_c(x - x_n) \prec\prec c)$ for all $n \geq n_c$. We denote it by $x_n \xrightarrow{l} x$ $(x_n \xrightarrow{r} x)$.

Remark 2. *A sequence (x_n) in (X, p_c) is convergent to $x \in X$ with respect to the cone norm p_c^s if and only if it is left p_c-convergent and right p_c-convergent to x.*

Lemma 8. *Let (x_n) be a sequence in an asymmetric cone normed space (X, p_c).*

1. If (x_n) is left p_c-convergent to $x \in X$ and right p_c-convergent to $y \in X$, then $p_c(y - x) = \theta_E$.
2. If (x_n) is left p_c-convergent to $x \in X$ and $p_c(x - y) = \theta_E$, then (x_n) is also left p_c-convergent to $y \in X$.

Proof. By hypothesis, given any $c \in E$ with $\theta_E \prec\prec c$ there exists $n_c \in \mathbb{N}$ such that $p_c(x_n - x) \prec\prec \frac{c}{2}$ and $p_c(y - x_n) \prec\prec \frac{c}{2}$ for all $n \geq n_c$. Hence, we obtain

$$\theta_E \preceq p_c(y - x) \preceq p_c(y - x_n) + p_c(x_n - x) \prec\prec c.$$

From Lemma 3, we conclude that $p_c(y - x) = \theta_E$.

By hypothesis, given any $c \in E$ with $\theta_E \prec\prec c$, there exists $n_c \in \mathbb{N}$ such that

$$p_c(x_n - y) \preceq p_c(x_n - x) + p_c(x - y) \prec\prec c$$

for all $n \geq n_c$ which means that (x_n) is left p_c-convergent to y.

Remark 3. As a result, it is clear that if a sequence (x_n) in an asymmetric cone normed space (X, p_c) is left p_c-convergent to $x \in X$, then x is not unique unlike in a cone normed space.

Lemma 9. Let (X, p_c) be an asymmetric cone normed space with a normal cone P on E and (x_n) be a sequence in X. (x_n) is left p_c-convergent to $x \in X$ if and only if $p_c(x_n - x) \to \theta_E$ as $n \to \infty$.

Proof. $\Rightarrow$ Let $x_n \xrightarrow{l} x$. Given any $\varepsilon > 0$, choose $c \in E$ with $\theta_E \prec\prec c$ such that $K \parallel c \parallel < \varepsilon$, where $K > 0$ is a normal constant. Then, there is $n_0 \in \mathbb{N}$ satisfying $\parallel p_c(x_n - x) \parallel \leq K \parallel c \parallel < \varepsilon$ for all $n \geq n_0$ which means $p_c(x_n - x) \to \theta_E$.

$\Leftarrow$ Suppose that $p_c(x_n - x) \to \theta_E$. From Lemma 1, for $c \in E$ with $\theta_E \prec\prec c$, there exists $\delta > 0$ such that $a \prec\prec c$ whenever $\parallel a \parallel < \delta$. For this $\delta > 0$, there is $n_0 \in \mathbb{N}$ satisfying $\parallel p_c(x_n - x) \parallel < \delta$ for all $n \geq n_0$. It follows that $p_c(x_n - x) \prec\prec c$ for all $n \geq n_0$ and so (x_n) is left p_c-convergent to x.

Example 3. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$ and $p_c : X \to E$ defined by

$$p_c(x) = \begin{cases} (x, x), & \text{if} x > 0 \\ (0,0), & \text{if} x \leq 0. \end{cases}$$

The sequence $((-1)^n)$ is left p_c-convergent to 1. Indeed, given any $(c_1, c_2) \in E$ with $(0,0) \prec\prec (c_1, c_2)$, we have $p_c((-1)^n - 1) = (0,0)$ for all $n \in \mathbb{N}$. (Clearly, this sequence is not only left p_c-convergent to 1 but also left p_c-convergent to all $x > 1$.) Also, this sequence is right p_c-convergent to -1 since $p_c(-1 - (-1)^n) = (0,0)$ for all $n \in \mathbb{N}$. By using Lemma 9, we conclude that it is not convergent with respect to the cone norm p_c^s since we have

$$p_c^s((-1)^n - x) = (|(-1)^n - x|, |(-1)^n - x|) \nrightarrow (0,0)$$

for any $x \in X$.

Definition 5. A sequence (x_n) in an asymmetric cone normed space (X, p_c) is said to be
1. p_c^s-Cauchy if for every $c \in E$ with $\theta_E \prec\prec c$ there exists $n_c \in \mathbb{N}$ such that $p_c(x_n - x_m) \prec\prec c$ for all $n, m \geq n_c$,
2. left (right) K-Cauchy if for every $c \in E$ with $\theta_E \prec\prec c$ there exists $n_c \in \mathbb{N}$ such that $p_c(x_n - x_m) \prec\prec c$ $(p_c(x_m - x_n) \prec\prec c)$ for all $n \geq m \geq n_c$,
3. weakly left (right) K-Cauchy if for every $c \in E$ with $\theta_E \prec\prec c$ there exists $n_c \in \mathbb{N}$ such that $p_c(x_n - x_{n_c}) \prec\prec c$ $(p_c(x_{n_c} - x_n) \prec\prec c)$ for all $n \geq n_c$,
4. left (right) p_c-Cauchy if for every $c \in E$ with $\theta_E \prec\prec c$ there exist $n_c \in \mathbb{N}$ and $x \in X$ such that $p_c(x_n - x) \prec\prec c$ $(p_c(x - x_n) \prec\prec c)$ for all $n \geq n_c$.

Remark 4. A sequence (x_n) in (X, p_c) is p_c^s-Cauchy if and only if it is left K-Cauchy and right K-Cauchy.

Remark 5. The following relations hold:
p_c^s-Cauchy $\Rightarrow$ left (right) K-Cauchy $\Rightarrow$ weakly left (right) K-Cauchy $\Rightarrow$ left (right) p_c-Cauchy

Example 4. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$ and $p_c : X \to E$ defined by

$$p_c(x) = \begin{cases} (1,1), & \text{if} x > 0 \\ (0,0), & \text{if} x \leq 0. \end{cases}$$

The sequence $(x_n) = (1,0,1,0, \dots)$ is weakly left K-Cauchy. Indeed, given any $(c_1, c_2) \in E$ with $(0,0) \prec\prec (c_1, c_2)$, we have $p_c(x_n - x_1) = (0,0) \prec\prec (c_1, c_2)$ for all $n \geq 1$. But, it is not left K-Cauchy due to the fact that for $n > m$, we have $p_c(x_n - x_m) = (1,1) \not\prec\not\prec (\frac{1}{2}, \frac{1}{2})$, where n is odd and m is even.

Definition 6. *Let (X, p_c) be an asymmetric cone normed space.*
1. If every left p_c-Cauchy sequence in X is left p_c-convergent, then X is called left p_c-sequentially complete asymmetric cone normed space.
2. If every weakly left K-Cauchy sequence in X is left p_c-convergent, then X is called weakly left K-sequentially complete asymmetric cone normed space.
3. If every left K-Cauchy sequence in X is left p_c-convergent, then X is called left K-sequentially complete asymmetric cone normed space.
4. If every p_c^s-Cauchy sequence in X is left p_c-convergent, then X is called p_c-sequentially complete asymmetric cone normed space.

Theorem 3. *Let (x_n) be a left K-Cauchy sequence in an asymmetric cone normed space (X, p_c).*
1. If (x_n) has a left p_c-convergent subsequence, then (x_n) is left p_c-convergent to the

same point.
2. If (x_n) has a right p_c-convergent subsequence, then (x_n) is right p_c-convergent to the same point.

Proof. 1. Suppose that (x_n) is a left K-Cauchy sequence in (X, p_c) and the subsequence (x_{n_k}) is left p_c-convergent to $x \in X$. Then, given any $c \in E$ with $\theta_E \prec\prec c$ there exists $n_c \in \mathbb{N}$ such that $p_c(x_n - x_m) \prec\prec \frac{c}{2}$ for all $n \geq m \geq n_c$. Choose $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq n_c$ and $p_c(x_{n_k} - x) \prec\prec \frac{c}{2}$ for all $k \geq k_0$. It follows that

$$p_c(x_n - x) \preceq p_c(x_n - x_{n_{k_0}}) + p_c(x_{n_{k_0}} - x) \prec\prec c$$

for $n \geq n_{k_0}$.

2. Now, suppose that the subsequence (x_{n_k}) is right p_c-convergent to $x \in X$. Choose $k_0 \in \mathbb{N}$ such that $n_k \geq n \geq n_c$ and $p_c(x - x_{n_k}) \prec\prec \frac{c}{2}$ for all $k \geq k_0$. Then, we have

$$p_c(x - x_n) \preceq p_c(x - x_{n_k}) + p_c(x_{n_k} - x_n) \prec\prec c.$$

Corollary 1. If (x_n) is a left K-Cauchy sequence in an asymmetric cone normed space (X, p_c) and the subsequence (x_{n_k}) is convergent to $x \in X$ with respect to the cone norm p_c^s, then (x_n) is convergent to $x \in X$ with respect to the cone norm p_c^s.

Proof. Let (x_n) be a left K-Cauchy sequence in (X, p_c). Suppose that there is a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is convergent to $x \in X$ with respect to the cone norm p_c^s. By Remark 2, we have $x_{n_k} \xrightarrow{l} x$ and $x_{n_k} \xrightarrow{r} x$. The last theorem implies that $x_n \xrightarrow{l} x$ and $x_n \xrightarrow{r} x$. Again by Remark 2, it follows that (x_n) is convergent to x with respect to p_c^s.

Theorem 4. Let (x_n) be a sequence in an asymmetric cone normed space (X, p_c). If

$$\sum_{n=1}^{\infty} \| p_c(x_{n+1} - x_n) \| < \infty$$

holds, then (x_n) is a left K-Cauchy sequence in X.

Proof. Let $c \in E$ with $\theta_E \prec\prec c$. From Lemma 1, we can find a $\delta > 0$ such that $a \prec\prec c$ holds for every $a \in E$ with $\| a \| < \delta$. By hypothesis, for this $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{i=0}^{\infty} \| p_c(x_{n_0+i+1} - x_{n_0+i}) \| < \delta.$$

Hence, we have

$$\| p_c(x_{n+k} - x_n) \| \leq \sum_{i=0}^{k-1} \| p_c(x_{n+i+1} - x_{n+i}) \| < \delta$$

which implies that $p_c(x_{n+k} - x_n) \prec\prec c$ for all $n \geq n_0$ and $k \in \mathbb{N}$. This means that (x_n) is a left K-Cauchy sequence in X.

Theorem 5. Let P be a normal cone with normal constant K. The asymmetric cone normed space (X, p_c) is left K-sequentially complete if and only if the sequence $(X_n) = (x_1 + x_2 + \ldots + x_n)$ is left p_c-convergent in X whenever (x_n) is a sequence in X such that $\sum_{n=1}^{\infty} \| p_c(x_n) \| < \infty$.

Proof.($\Rightarrow$) Suppose that X is left K-sequentially complete. Given any $c \in E$ with $\theta_E \prec\prec c$, choose $\delta > 0$ as in the proof of the previous theorem. If (x_n) is a sequence in X such that $\sum_{n=1}^{\infty} \| p_c(x_n) \| < \infty$ holds, then there exists $n_0 \in \mathbb{N}$ satisfying $\sum_{i=0}^{\infty} \| p_c(x_{n_0+i}) \| < \delta$. Hence, we have

$$\| p_c(X_{n+k} - X_n) \| \leq \sum_{i=1}^{k} \| p_c(x_{n+i}) \| < \delta$$

which implies that $p_c(X_{n+k} - X_n) \prec\prec c$ for all $n \geq n_0$ and $k \in \mathbb{N}$. This means that (X_n) is a left K-Cauchy sequence in X. By left K-sequentially completeness of X, the sequence (X_n) is left p_c-convergent in X.

($\Leftarrow$) Take a left K-Cauchy sequence (x_n) in X. Then, for $e \in \mathrm{Int}P$ with $\| e \| = 1$, there exists $n_1 \in \mathbb{N}$ such that $p_c(x_n - x_m) \prec\prec \frac{e}{2}$ for all $n \geq m \geq n_1$. Choose $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $p_c(x_n - x_m) \prec\prec \frac{e}{2^2}$ for all $n \geq m \geq n_2$. Continuing in this manner, we obtain an increasing sequence of natural numbers $n_1 < n_2 < \ldots$ satisfying $p_c(x_{n_{i+1}} - x_{n_i}) \prec\prec \frac{e}{2^i}$ for all $i \in \mathbb{N}$. Hence, we obtain $\sum_{i=1}^{\infty} \| p_c(x_{n_{i+1}} - x_{n_i}) \| \leq K < \infty$. By hypothesis, the sequence $((x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \ldots + (x_{n_{k+1}} - x_{n_k})) = (x_{n_{k+1}} - x_{n_1})$ is left p_c-convergent to some $y \in X$ which implies that the subsequence (x_{n_k}) is left p_c-convergent to $y + x_{n_1}$. From Theorem 3, (x_n) is also left p_c-convergent to $y + x_{n_1}$ and so X is left K-sequentially complete.

Theorem 6. An asymmetric cone normed space (X, p_c) is weakly left K-sequentially complete if and only if it is left K-sequentially complete.

Proof. ($\Rightarrow$) Since a left K-Cauchy sequence is also a weakly left K-Cauchy sequence, then it is obvious that weakly left K-sequentially completeness of X implies its left K-sequentially completeness.

(⇐) Now, suppose that X is left K-sequentially complete asymmetric cone normed space. Let (x_n) be a weakly left K-Cauchy sequence in X. For $e \in \mathrm{Int}P$ with $\| e \| = 1$, choose the smallest natural number n_1 satisfying $p_c(x_n - x_{n_1}) \prec\prec e$ for all $n \geq n_1$. If $p_c(x_n - x_{n_1}) = \theta_E$ for all $n \geq n_1$, then we have $x_n \xrightarrow{l} x_{n_1}$ which completes the proof.

If $\theta_E \prec\prec p_c(x_{m_1} - x_{n_1})$ for some $m_1 > n_1$, then from Lemma 2 there exists $k_2 \in \mathbb{N}$ such that

$$\frac{e}{k_2} \prec\prec p_c(x_{m_1} - x_{n_1}) \prec\prec e.$$

Let n_2 be the smallest natural number satisfying $p_c(x_n - x_{n_2}) \prec\prec \frac{e}{k_2}$ for all $n \geq n_2$. Similarly, suppose that $\theta_E \prec\prec p_c(x_{m_2} - x_{n_2})$ for some $m_2 > n_2$, then from Lemma 2 there exists $k_3 \in \mathbb{N}$ such that

$$\frac{e}{k_3} \prec\prec p_c(x_{m_2} - x_{n_2}) \prec\prec \frac{e}{k_2}.$$

By continuing the same process, we obtain increasing sequences of natural numbers $1 = k_1 < k_2 < \ldots.$ and $n_1 < n_2 < \ldots$ such that $p_c(x_n - x_{n_i}) \prec\prec \frac{e}{k_i}$ for all $n \geq n_i$ and $i \in \mathbb{N}$. The subsequence (x_{n_i}) constructed in this way is a left K-Cauchy sequence. In fact, given any $c \in E$ with $\theta_E \prec\prec c$, we can find $i_0 \in \mathbb{N}$ such that $\frac{e}{k_{i_0}} \prec\prec c$ and so $p_c(x_{n_i} - x_{n_j}) \prec\prec c$ for all $i \geq j \geq i_0$. Since X is left K-sequentially complete, (x_{n_i}) is left p_c-convergent to some $x \in X$. Also,

$$p_c(x_n - x) \preceq p_c(x_n - x_{n_i}) + p_c(x_{n_i} - x) \prec\prec c$$

holds for sufficiently large $i \in \mathbb{N}$. Hence, we conclude that given any weakly left K-Cauchy sequence in X is left p_c-convergent which means X is weakly left K-sequentially complete.

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