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Philosophy of Hilbert's Formalization Program

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Mathematics is well known for its absoluteness, purity, abstractness, and beauty. For what is known so far, theorem-proof style formal mathematics begins with the Pythagorean Theorem, although Babylonians introduced a special case of the Pythagorean Theorem before Pythagoras. One can observe from the philosophical works of ancient Greeks such as Pythagoras, Plato, Zeno, Aristotle etc. that they really relied on pure reasoning and that mathematics was the language of nature, would show the absolute truth and that there was nothing "inexpressible" by natural numbers. Influenced by the idea of absoluteness in mathematics, David Hilbert, a famous German mathematician, proposed a program for *formalizing* mathematics to get rid of the vagueness and contradictions that have ever been encountered in the history of mathematics. In this paper I discuss the purpose and consequences of Hilbert's program in mathematics. I first discuss some of the crises in the philosophy of mathematics but mainly the reason why Hilbert set such a program for mathematicians and philosophers. I also discuss Gödel's incompleteness theorems and the reason behind them which let another mathematician, Alan Turing, discover what could be achieved from the failure of Hilbert's program.

1. Gödel Incompleteness for Ancient Greece

It is known that there have been some crises in the history of mathematics which had significant impacts on it. This shows that mathematics is not eternal and static. One of the first crises was the fact that $\sqrt{2}$ is irrational. This means that $\sqrt{2}$ cannot be expressed in the form for some natural numbers a and non-zero b. Irrationality was first thought by Indian mathematicians back in the 7th century B.C, as Carl Benjamin

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Boyer quotes:

> "It has been claimed also that the first recognition of incommensurables is to be found in India during the Sulbasutra period, but such claims are not well substantiated. The case for early Hindu awareness of incommensurable magnitudes is rendered most unlikely by the lack of evidence that Indian mathematicians of that period had come to grips with fundamental concepts."[1]

It was however first proven in Ancient Greece by Pythagoras that irrational numbers do exist.[2] Pythagorean's doctrine is the idea that everything in the world can be expressed through natural numbers and their ratios. In other words, Pythagoras and his students believed that the essential unity of things were not in a physical substrate. For them, the one thing that formed the substrate of all things in the universe was natural numbers and their ratios. It turned out that they were wrong. Gregory Chaitin, famous with his work in the field of algorithmic information theory, says in his lecture at Cargenie Mellon University the following:

> "..the Greeks thought that rationality was the supreme goal. Plato! Reason! If a number is called irrational that means this was the Gödel incompleteness theorem of Ancient Greece."[3]

So the existence of irrational numbers must have caused a serious crisis in Ancient Greece because one finds an irrational number in a world that is rational! I think the shock is due to the confusion between the notion of discreteness and continuity. Irrational numbers have an infinite decimal expansion. So one can imagine that we need infinite number of steps to *measure* the square root of two. In a sense, natural numbers and their ratios can only be associated with *discreteness* and *finiteness.* This gives us the problem that Zeno of Elea had later when Pythagoras proved his theorem. Zeno was an Ancient Greek philosopher from southern Italy famous with his paradoxes regarding motion. Bertrand Russell describes him as follows:

> "In this capricious world nothing is more capricious than posthumous fame. One of the most notable victims of posterity's lack of

1 Boyer, *A History of Mathematics*, 1991, p. 208.

2 Ancient Greeks expressed this irrational number as *alogos*, which can be translated as *illogical* or *inexpressible*.

3 Chaitin, *Conversations with a mathematician,* 2002, p. 131.

> judgement is the Eleatic Zeno. Having invented four arguments all immeasurably subtle and profound, the grossness of subsequent philosophers pronounced him to be a mere ingenious juggler, and his arguments to be one and all sophisms. After two thousand years of continual refutation, these sophisms were reinstated, and made the foundation of a mathematical renaissance..."[4]

The problem Zeno had was that whether or not motion was really possible. This was also mentioned in Aristotle's work.[5] Zeno argued that for an arrow to arrive its destination, it first needs to travel half the way. To travel half the way, it needs to travel quarter the way and so on. Thus for an arrow to move from here to there, infinitely many steps should be performed. It requires an infinite measure to see the arrow moving. That is, this phenomenon cannot be described by a natural number. From this point of view, Zeno's paradox and the existence of irrational numbers are quite related to each other.

2. Calculus As a Rigorous Foundation

It took people many years to figure out what was wrong with Zeno's argument, until Calculus became a rigorous foundation of mathematics. However, another crisis was caused by the calculus itself. Perhaps one of the most well-known person who criticized calculus was Bishop Berkeley, Bishop of Cloyne. Judith V. Grabiner writes a quote by Berkeley saying:

> "Scientists, attack religion for being unreasonable; well, let them improve their own reasoning first. A quantity is either zero or not; there is nothing in between"[6]

Berkeley characterizes the mathematicians of his time as men *"rather accustomed to compute, than to think"*.[7] In another passage, he criticizes Newton's method of fluxions[8] and he quotes:

4 Russell, *The Principles of Mathematics*, (Reprinted) 1996, p. 347.
5 Aristotle, *The Collected Works of Aristotle*, Editor: Jonathan Barnes, Oxford U. Press, Book VI, 239b5-239b9, 1991, p. 404.
6 Grabiner, "The Changing Concept of Change: The Derivative from Fermat to Weierstrass", 1983, p. 200.
7 Berkeley, *The Analyst, or a Discourse Addressed to an Infidel Mathematician*, Kessinger Publishing Co, 2004, p. 6.
8 Method of fluxions is also known as Newton's terminology for differential calculus.

> "It must, indeed, be acknowledged, that [Newton] used Fluxions, like the Scaffold of a building, as things to be laid aside or got rid of, as soon as finite Lines were found proportional to them. But then these finite Exponents are found by the help of Fluxions. Whatever therefore is got by such Exponents and Proportions is to be ascribed to Fluxions: which must therefore be previously understood. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?"[9]

Although Berkeley was a theologian, this must have something to do with the religion versus science conflict as this was the case back in the 17th century western world after the Renaissance movements. So it is possible to say that calculus caused more trouble in the religious community than that in the mathematical community. The danger in Ancient Greece with the existence of irrational number was the denial of reason and simplicity. Around the 16th and 17th century of the western world, it appears that the danger that calculus brought, with the study of infinitesimals, seems to be the denial of religious thoughts. One can see that there is indeed a strong relation between theology and mathematics.

3. Sizes of Infinity and Theory of Sets

The next crisis I am going to look at had an impact on the foundations of mathematics. This crisis goes back to a little more than a hundred years to the work of Georg Cantor on set theory. Cantor asked himself how big the infinite was. Of course Cantor was not the first person who worked on the notion of infinite. Ancient Greeks, Galileo Galilei, and possibly many others worked on this notion earlier. However, Cantor treated the infinite more rigorously and mathematically. For Cantor, the infinite had a special meaning. Although Cantor was not very a religious person, he identified the absolute infinite with God and he considered his work on transfinite numbers to have been directly communicated to him by God, who had chosen Cantor to reveal them to the world.[10] Cantor started to work on the size of infinite sets. How can one measure the size (cardinal)

9 Berkeley, 2004, p. 20.
10 Hallett, *Cantorian Set Theory and Limitation of Size*, 1986, p. 13.

of an infinite set? It is easy to tell if the set is finite. One can count the number of elements in it and that gives us its cardinal. For any set *A*, let us denote the cardinality of *A* by $|A|$.. For infinite sets it is quite different to show the cardinality. If *A* and *B* are two sets and if there is a one to one and onto mapping between them, then we can say that their cardinalities are equal. The sets of those having cardinality less than or equal to the cardinality of the set of natural numbers are called *countable* sets. Cantor called the least infinite cardinal as $\aleph_0$ which is equal to the size of the set of natural numbers. It was first shown that the set of rational numbers had the cardinality $\aleph_0$. Cantor's *pairing function* $f(x,y)=((x+y)(x+y+1)/(2)+y)$ can be used to show that the set rational numbers has the same cardinality as that of the set of natural numbers. One result that Cantor found shook the mathematical community. He showed that the set of real numbers had a strictly larger cardinality than that of the set of natural numbers. To prove this we must show that the set of real numbers cannot be countable. We call such sets *uncountable*. Cantor's method for proving this is called the *diagonal* method.[11] Suppose that there is a one to one and onto mapping $f:N\rightarrow[0,1]$ between the set of natural numbers, denoted by *N*, and the set of real numbers in the interval [0,1].. If we show that *f* cannot exist for this interval it clearly cannot exist for the entire set of real numbers. Suppose that *f* is given. Then we can enumerate the elements of [0,1] as $x_1=f(1),x_2=f(2),\ldots$ where each x_k is expressed in decimal notation $x_k=0.a_1^k\ a_{-2}^k\ a_n^k$ such that $0\leq a_n^k\leq 9$. We can construct a real number $y=0.y_1\ y_2$ such that $y\neq a_r^r$ for $r=1,2,\ldots$ by letting $y_r=(a_r^r+1)\mathrm{mod}\ 10$. So we can conclude that *f* cannot be onto and consequently the real numbers are not countable. One can argue that we can add this new number to our list but then we can follow the same argument to find another number which will not be on the list. This will go on forever and we will never be able to write a complete list for the set of real numbers.

Cantor in fact showed here that there was a larger infinity. This naturally shook the mathematical community because only one infinite had

11 I think this is the first application of the Cretan paradox ("All Cretans are liars", says a Cretan) to mathematics. However, diagonalization method was really used before Cantor by Decartes in his method of doubt.

been known until then. But Cantor showed the existence of a larger infinite and possibly more in a hierarchy. He also has a famous theorem which says that for every set there is a larger set which is the set of all subsets of it. Soon after he proved that there was a larger infinite, he worked on *ordinal numbers*. Cardinal numbers show the size of the set, on the other hand ordinal numbers show the order type. Roughly speaking, if cardinals were 1,2,3,···, ordinals would be 1st,2nd,3rd,···. In other words, ordinal numbers should be considered like the *length* of an ordered set of a particular kind. Cantor was not initially working on set theory but he later found himself working on transfinite numbers. Cantor had the idea that if we have1,2,3,···, then we should not stop there. So he puts another "number", denoted by ω, after all the finite numbers. The sequence then becomes 1,2,3,··· ω. We can still go on and write1,2,3,···,ω,ω+1,ω+2,···.If we continue, we can write, $1,2,3,\cdots,\omega,\cdots,\omega 2,\cdots,\omega^{\omega},\cdots,\omega^{\omega^{\omega\cdots}}$,···. We can continue like this forever, and if we take the limit supremum of this last list, it gives us the smallest solution of the equation $\varepsilon_0=\omega^{\varepsilon_0}$ .The ordinal ε_0 is also known to be the proof theoretic ordinal strength of Peano arithmetic. One can see that this is becoming theological, although it is very imaginative. The ideas were new and obscure for that era. So there was certainly a crisis. Some people like Henri Poincaré regarded set theory as a "grave disease". J.W. Dauben writes a quote by Henri Poincaré saying:

> "Most of the ideas of Cantorian set theory should be banished from mathematics once and for all."[12]

Dauben writes another quote by Kronecker refering to Cantor as *"a scientific charlatan"* and *"a corrupter of youth"*. However, David Hilbert defends Cantor from the critics in his lecture, saying *"No one shall expel us from the Paradise that Cantor has created"*.[13]

At the beginning we said that Cantor's work on the theory of infinite sets caused a crisis in the foundation of mathematics. I think Cantor put mathematics on an abstract "structural" analysis from the 17th century hard analysis, possibly involving special cases and formulas. One can say that 20th century mathematics was set theoretical rather than concrete

12 Dauben, *Georg Cantor: His mathematics and the philosophy of the infinite*, 1979, p. 266.
13 Hilbert, "Über das Unendliche", 1926. In Constance Reid, *Hilbert*, New York, 1996, p. 177.

analysis. This caught the interest of some logicians. Gottlob Frege wanted to establish a system of formal mathematics influenced by Cantor's set theory. Frege's aim was to defend that mathematics grows out of logic and that arithmetic was a branch of logic. Bertrand Russell, however, found a fatal flaw in one of the laws in Frege's system which was a turning point in mathematical logic. Russell considered the set of all sets that are not the members of themselves. That is, he considered

$$R=\{A:A\notin A\}.$$

Now if A is not in A, then it must be in R. Russell asks whether R is in R or not. It turns out that, by definition, $R\in R$ if and only if $R\notin R$. This is called *Russell's paradox*. So it appears that we have a problem because of the nature of self-reference and circular definitions. Paradoxes are inevitable when such definitions are used in formal systems. Hence, not everything can be a set.

Another controversial result at the end of the 19th century was the fact that for every set there is a larger set which is the set of all its subsets. If one considers the *universal set*, i.e, the "set" of everything, and if we apply this fact to the universal set, we get a set which is larger than the universal set. This is paradoxical. Are these sets equal or is there something wrong with the definition of the universal set? One can get rid of this by saying that the collection which contains everything is not a set theoretic object but say, a *class*. But then we can apply Russell's argument to classes and get another paradoxical statement. Certainly, Cantor's set theory was a bit controversial but it was a breakthrough which put mathematics into abstract set theoretical basis.

4. Hilbert's Formalization Program and Gödel's Theorems

Russell's paradox was certainly a warning for future generations that it was dangerous to play with the foundations of mathematics. Yet, in 1908, Ernst Zermelo proposed a formal axiomatic system for Cantor's intuitive set theory.[14] A formal system basically includes a formal language, a set of postulates, and logical rules of inference. Zermelo's system was later improved by some other people and later became an *almost* complete and

14 Zermelo, "Untersuchungen über die Grunlagen der Mengenlehre", 1908, p. 261.

well known system for formalizing mathematics. This system is known as *Zermelo-Fraenkel (ZF) set theory*.

David Hilbert proposed a program for logicians to get rid of all the contradictions once and for all. Hilbert himself was a *formalist*. Formalism is a philosophy of mathematics where logic and mathematics is, roughly speaking, considered as a string manipulation activity involving symbols in a formal language and with formal rules of inference where we derive statements using them. Since these symbols do not have any meaning, formalism ignore semantics. The aim of so called *Hilbert's program* was to save mathematics from inconsistencies, bring absolute truth, and to establish a firm basis by finding a complete formal system for all of mathematics. The real world as we know is complicated, but one realm where things are black or white is pure mathematics. He believed that all of mathematics could be derived from a carefully written set of axioms, and that there would be a mechanical way to prove the consistency of this set of axioms. This means that we would be able to prove every mathematical statement within the formal system which was meant to be established. In other words, mathematics would be mechanized and, at the end of an algorithmic procedure, we would be able to answer whether a statement is true or not.

Hilbert also believed in the completeness of scientific knowledge. In his famous radio address in 1930 to the Society of German Scientists and Physicians, in Königsberg, he says:

> "We must not believe those, who today, with philosophical bearing and deliberative tone, prophesy the fall of culture and accept the ignorabimus. For us there is no ignorabimus, and in my opinion none whatever in natural science. In opposition to the foolish ignorabimus our slogan shall be: We must know, we will know!"

Many mathematicians, including John Von Neumann, worked on this program.[15] Hilbert's program seemed interesting until Kurt Gödel, an Austrian logician, came along. Mathematics in the early 20th century was mostly concerned with foundational problems and finding a firm basis for mathematics. Soon after Hilbert gave his speech, Gödel announced

15 Neumann, "Zur Hilbertschen Beweistheorie", 1927, p. 1.

something that caused a crisis in the entire mathematical community. He announced that Hilbert's program was not possible to achieve. That is, all of mathematics could not be formalized and that there would always be something that we would never know unlike Hilbert thought. It was a mathematical result with deep philosophical implications. Gödel called his results as *incompleteness theorems*.[16] Although Gödel's theorems had a negative impact on Hilbert's program, his theorems form a theoretical basis for what later came to be known as *computability theory*. We shall very briefly give the essence of incompleteness and then we shall discuss its consequences on mathematics and philosophy.

Let us say that a *theory* is a set of sentences in a formal language. A theory T *proves* a statement *S* if we can derive *S*, using the rules of inference, from T. A theory is *consistent* if no contradiction can be derived from it. Otherwise, we say that the theory is *inconsistent.* Let T be a consistent theory; if we can prove or disprove every sentence , in the language of T, then we say that T is *complete*. Otherwise, is *incomplete*. A predicate is called *decidable* if we can algorithmically decide the truth value of it.[17] A predicate is called *semi-decidable* if there is an algorithm which always tells correctly when the predicate is true, but may not give a negative answer or no answer at all otherwise.

Gödel's theorems consider *sufficiently strong* theories in first-order logic. By sufficiently strong we mean theories that are strong enough to represent enough arithmetic, but more precisely theories that capture the notion of *semi-decidability*. In other words, we need the theory to capture all semi-decidable predicates. This is necessary because Gödel originally uses the *provability* predicate in his proof and we know that this kind of predicate is placed mathematically highly enough in the hierarchy of sets, according to their degree of unsolvability, to compute any semi-decidable predicate.

16 Gödel, "Über formal unentscheidbare Satze der Principia Mathematica und verwandtler Systeme I", 1931, p. 173.

17 Note that Gödel could not use this definition since there was no mathematical model for algorithms back then. He instead first defined the notion of decidability (recursiveness) using his own formal language by defining primitive recursive functions.

The basic idea relies on the liar paradox. Instead of saying "This statement is false", Gödel considers "This statement is unprovable". Of course he constructs this statement in his formal language, by using Gödel numbering, i.e. assigning each symbol in the language a prime number and obtain a unique code for each sentence. The construction of the *Gödel sentence* is based on Cantor's diagonal argument. So this is really a sentence about arithmetic. But this Gödel sentence refers to itself and saying that the statement itself is unprovable. Now such sentence is either provable or unprovable. Assume that our theory is consistent and complete. If the theory proves the Gödel sentence, then it proves something false because the sentence says that it is unprovable. So our theory in this case is inconsistent. If it is unprovable then it is true that the Gödel sentence is unprovable, our theory is consistent, but we have incompleteness. In this case we have such statement which is true but unprovable, hence contradicting completeness of our formal system. The first incompleteness theorem then says the following:

"Every sufficiently strong theory is either inconsistent or incomplete."

Similar to Cantor's argument, even though we add the Gödel sentence to our theory (hence we get a new theory), we can again construct another Gödel sentence in this new theory so the first incompleteness theorem still applies.

One important goal of Hilbert's program was to prove mechanically that mathematics is consistent within the formal axiomatic system that Hilbert was hoping to have. Perhaps Gödel's second incompleteness theorem is more straight to the point. It follows from the first incompleteness theorem and it says:

"Sufficiently strong theories cannot prove their own consistency."

In fact, only stronger theories can prove the consistency of weaker theories if we assume the consistency of the stronger theory. Therefore, the second incompleteness theorem implies that absolute consistency is impossible and that we can only have relative consistency in sufficiently strong theories.

The incompleteness theorem must have looked somewhat obscure at that time because the notion of mechanical computability was not very well known. Gödel's theorems show the limitations of formal systems, reasoning, and algorithmic computability. They were of course controversial. One may say that this is not only because there were new notions in his paper, but mostly because of that he used mathematics to show the limitations of mathematics. Ernst Zermelo was one of few who felt skeptical. In a letter that he wrote to Gödel, he claimed there was an *"essential gap"* in Gödel's arguments, but then Gödel replied with a 10 page letter.[18]

5. **Halting Problem and Algorithmic Randomness**

Although some people did not understand what Gödel said, some tried to go more deeply. Alan Turing, a British mathematician, was one of them. After incompleteness, Turing started to talk about computers in his famous paper.[19] In fact, what Turing showed was the reason why we had incompleteness. Gödel showed the existence of undecidability, Turing showed why we had that. Hilbert was hoping to find a procedure for proof checking. What Hilbert really means is that there should be a computer program for checking proofs. Then Turing describes what a computer is, and that is a *Turing machine*. Turing really thought that the way of finite thinking of the human mind had limitations and this was the reason why we had incompleteness. So Turing modelled a proving mathematician (whom he calls a computer).[20] Basically, a Turing machine is an abstract computational model which contains an *infinite tape* divided into cells with a *tape head* to read/write symbols on the tape cell, a *tape alphabet*, a finite set of *states*, and a finite set of *instructions*. The computation starts by reading the leftmost symbol of the input written on the tape, then following the instructions, we move the tape head, write a symbol on the cell, change the state of the machine if necessary. The important thing here is that Turing finds an almost concrete statement which escapes the power of logic and

18 Dawson, *Logical Dilemmas: The life and work of Kurt Gödel*, 1979, p. 76.

19 Turing, "On computable numbers with an application to the Entscheidungsproblem", 1936, p. 230.

20 Turing in his paper refers to computer as "he" instead of "it". For him, a computer is anyone who computes. In this case, a proving mathematician was considered to be a computer.

reasoning. It's called the *halting problem.* It is the problem of deciding whether an algorithmic procedure (or a program), on a given input, will ever halt or not. It turns out that when we put no time limit, there is no way of deciding this in a mechanical way. Hence, there is no set of axioms, for Hilbert's formal axiomatic system, that enables us to prove whether a program will halt or not. The idea of the proof again uses Cantor's diagonal argument and Russell's paradox.

One important question to ask is whether or not Turing machines really capture the intuitive notion of algorithmic computation. Now on one hand we have a mathematical object called Turing machine. On the other hand, we have an intuitive notion of "algorithmic" computation. Therefore we cannot answer this question mathematically. Alonzo Church and Alan Turing claimed that whatever algorithmically computable was also Turing machine computable and vice versa. This is called the *Church-Turing hypothesis* and no counter example has emerged in almost 80 years. What this hypothesis really says is that mechanical thinking has limits and it cannot exceed the computational power of Turing machines. The halting problem is one concrete problem that cannot be solved by mechanical thinking. This is the reason why we have incompleteness in mathematics. One may say that Turing's paper is more concrete than Gödel's because Turing now talks about physical devices, i.e. computers. However there is more to say. Halting problem itself does not sound like a satisfactory reason why we have incompleteness in reasoning. Why do we have the halting problem? What problem do we have at the very base of the foundations of mathematics?

Gregory Chaitin, an American mathematician, made some important contributions to this problem for finding the ultimate reason behind incompleteness and the halting problem. Examining the details of his work is beyond the scope of our paper. His results conclude there exist mathematical facts that are true for no reason, i.e. true by accident, hence they are random mathematical facts. This is of course deeper than what Turing found. Chaitin claims that the reason we have incompleteness and the halting problem is because there exists *irreducible* information, i.e. an information content which cannot be expressed in simpler terms.[21] These are things that have no mathematical pattern or structure. We can generate a sequence of 0's and 1's which has some mathematical pattern in it. For example 01010101… is a sequence which has some structure. Because we can see that it is an alternating 0-1 sequence with a rule that the digit is 0 if it is in an even position, 1 otherwise. The sequence 001001001… is another example of a reducible information with some pattern. Or a sequence like 010110111011111… still has some pattern. However the sequence 01101010010… for example looks more *random* than the others. This can be thought of as an independent sequence of tossing of a fair coin so that no previous toss gives any information about the outcome of the next toss. Chaitin constructed a number, so called the *Omega number*, which informally represents the halting probability of a randomly constructed algorithm. Omega number is a real number expressed in binary number system such that each bit is *incompressible* and has no structure. By incompressible we mean that to compute the first *n* bits of the Omega number we need at least *n* bits of information. According to Chaitin, this is a random mathematical truth for no reason. It is *true* as it is. Hence, according to Chaitin, the reason we have the halting problem is because there is no structure in some mathematical facts and therefore this is why we fail to algorithmically find a justification of some mathematical statements such as the halting problem.

One can see that there is a strong link between Cantor's theory of infinite sets, Gödel's incompleteness, Turing's halting problem, and Chaitin's theory of irreducible information. David Hilbert wanted to formalize all of mathematics. However, Gödel was the first person who gave a negative answer to Hilbert. It became more natural with Turing and Chaitin's work why Hilbert's program could not be entirely achieved. The attempt to formalize all of mathematics was a failure, but the failure of Hilbert's program gave birth to the philosophical notion of computers. One consequence concerns the methodology of mathematics. Most people consider Gödel's theorems as a negative result in mathematics. However,

21 Chaitin, *Meta Math! The Quest for Omega*, 2005, p. 101.

this really means that mathematics cannot be mechanized. The traditional notion of mathematics is that we use reasoning from a set of axioms and derive new theorems from the axioms. This notion of what mathematics is about sometimes does not work, i.e., reducing things to axioms, compression. Otherwise mathematics could be done by computers. If axioms are as complicated as the result then there is no use of reasoning. Because theory implies compression. Hilbert took this tradition to its extreme and believed that a single formal axiomatic system of finite complexity, i.e. a finite number of bits of information, would suffice to generate *all* of mathematical truth. This ambitious project was not successful however. So Hilbert's program was originally set for mechanizing all of mathematics by finding a complete and consistent formal axiomatic system, but it resulted in a failure which later gave birth to the notion of computers as a philosophical concept. It can be said that the idea of formalization is not good for reasoning or doing mathematics, as Gödel's theorems imply this so, but it is good for computing and programming which must have been a great achievement of the previous century.

Öz

Hilbert'in Biçimselleştirme Programının Felsefesi

Bu yazıda, Hilbert'in biçimselleştirme programına kadar giden matematik felsefesindeki önemli krizlerden bahsedilmiştir. Hilbert'in programının gerçekleşemeyeceğini gösteren Gödel'in eksiklik teoremi gibi programın sonuçları ve nedenleri tartışılmıştır. Daha sonra matematikteki eksikliğin neden varolduğu anlatılmaya çalışılmıştır. Son olarak, Hilbert'in programından çıkan sonuç irdelenmiştir.

Anahtar Kelimeler: Matematik felsefesi, matematiksel mantık, Gödel'in eksiklik teoremi, Hilbert'in programı, biçimselleştirme, hesaplanabilirlik, rastgelelik.

Abstract

Philosophy of Hilbert's Formalization Program

I discuss some of the important crises in the philosophy of mathematics which led to Hilbert's program for formalizing mathematics. I discuss its purpose and consequences such as Gödel's incompleteness theorems which showed that Hilbert's program was unachievable. I also argue the reason why we have incompleteness in mathematics. Finally, I argue what followed from the failure of Hilbert's formalization program.

Keywords: Philosophy of mathematics, mathematical logic, Gödel's incompleteness theorem, Hilbert's program, formalization, computability, randomness.

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