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Nearness Γ-Rings

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Abstract: In this paper, nearness Γ-rings are introduced. Also, subnearness Γ-rings, and nearness Γ-ideals are given. Moreover, some properties of these structures are investigated.

Key words: Nearness rings, Γ-rings, nearness Γ-rings, nearness Γ-ideals.

1. Introduction

Nearness approximation spaces and near sets are introduced in 2007 as a generalization of rough sets [12, 17]. Near set theory begins with the selection of probe functions that provide a basis for describing and discerning affinities between objects in distinct perceptual granules. A probe function is a real-valued function that represents a feature of objects such as images.

In the concept of ordinary algebraic structures, such a structure that consists of a nonempty set of abstract points with binary operations, which are required to satisfy certain axioms. A groupoid $(A, \circ)$ is an algebraic structure consisting of a nonempty set A and a binary operation "$\circ$" defined on A [2]. In the nearness approximation space, however, the sets are composed of perceptual objects (non-abstract points) instead of abstract points. Perceptual objects are points that have features. These points describable with feature vectors [12]. Upper approximation of a set is determined by matching descriptions of objects in the set of perceptual objects. In the algebraic structures constructed on nearness approximation spaces, the basic tool is the consideration of upper approximations of the subsets of perceptual objects. In a nearness groupoid, the binary operation must be closed in upper approximation of the set instead of the set itself.

Nobusawa [9] introduced the notion of a Γ-ring, as more general than a ring. Barnes [1] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Barnes [1], Kyuno [6] and Luh [7] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in the ring theory.

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In 2012, İnan and Öztürk [3, 4] investigated the concept of groups on nearness approximation spaces. In 2013, Öztürk et al. [11] introduced nearness group of weak cosets. In 2015, İnan and Öztürk [5] also investigated the nearness semigroups. In 2019, Öztürk and İnan introduced nearness rings as well [10].

The aim of this study is to introduce nearness Γ-rings. Also, subnearness Γ-rings and nearness Γ-ideals are given. Moreover, some properties of these structures are investigated.

2. Preliminaries

Perceptual objects are points that are describable with feature vectors. Let $\mathcal{O}$ be a set of perceptual objects, $X \subseteq \mathcal{O}$, $\mathcal{F}$ be a set of probe functions and $\Phi : \mathcal{O} \longrightarrow \mathbb{R}^L$ be a mapping where the description length is $|\Phi| = L$.

$\Phi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \cdots, \varphi_i(x), \cdots, \varphi_L(x))$ is an object description of $x \in X$ such that each $\varphi_i \in B \subseteq \mathcal{F}$ $(\varphi_i : \mathcal{O} \longrightarrow \mathbb{R})$ is a probe function that represents features of sample objects $X \subseteq \mathcal{O}$ [12].

Sample objects are near each other if and only if the objects have similar descriptions. Recall that each φ_i defines a description of an object. Δ_{φ_i} is defined by $\Delta_{\varphi_i} = |\varphi_i(x') - \varphi_i(x)|$, where $x, x' \in \mathcal{O}$.

Let $x, x' \in \mathcal{O}$ and $B \subseteq \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \Delta_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called an indiscernibility relation on $\mathcal{O}$, where description length is $i \leq |\Phi|$ [12].

Definition 2.1 *[8] Let $\mathcal{O}$ be a set of perceptual objects, Φ be an object description and $A \subseteq \mathcal{O}$. Then the set description of A is defined as*

$$Q(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2.2 *[8, 14] Let $\mathcal{O}$ be a set of perceptual objects and A, $B \subseteq \mathcal{O}$. Then the descriptive (set) intersection of A and B is defined as*

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

If $Q(A) \cap Q(B) \neq \emptyset$, then A is called a descriptively near B and denoted by $A\delta_\Phi B$. Also, $\xi_\Phi(A) = \{B \in \mathcal{P}(\mathcal{O}) \mid A\delta_\Phi B\}$ is called a descriptive nearness collection [13].

Definition 2.3 *[12] Let $X \subseteq \mathcal{O}$ and $x \in X$.*

$$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$$

is called a nearness class of $x \in X$ *.*

Definition 2.4 *[12] Let* $X \subseteq \mathcal{O}$ *.*

$$N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$$

is called an upper approximation of X *.*

A nearness approximation space is $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ where $\mathcal{O}$ is a set of perceptual objects, $\mathcal{F}$ is a set of probe functions, "$\sim_{B_r}$" is an indiscernibility relation relative to $B_r \subseteq B \subseteq \mathcal{F}$, $N_r(B)$ is a collection of partitions and $\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \longrightarrow [0,1]$ is an overlap function that maps a pair of sets to $[0,1]$ representing the degree of nearness between sets. The subscript r denotes the cardinality of the restricted subset B_r.

Definition 2.5 *[3] Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ *be a nearness approximation space and "*$\cdot$*" be a binary operation defined on* $\mathcal{O}$*.* $G \subseteq \mathcal{O}$ *is called a nearness group if the following properties are satisfied:*

(NG_1) *For all* $x, y \in G$, $x \cdot y \in N_r(B)^* G$,

(NG_2) *For all* $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *property holds in* $N_r(B)^* G$,

(NG_3) *There exists* $e_G \in N_r(B)^* G$ *such that* $x \cdot e_G = e_G \cdot x = x$ *for all* $x \in G$ *(*e_G *is called a near identity element of* G*),*

(NG_4) *There exists* $y \in G$ *such that* $x \cdot y = y \cdot x = e_G$ *for all* $x \in G$ *(*y *is called a near inverse of* x *in* G *and denoted as* x^{-1}*).*

Additionally, if $x \cdot y = y \cdot x$ property is satisfied in $N_r(B)^* G$ for all $x, y \in G$, then G is said to be a commutative nearness group.

Also, $S \subseteq \mathcal{O}$ is called a nearness semigroup if $x \cdot y \in N_r(B)^* S$ for all $x, y \in S$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property is satisfied in $N_r(B)^*(S)$ for all $x, y, z \in S$.

Theorem 2.6 *[4] Let* G *be a nearness group,* H *be a nonempty subset of* G *and* $N_r(B)^* H$ *be a groupoid. Then* $H \subseteq G$ *is a subnearness group of* G *if and only if* $x^{-1} \in H$ *for all* $x \in H$*.*

Definition 2.7 *[10] Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ *be a nearness approximation space and "*$+$*" and "*$\cdot$*" be binary operations defined on* $\mathcal{O}$*.* $R \subseteq \mathcal{O}$ *is called a nearness ring if the following properties are satisfied:*

(NR_1) R *is an abelian nearness group with binary operation "*$+$*",*

(NR_2) R *is a nearness semigroup with binary operation "*$\cdot$*",*

(NR_3) *For all* $x, y, z \in R$,

$$x \cdot (y+z) = (x \cdot y) + (x \cdot z) \text{ and } (x+y) \cdot z = (x \cdot z) + (y \cdot z)$$

properties hold in $N_r(B)^* R$.

In addition,

(NR_4) R *is said to be a commutative nearness ring if* $x \cdot y = y \cdot x$ *for all* $x, y \in R$,

(NR_5) R *is said to be a nearness ring with identity if* $N_r(B)^* R$ *contains an element* 1_R *such that* $1_R \cdot x = x \cdot 1_R = x$ *for all* $x \in R$.

Definition 2.8 *[1] An* Γ*-ring (in the sense of Barnes) is a pair* (M, Γ) *where* M *and* Γ *are (additive) abelian groups for which exists a* $(-,-,-) : M \times \Gamma \times M \to M$, *the image of* (a, α, b) *being denoted by* $a\alpha b$ *for* $a, b \in M$ *and* $\alpha \in \Gamma$, *satisfying for all* $a, b, c \in M$ *and all* $\alpha, \beta \in \Gamma$*:*

- $(a+b)\alpha c = a\alpha c + b\alpha c$,
- $a(\alpha+\beta)b = a\alpha b + a\beta b$,
- $a\alpha(b+c) = a\alpha b + a\alpha c$,
- $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Definition 2.9 *[1] Let* M *be an* Γ*-ring. A left (right) ideal of* M *is an additive subgroup* U *of* M *such that* $M\Gamma U \subseteq U$ $(U\Gamma M \subseteq U)$*. If* U *is both a left and a right ideal, then we say that* U *is an ideal of* M.

Definition 2.10 *[1] A mapping* $\theta : M \to N$ *of* Γ*-rings is called an* Γ*-homomorphism if* $\theta(a+b) = \theta(a) + \theta(b)$ *and* $\theta(a\alpha b) = \theta(a)\alpha\theta(b)$ *for all* $a, b \in M$ *and all* $\alpha \in \Gamma$.

3. Nearness Γ-Rings

Definition 3.1 *Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ *be a nearness approximation space and* $M, \Gamma \subseteq \mathcal{O}$ *be an additive abelian nearness groups in* $\mathcal{O}$. $M \subseteq \mathcal{O}$ *is called an* Γ*-ring on nearness approximation space or shortly, nearness* Γ*-ring if the following properties are satisfied:*

$(N\Gamma_1)$ $a\alpha b \in N_r(B)^* M$,

$(N\Gamma_2)$ $(a\alpha b)\beta c = a\alpha(b\beta c)$ *property verify on* $N_r(B)^* M$,

$(N\Gamma_3)$ $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$ *properties verify on* $N_r(B)^* M$

for all $a, b, c \in M$ *and all* $\alpha, \beta \in \Gamma$.

In addition, M *is called a commutative nearness* Γ*-ring if* $a\alpha b = b\alpha a$ *for all* $a, b \in M$ *and all* $\alpha \in \Gamma$.

M *is called a nearness* Γ*-ring with identity if* $N_r(B)^* M$ *contains* 1_M *such that* $1_M\alpha a = a\alpha 1_M = a$ *for all* $a \in M$ *and all* $\alpha \in \Gamma$.

$(N\Gamma_1)-(N\Gamma_3)$ properties must be satisfied in $N_r(B)^* M$. Sometimes these properties may be hold in $\mathcal{O} \setminus N_r(B)^* M$, in which case M is not a nearness Γ-ring. Addition or multiplying of finite number of elements in M may not always belong to $N_r(B)^* M$. As a result, we can not always say that $na \in N_r(B)^* M$ or $a^n \in N_r(B)^* M$ for all $a \in M$, all $\alpha \in \Gamma$ and some $n \in \mathcal{Z}^+$.

If $\left(N_r(B)^* M, +\right)$ is a groupoid and $N_r(B)^* M$ is an Γ-groupoid, then we can say that $na \in N_r(B)^* M$ for all $a \in M$ and all $n \in \mathcal{Z}$ or $a^n \in N_r(B)^* M$ for all $a \in M$, all $\alpha \in \Gamma$ and all $n \in \mathcal{Z}^+$.

Let M be a nearness Γ-ring with near identity. $a \in M$ is called a left (resp. right) near invertible if there exists $b \in N_r(B)^* M$ (resp. $c \in N_r(B)^* M$) such that $b\alpha a = 1_M$ (resp. $a\alpha c = 1_M$). b (resp. c) is called a left (resp. right) near inverse of a. If $a \in M$ is both a left and a right near invertible, then a is called a near invertible.

Lemma 3.2 *Every Γ-ring is a nearness Γ-ring.*

Proof Let $M \subseteq \mathcal{O}$ be an Γ-ring. Since $M \subseteq N_r(B)^* M$, then the properties $(N\Gamma_1)-(N\Gamma_3)$ hold in $N_r(B)^* M$. Therefore M is a nearness Γ-ring. □

Example 3.3 *$\mathcal{O} = \{a_{ij} \mid 0 \leq i, j \leq 4\}$ be a set of perceptual objects and $B = \{\varphi\} \subseteq \mathcal{F}$ be a set of probe function. Probe function*

$$\varphi : \mathcal{O} \longrightarrow V_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$$

is given in Table 1.

Table 1: Probe function

	a_{00}	a_{01}	a_{02}	a_{03}	a_{04}	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
φ	x_1	x_2	x_1	x_3	x_3	x_1	x_2	x_3	x_4	x_5

	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}	a_{30}	a_{31}	a_{32}	a_{33}	a_{34}
φ	x_4	x_3	x_6	x_4	x_7	x_3	x_4	x_7	x_8	x_9

	a_{40}	a_{41}	a_{42}	a_{43}	a_{44}
φ	x_3	x_4	x_1	x_9	x_5

Thus

$$\begin{aligned} [a_{00}]_\varphi &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{00}) = x_1\} \\ &= \{a_{00}, a_{02}, a_{10}, a_{42}\} = [a_{02}]_\varphi = [a_{10}]_\varphi = [a_{42}]_\varphi, \end{aligned}$$

$$\begin{aligned} [a_{01}]_\varphi &= \{a \in \mathcal{O} \mid \varphi(a) = \varphi(a_{01}) = x_2\} \\ &= \{a_{01}, a_{11}\} = [a_{11}]_\varphi, \end{aligned}$$

$$\begin{aligned}[a_{03}]_{\varphi} &= \{a \in \mathcal{O} \,|\varphi(a) = \varphi(a_{03}) = x_3\} \\ &= \{a_{03}, a_{04}, a_{12}, a_{21}, a_{30}, a_{40}\} \\ &= [a_{04}]_{\varphi} = [a_{12}]_{\varphi} = [a_{21}]_{\varphi} = [a_{30}]_{\varphi} = [a_{40}]_{\varphi}\,,\end{aligned}$$

$$\begin{aligned}[a_{13}]_{\varphi} &= \{a \in \mathcal{O} \,|\varphi(a) = \varphi(a_{13}) = x_4\} \\ &= \{a_{13}, a_{20}, a_{23}, a_{31}, a_{41}\} \\ &= [a_{20}]_{\varphi} = [a_{23}]_{\varphi} = [a_{31}]_{\varphi} = [a_{41}]_{\varphi}\,,\end{aligned}$$

$$\begin{aligned}[a_{14}]_{\varphi} &= \{a \in \mathcal{O} \,|\varphi(a) = \varphi(a_{14}) = x_5\} \\ &= \{a_{14}, a_{44}\} = [a_{44}]_{\varphi}\,,\end{aligned}$$

$$\begin{aligned}[a_{22}]_{\varphi} &= \{a \in \mathcal{O} \,|\varphi(a) = \varphi(a_{22}) = x_6\} \\ &= \{a_{22}\}\,,\end{aligned}$$

$$\begin{aligned}[a_{24}]_{\varphi} &= \{a \in \mathcal{O} \,|\varphi(a) = \varphi(a_{24}) = x_7\} \\ &= \{a_{24}, a_{32}\} = [a_{32}]_{\varphi}\,,\end{aligned}$$

$$\begin{aligned}[a_{33}]_{\varphi} &= \{a \in \mathcal{O} \,|\varphi(a) = \varphi(a_{33}) = x_8\} \\ &= \{a_{33}\}\,,\end{aligned}$$

$$\begin{aligned}[a_{34}]_{\varphi} &= \{a \in \mathcal{O} \,|\varphi(a) = \varphi(a_{34}) = x_9\} \\ &= \{a_{34}, a_{43}\} = [a_{43}]_{\varphi}\,.\end{aligned}$$

Therefore

$$\xi_{\varphi} = \left\{[a_{00}]_{\varphi}, [a_{01}]_{\varphi}, [a_{03}]_{\varphi}, [a_{13}]_{\varphi}, [a_{14}]_{\varphi}, [a_{22}]_{\varphi}, [a_{24}]_{\varphi}, [a_{33}]_{\varphi}, [a_{34}]_{\varphi}\right\}.$$

Hence, a set of partitions of $\mathcal{O}$ *is* $N_1(B) = \{\xi_{\varphi}\}$ *for* $r = 1$*. Thus*

$$\begin{aligned}N_1(B)^* M &= \bigcup_{[a]_{\varphi} \cap M \neq \varnothing} [a]_{\varphi} \\ &= \{a_{00}, a_{02}, a_{10}, a_{42}\} \cup \{a_{01}, a_{11}\} \\ &= \{a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{42}\}\end{aligned}$$

and

$$\begin{aligned}N_1(B)^* \Gamma &= \bigcup_{[a]_{\varphi} \cap \Gamma \neq \varnothing} [a]_{\varphi} \\ &= \{a_{00}, a_{02}, a_{10}, a_{42}\}\end{aligned}$$

where $M = \{a_{01}, a_{10}\}$, $\Gamma = \{a_{42}\} \subseteq \mathcal{O}$.

Let

$$\begin{aligned}+_1 \;:\; \mathcal{O} \times \mathcal{O} &\longrightarrow \mathcal{O} \\ (a_{ij}, a_{mn}) &\longmapsto a_{ij} +_1 a_{mn}\end{aligned}$$

be a binary operation on $M = \{a_{01}, a_{10}\} \subseteq \mathcal{O}$ *such that*

$$a_{ij} +_1 a_{mn} = a_{pr}, \quad i + m \equiv p \pmod 2 \text{ ve } j + n \equiv r \pmod 2.$$

Then $(M, +_1)$ *is an abelian nearness group.*

Furthermore, let

$$+_2 \quad : \quad \begin{array}{ll} \mathcal{O} \times \mathcal{O} & \longrightarrow \mathcal{O} \\ (a_{ij}, a_{mn}) & \longmapsto a_{ij} +_2 a_{mn} \end{array}$$

be a binary operation on $\Gamma = \{a_{42}\} \subseteq \mathcal{O}$ *such that*

$$a_{ij} +_2 a_{mn} = a_{st}, \quad i + m \equiv s \pmod 4 \text{ ve } j + n \equiv t \pmod 4.$$

Then $(\Gamma, +_2)$ *is an abelian nearness group.*

Since $a_{01} + a_{10} = a_{11} \notin M$, $M \subseteq \mathcal{O}$ *is not a group with binary operation "*$+_1$*" and so* M *is not an* Γ*-ring.*

Let

$$\begin{array}{ll} \mathcal{O} \times \Gamma \times \mathcal{O} & \longrightarrow \mathcal{O} \\ (a_{ij}, a_{kl}, a_{mn}) & \longmapsto a_{ij} a_{kl} a_{mn} \end{array}$$

be an operation such that

$$a_{ij} a_{kl} a_{mn} = a_{uv}, \quad u = \min\{i, k, m\} \text{ ve } v = \min\{j, l, n\}.$$

From Definition 3.1, it is easily shown that

$(N\Gamma_1)$ $a\alpha b \in N_r(B)^* M$,

$(N\Gamma_2)$ $(a\alpha b)\beta c = a\alpha(b\beta c)$ *property verify on* $N_r(B)^* M$,

$(N\Gamma_3)$ $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$ *properties verify on* $N_r(B)^* M$

for all $a, b, c \in M$ *and all* $\alpha, \beta \in \Gamma$.

Consequently, M *is a nearness* Γ*-ring.*

Lemma 3.4 *Let* $M \subseteq \mathcal{O}$ *be a nearness* Γ*-ring and* $0_M \in M$. *If* $0_M \alpha a$, $a\alpha 0_M \in M$ *for all* $a \in M$ *and all* $\alpha \in \Gamma$, *then*

(i) $a\alpha 0_M = a 0_\Gamma b = 0_M \alpha a = 0_M$,

(ii) $a\alpha(-b) = (-a)\alpha b = -(a\alpha b)$,

(iii) $(-a)\alpha(-b) = a\alpha b$

for all $a, b \in M$ *and all* $\alpha \in \Gamma$.

Proof (i) Let $a\alpha 0_M \in M$, $a \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} a\alpha 0_M &= a\alpha(0_M + 0_M) \\ &= a\alpha 0_M + a\alpha 0_M. \end{aligned}$$

Let we consider $-(a\alpha 0_M) \in M$. Hence

$$\begin{aligned} -(a\alpha 0_M) + a\alpha 0_M &= -(a\alpha 0_M) + a\alpha 0_M + a\alpha 0_M \\ &\Rightarrow 0_M = 0_M + a\alpha 0_M \\ &\Rightarrow 0_M = a\alpha 0_M. \end{aligned}$$

Similarly, $a0_\Gamma b = 0_M$ and $0_M \alpha a = 0_M$. Thus $a\alpha 0_M = a0_\Gamma b = 0_M \alpha a = 0_M$.

(ii) For all $a, b \in M$ and all $\alpha \in \Gamma$,

$$\begin{aligned} & a\alpha 0_M = 0_M \\ \Rightarrow \quad & a\alpha (b + (-b)) = 0_M \\ \Rightarrow \quad & a\alpha b + (a\alpha (-b)) = 0_M. \end{aligned}$$

Since $(M, +)$ is a nearness group, there exists the near inverse of $a\alpha b$. Thus $a\alpha (-b) = -(a\alpha b)$ from the near inverse element is unique. Similarly, $(-a)\alpha b = -(a\alpha b)$. As a results, $a\alpha(-b) = (-a)\alpha b = -(a\alpha b)$ for all $a, b \in M$ and all $\alpha \in \Gamma$.

(iii) From (ii)

$$(-a)\alpha(-b) = -(a\alpha(-b)) = -(-(a\alpha b)),$$

$$(-a)\alpha(-b) = -((-a)\alpha b) = -(-(a\alpha b))$$

for all $a, b \in M$ and all $\alpha \in \Gamma$.

Since M is a nearness Γ-ring, there exists the near inverse of $a\alpha b$, that is, $-(a\alpha b) \in M$. Hence

$$a\alpha b + (-(a\alpha b)) = -(a\alpha b) + a\alpha b = 0_M.$$

Similarly,

$$-(a\alpha b) + [-(-(a\alpha b))] = -(-(a\alpha b)) + (-(a\alpha b)) = 0_M.$$

Therefore

$$\begin{aligned} -(-(a\alpha b)) &= -(-(a\alpha b)) + 0_M \\ &= -(-(a\alpha b)) + (-(a\alpha b)) + a\alpha b \\ &= 0_M + a\alpha b \\ &= a\alpha b \\ \Rightarrow -(-(a\alpha b)) &= a\alpha b \end{aligned}$$

for all $a, b \in M$ and all $\alpha \in \Gamma$. Consequently, $(-a)\alpha(-b) = a\alpha b$. □

Definition 3.5 *Let $M, \Gamma \subseteq \mathcal{O}$, M be a nearness Γ-ring and $K \subseteq M$. If K is an additive abelian nearness group and satisfy the conditions $(N\Gamma_1) - (N\Gamma_3)$, K is called a subnearness Γ-ring of M.*

Theorem 3.6 *Let* $M, \Gamma \subseteq \mathcal{O}$, M *be a nearness* Γ*-ring,* $K \subseteq M$ *and* $\left(N_r(B)^* K, +\right)$ *be a groupoid and* $N_r(B)^* K$ *be an* Γ*-groupoid. Then* K *is a subnearness* Γ*-ring of* M *if and only if* $-k \in K$ *for all* $k \in K$*.*

Proof ($\Rightarrow$) Let K is a subnearness Γ-ring of M. From Definition 3.5, $(K, +)$ is a nearness group and hence $-k \in K$ for all $k \in K$.

($\Leftarrow$) Let $-k \in K$ for all $k \in K$. Since $\left(N_r(B)^* K, +\right)$ is a groupoid, $(K, +)$ is an abelian nearness group from Theorem 2.6. Therefore, since $N_r(B)^* K$ is an Γ-groupoid and $K \subseteq M$, $k\alpha l \in N_r(B)^* K$ and $(k\alpha l)\beta m = k\alpha(l\beta m)$ property holds in $N_r(B)^* K$ for all $k, l, m \in K$ and all $\alpha, \beta \in \Gamma$. For all $k, l, m \in K$ and all $\alpha \in \Gamma$, $k+l$, $k\alpha m$, $l\alpha m$, $k\alpha m + l\alpha m \in N_r(B)^* K$. Since M is a nearness Γ-ring, $(k+l)\alpha m = k\alpha m + l\alpha m$ property holds in $N_r(B)^* K$ for all $k, l, m \in K$ and all $\alpha \in \Gamma$. Similarly, $k(\alpha+\beta)l = k\alpha l + k\beta l$ and $k\alpha(l+m) = k\alpha l + k\alpha m$ properties hold in $N_r(B)^* K$ for all $k, l, m \in K$ and all $\alpha, \beta \in \Gamma$. Consequently, K is a subnearness Γ-ring of M.
□

Example 3.7 *From Example 3.3, let* $K = \{a_{10}\}$ *be a subset of nearness* Γ*-ring* $M = \{a_{01}, a_{10}\}$*. Then*

$$\begin{aligned} N_1(B)^* K &= \bigcup_{[a]_{\varphi_i} \cap K \neq \varnothing} [a]_{\varphi_i} \\ &= \{a_{00}, a_{02}, a_{10}, a_{42}\}. \end{aligned}$$

Thus $N_r(B)^* K$ *is a groupoid with the binary operation "*$+_1$*" and* $N_r(B)^* K$ *is an* Γ*-groupoid. Furthermore, since* $-a_{10} = a_{10} \in K$*,* K *is a subnearness* Γ*-ring of* M *from Teorem 3.6.*

Theorem 3.8 *Let* M *be a nearness* Γ*-ring and* K_1*,* K_2 *be two subnearness* Γ*-rings of* M*. Let* $\left(N_r(B)^* K_1, +\right)$*,* $\left(N_r(B)^* K_2, +\right)$ *be groupoids and* $N_r(B)^* K_1$*,* $N_r(B)^* K_2$ *be* Γ*-groupoids. If*

$$\left(N_r(B)^* K_1\right) \cap \left(N_r(B)^* K_2\right) = N_r(B)^* (K_1 \cap K_2),$$

then $K_1 \cap K_2$ *is a subnearness* Γ*-ring of* M*.*

Proof Let K_1, K_2 be two subnearness Γ-rings of M. Obviously, $K_1 \cap K_2 \subseteq M$. Since $\left(N_r(B)^* K_1\right) \cap \left(N_r(B)^* K_2\right) = N_r(B)^* (K_1 \cap K_2)$, $N_r(B)^* (K_1 \cap K_2)$ is a groupoid and an Γ-groupoid from $\left(N_r(B)^* K_1, +\right)$, $\left(N_r(B)^* K_2, +\right)$ are groupoids and $N_r(B)^* K_1$, $N_r(B)^* K_2$ are Γ-groupoids. Let $k \in K_1 \cap K_2$. Since K_1, K_2 be two subnearness Γ-rings of M, $-k \in K_1$ and $-k \in K_2$, that is, $-k \in K_1 \cap K_2$. As a results, $K_1 \cap K_2$ is a subnearness Γ-ring of M from Teorem 3.6. □

Corollary 3.9 *Let M be a nearness Γ-ring, $\{K_i : i \in \Delta\}$ be a nonempty family of subnearness Γ-rings of M. Let $\left(N_r(B)^* K_i, +\right)$ be groupoids and $N_r(B)^* K_i$ be Γ-groupoids for all $i \in \Delta$. If*

$$\bigcap_{i\in\Delta} N_r(B)^* K_i = N_r(B)^* \left(\bigcap_{i\in\Delta} K_i\right),$$

then $\bigcap_{i\in\Delta} K_i$ is a subnearness Γ-ring of M.

Definition 3.10 *Let M be a nearness Γ-ring and $I \subseteq M$. I is called a left (right) nearness Γ-ideal of M if the following properties satisfied:*

(i) $x + y \in N_r(B)^* I$,

(ii) $-x \in I$,

(iii) $m\alpha x \in N_r(B)^* I$ $\left(x\alpha m \in N_r(B)^* I\right)$

for all $x, y \in I$, all $\alpha \in \Gamma$ and all $m \in M$. If I is both a left and a right nearness Γ-ideal, then I is a nearness Γ-ideal of M.

Remark 3.11 *There is only one trivial nearness Γ-ideal of nearness Γ-ring M, that is, M itself. Moreover, $\{0_M\}$ is a trivial nearness Γ-ideal of M if and only if $0_M \in M$.*

Lemma 3.12 *Let I be a nearness Γ-ideal of nearness Γ-ring M. If $N_r(B)^* I$ is a groupoid and an Γ-groupoid, then I is a subnearness Γ-ring of M.*

Proof It is obvious from Theorem 3.6. □

Example 3.13 *From Example 3.3 and Example 3.7, let we consider nearness Γ-ring $M = \{a_{01}, a_{10}\}$ and subnearness Γ-ring $K = \{a_{10}\} \subseteq M$. Then $x + y \in N_r(B)^* K$, $-x \in K$, $m\alpha x \in N_r(B)^* K$ and $x\alpha m \in N_r(B)^* K$ for all $x, y \in K$, all $m \in M$ and all $\alpha \in \Gamma$. Therefore K is a nearness Γ-ideal of M from Definition 3.10.*

Theorem 3.14 *Let M be a nearness Γ-ring and I_1, I_2 be two nearness Γ-ideals of M. Let $\left(N_r(B)^* I_1, +\right)$, $\left(N_r(B)^* I_2, +\right)$ be groupoids and $N_r(B)^* I_1$, $N_r(B)^* I_2$ be Γ-groupoids. If*

$$\left(N_r(B)^* I_1\right) \cap \left(N_r(B)^* I_2\right) = N_r(B)^* (I_1 \cap I_2),$$

then $I_1 \cap I_2$ is a nearness Γ-ideal of M.

Proof Let I_1, I_2 be two nearness Γ-ideals of M. Obviously, $I_1 \cap I_2 \subseteq M$. Since I_1, I_2 are nearness Γ-ideals,

$$x + y \in N_r(B)^* I_1, \ -x \in I_1 \text{ and } m\alpha x \in N_r(B)^* I_1,$$

$$x + y \in N_r(B)^* I_2, \ -x \in I_2 \text{ and } m\alpha x \in N_r(B)^* I_2$$

for all $x, y \in I_1 \cap I_2$, all $m \in M$ and all $\alpha \in \Gamma$. Then $x + y \in \left(N_r(B)^* I_1\right) \cap \left(N_r(B)^* I_2\right)$, $-x \in I_1 \cap I_2$ and $m\alpha x \in \left(N_r(B)^* I_1\right) \cap \left(N_r(B)^* I_2\right)$. Since $\left(N_r(B)^* I_1\right) \cap \left(N_r(B)^* I_2\right) = N_r(B)^*(I_1 \cap I_2)$,

$$x + y \in N_r(B)^*(I_1 \cap I_2),\ -x \in I_1 \cap I_2 \text{ and } m\alpha x \in N_r(B)^*(I_1 \cap I_2)$$

for all $x, y \in I_1 \cap I_2$, all $m \in M$ and all $\alpha \in \Gamma$. Thus $I_1 \cap I_2$ is a left nearness Γ-ideal of M.

Similarly, it is easily shown that $I_1 \cap I_2$ is a right nearness Γ-ideal of M. As a results, $I_1 \cap I_2$ is a nearness Γ-ideal of M from Definition 3.10. □

Corollary 3.15 *Let M be a nearness Γ-ring, $\{I_i : i \in \Delta\}$ be a nonempty family of nearness Γ-ideals of M. Let $\left(N_r(B)^* I_i, +\right)$ be groupoids and $N_r(B)^* I_i$ be Γ-groupoids for all $i \in \Delta$. If*

$$\bigcap_{i\in\Delta} N_r(B)^* I_i = N_r(B)^* \left(\bigcap_{i\in\Delta} I_i\right),$$

then $\bigcap_{i\in\Delta} I_i$ is a nearness Γ-ideal of M.

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