Avrupa Bilim ve Teknoloji Dergisi
Özel Sayı, 34, S. 745-749, Mart 2022

Araştırma Makalesi

www.ejosat.com ISSN:2148-2683

European Journal of Science and Technology
Special Issue 34, pp. 745-749, March 2022

Research Article

Strophoidal Surfaces

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(2nd International Conference on Applied Engineering and Natural Sciences ICAENS 2022, March 10-13, 2022)

(**DOI:** 10.31590/ejosat.1085758)

ATIF/REFERENCE: Güler, E. & Kişi, Ö. (2022). Strophoidal Surfaces. *European Journal of Science and Technology*, (34), 745-749.

Abstract

The mathematicians for centuries have researched the surfaces theory. In this paper, we consider strophoidal surface in three dimensional Euclidean space $\mathbb{E}^3$. We present notations of a Euclidean geometry. In addition, stating a helicoidal surface, we define strophoidal surface, and calculate its Gauss map, Gaussian curvature, mean curvature. Finally, we give some relations of the Gaussian curvature and the mean curvature of that kind surfaces.

Keywords: 3-space, strophoidal surface, Gauss map, Gaussian curvature, mean curvature.

Strophoidal Yüzeyler

Öz

Matematikçiler yüzyıllardır yüzeyler teorisini araştırmışlardır. Bu çalışmada, üç boyutlu Öklid uzayı $\mathbb{E}^3$'de strophoidal yüzeyi ele aldık. Öklid geometrisinin notasyonlarını sunduk. İlave olarak, bir helisoidal yüzeyi vererek, strophoidal yüzeyi tanımladık ve Gauss tasvirini, Gauss eğriliğini, ortalama eğriliğini hesapladık. Son olarak, bu tür yüzeylerin ortalama eğriliği ve Gauss eğriliği ile ilgili bazı bağıntıları verdik.

Anahtar Kelimeler: 3-boyut, strophoidal yüzey, Gauss tasviri, Gauss eğriliği, ortalama eğrilik.

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1. Introduction

The surface theory has been worked for a long years. We see some books for the topic in the literature, such as [1-6].

We consider the strophoidal surface in 3-space. We indicate the notions of 3-space in Section 1. In Section 2, we give helicoidal surface and then, we reveal strophoidal surface, compute its Gauss and mean curvatures. We give some relations for the curvatures of the surface. We serve a conclusion in the last section.

We consider identify a vector $(\mathrm{p}, \mathrm{q}, \mathrm{r})$ with transpose of it. Next, in $\mathbb{E}^3$, we describe the fundamental forms I, II, shape operator matrix $\boldsymbol{\mathcal{S}}$, Gauss curvature K, mean curvature H of the surface $\mathbf{s} = \mathbf{s}(u, v)$.

Let $\mathbf{s}$ be a surface M^2 in $\mathbb{E}^3$. The outer product of $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ on $\mathbb{E}^3$ is defined by

$$\vec{\alpha} \times \vec{\beta} = det \begin{pmatrix} e_1 & e_2 & e_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}.$$

We consider the following matrices

$$I = (g_{ij})_{2\times 2},$$

and

$$II = (h_{ij})_{2\times 2},$$

where

$$g_{11} = \mathbf{s}_u \cdot \mathbf{s}_u,$$

$$g_{12} = \mathbf{s}_u \cdot \mathbf{s}_v = g_{21},$$

$$g_{22} = \mathbf{s}_v \cdot \mathbf{s}_v,$$

$$h_{11} = \mathbf{s}_{uu} \cdot n,$$

$$h_{12} = \mathbf{s}_{uv} \cdot n = h_{21},$$

$$h_{22} = \mathbf{s}_{vv} \cdot n,$$

" $\cdot$ " is a Euclidean inner product, the unit normal (i.e. the Gauss map) of the surface is given by

$$n = \frac{\mathbf{s}_u \times \mathbf{s}_v}{\|\mathbf{s}_u \times \mathbf{s}_v\|}.$$

We have $I^{-1}.II$, and it gives the following shape operator matrix

$$\boldsymbol{\mathcal{S}} = \frac{1}{detI} \begin{pmatrix} g_{22}h_{11} - g_{12}h_{12} & g_{22}h_{12} - g_{12}h_{22} \\ g_{11}h_{12} - g_{12}h_{11} & g_{11}h_{22} - g_{12}h_{12} \end{pmatrix}.$$

So, we obtain the following Gaussian curvature K and mean curvature H formulas

$$K = det(\boldsymbol{\mathcal{S}})$$

$$= \frac{h_{11}h_{22} - {h_{12}}^2}{g_{11}g_{22} - {g_{12}}^2}$$

and

$$H = \frac{1}{2} tr(\boldsymbol{\mathcal{S}})$$

$$= \frac{g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}}{2(g_{11}g_{22} - {g_{12}}^2)}.$$

The surface $\mathbf{s}$ is flat when $K = 0$, and it is minimal $H = 0$.

2. Strophoidal Surface

In $\mathbb{E}^3$, we will give the surface of rotation and the helicoidal surface.

Consider open interval I, let $\gamma : \mathrm{I} \subset \mathbb{R} \longrightarrow \Pi$ be a curve, and ℓ be a line in Π.

We state the rotational surface as a surface rotating the profile curve γ about the axis ℓ.

The profile curve rotates about ℓ, it replaces parallel lines orthogonal to ℓ, then the accelerate of replacement is in proportion to the accelerate of rotation.

Therefore, the above surface is named the *helicoidal surface* having axis ℓ, pitch $\wp \in \mathbb{R}^+$.

The orthogonal matrix is given by

$$M(v) = \begin{pmatrix} cosv & -sinv & 0 \\ sinv & cosv & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here, $v \in \mathbb{R}$. The matrix M supplies the following, simultaneously,

$$M.\ell = \ell, \quad M^t.M = M.M^t = \mathfrak{I}_3, \ detM = 1,$$

where $\mathfrak{I}_3$ is the identity matrix.

When the rotation axis be ℓ, there is a transformation transformed ℓ to the axis x_3.

The generating curve is given by

$$\gamma(u) = (\mathscr{f}(u), 0, \hbar(u)),$$

where $\mathscr{f}(u), \hbar(u) \in C^k(\mathrm{I}, \mathbb{R})$.

Hence, the helicoidal surface spanned by the $(0,0,1)$ having pitch $\wp$, is defined by

$$\boldsymbol{\mathcal{H}}(u,v) = M(u).\gamma(u) + p\ v\ \ell^t,$$

where $u \in \mathrm{I}$, $v \in [0, 2\pi)$.

So, we have the following helicoidal surface

$$\boldsymbol{\mathcal{H}}(u,v) = \begin{pmatrix} \mathscr{f}(u)cosv \\ \mathscr{f}(u)sinv \\ \hbar(u) + \wp v \end{pmatrix}.$$

When $\wp = 0$, the helicoidal surface transforms to the rotational surface.

In $\mathbb{E}^2$, a strophoid curve is given by

$$c(u) = (\delta, u\delta).$$

where

$$\delta = \frac{u^2 - 1}{u^2 + 1}.$$

In $\mathbb{E}^3$, a strophoidal surface (see Figure 1) spanned by the $(0,0,1)$, has pitch $\wp \in \mathbb{R}^+$, (see Figure 2 for $\wp = 0$) is defined by

$$\mathcal{H}(u,v) = \begin{pmatrix} \delta(cosv - usinv) \\ \delta(sinv + ucosv) \\ \varphi(u) + \wp v \end{pmatrix},$$

where the generating space curve is presented by

$$\gamma(u) = \left(\delta, u\delta, \varphi(u)\right),$$

where, $\varphi \in C^k(\mathrm{I}, \mathbb{R})$, $u \in \mathrm{I}$, $\wp \in \mathbb{R}^+$, $v \in [0, 2\pi)$.

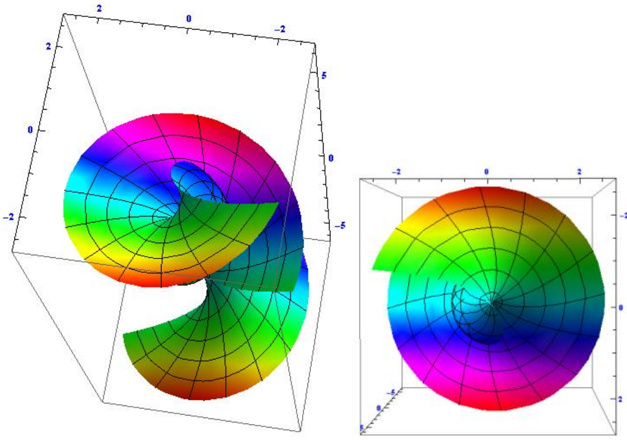

Figure 1. Left: Strophoidal surface, Right: Its top view

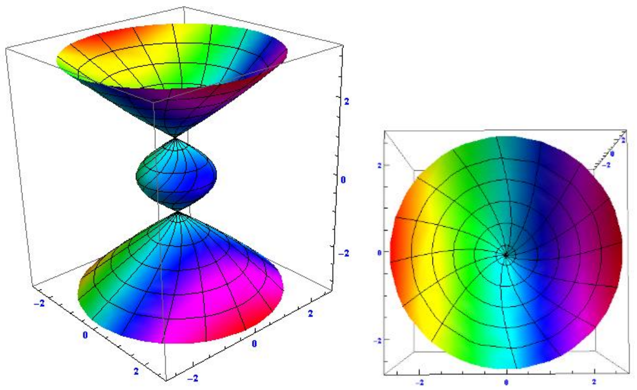

Figure 2. Left: Stropho-rotational surface, Right: Its top view

With the first differentials of the strophoidal surface $\mathcal{H}(u,v)$ depends on u and v, we obtain the following quantities

$$g_{11} = {\varphi'}^2 + \mu^2 + 1,$$

$$g_{12} = \delta^2 + \wp\varphi',$$

$$g_{22} = \xi^2 + \wp^2,$$

and then, we get

$$\det I = \xi^2\varphi'^2 - 2\wp\delta^2\varphi' + (\mu^2+1)(\xi^2+\wp^2) - \delta^4,$$

where

$$\mu = \frac{2u}{u^2+1},$$

$$\xi = \frac{u^2-1}{(u^2+1)^{1/2}},$$

$$\delta = \frac{u^2-1}{u^2+1}.$$

The Gauss map of the surface is given by

$$n = \frac{1}{(\det I)^{1/2}}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix},$$

where

$$n_1 = \frac{1}{(u^2+1)^2}\{(1-u^4)(\cos v - u\sin v)\varphi' + \left((u^4+4u^2-1)\cos v + 4u\sin v\right)\wp\},$$

$$n_2 = \frac{1}{(u^2+1)^2}\{(1-u^4)(\sin v - u\cos v)\varphi' + \left((u^4+4u^2-1)\sin v - 4u\cos v\right)\wp\},$$

$$n_3 = u - \frac{4u}{(u^2+1)^2}.$$

In the end, the mean curvature of the strophoidal surface is given by

$$H = \frac{\mathfrak{H}(u)}{2\aleph(u)^{3/2}},$$

where

$$\begin{aligned}\mathfrak{H}(u) = -12\Bigg(&-\frac{1}{12}(u^2-1)^2(u^2+1)^3\varphi'^3 \\ &+ \wp\frac{1}{4}(u^8-2u^4+1)\varphi'^2 \\ &- \frac{1}{6}\left(\frac{1}{2}u^4 + (\wp^2-1)u^2 + \frac{1}{2} + \wp^2\right)(u^6 \\ &+ 9u^4 + 3u^2 + 3)\varphi' \\ &- \frac{1}{12}u(u^4-1)(u^2+3)(u^4+(\wp^2-2)u^2 \\ &+ \wp^2+1)\varphi'' \\ &+ \left(-\frac{1}{12}u^8 + \frac{2}{3}u^6 + \left(\wp^2 - \frac{5}{6}\right)u^4 \\ &+ \frac{1}{3}(4u^2+1)\wp^2 + \frac{1}{4}\right)\wp\Bigg)(u^2+1)^2,\end{aligned}$$

and the Gaussian curvature of the strophoidal surface is given by

$$K = \frac{\mathfrak{K}(u)}{\aleph(u)^2},$$

where

$$\begin{aligned}\mathfrak{K}(u) - (u^2+1)^3\Bigg(&-3(u^2+1)(u-1)^4(u+1)^4\varphi'^2 \\ &- 2(u-1)^3\left(\frac{1}{2}u(u^2+3)(u^2+1)^2\varphi'' \\ &+ \wp(u^4+3)\right)(u+1)^3\varphi' \\ &+ \wp\Big((u^{11}+u^9-6u^7+2u^5+5u^3-3u)\varphi'' \\ &+ \wp(u^{10}+13u^8+38u^6+70u^4+9u^2 \\ &- 3)\Big)\Bigg)\end{aligned}$$

and

$$\begin{aligned}\aleph(u) = {}&(u^2-1)^2(u^2+1)^3\varphi'^2 + 2(-u^8+2u^4-1)\wp\varphi' \\ &+ u^{10} + (\wp^2+4)u^8 + (8\wp^2-2)u^6 \\ &+ (14\wp^2-12)u^4 + (8\wp^2+9)u^2 + \wp^2.\end{aligned}$$

3. Conclusion

Corollary 3.1. *Let $\mathcal{H} : M^2 \longrightarrow \mathbb{E}^3$ be an immersion given by $\mathcal{H}(u,v)$. M^2 is minimal iff*

$$\mathfrak{H}(u) = 0.$$

Corollary 3.2. *Let $\mathcal{H} : M^2 \longrightarrow \mathbb{E}^3$ be an immersion given by $\mathcal{H}(u,v)$. M^2 is flat iff*

$$\mathfrak{K}(u) = 0.$$

Corollary 3.3. *Let $\mathcal{H} : M^2 \longrightarrow \mathbb{E}^3$ be an immersion given by $\mathcal{H}(u,v)$. M^2 has the following relation of the Weingarten*

$$H^2 = \frac{\mathfrak{H}^2}{4\mathfrak{K}} K.$$

Corollary 3.4. *Let $\mathcal{H} : M^2 \longrightarrow \mathbb{E}^3$ be an immersion given by $\mathcal{H}(u,v)$. M^2 has umbilic point if and only if*

$$\mathfrak{H} = \mp 2\mathfrak{K}^{1/2}.$$

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