Commun. Fac. Sci. Univ. Ank. Series A_1
V. 44. pp. 85-94 (1995)

A CLASS OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE AND FIXED FINITELY MANY COEFFICIENTS

M.K. AOUF, and H.M. HOSSEN

Department of Mathematics Faculty of Science University of Mansoura, Mansoura, Egypt

(Received June 30, 1994; Accepted Dec. 7, 1995)

ABSTRACT

The object of the present paper is to derive several interesting properties of the class Σp, c_k (α, β, γ) consisting of regular and univalent meromorphic functions with positive and fixed finitely many coefficients. These include coefficient estimates, closure theorems, and radius of convexity for functions belonging to the class Σp, c_k (α, β, γ).

1. INTRODUCTION

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $U^* = \{z: 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. And let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.2}$$

that are analytic and univalent in U^*. Recently Cho, Lee and Owa [2] investigated the class Σ_p (α, β, γ) which is a subclass of Σ_p, defined as follows:

A function $f(z)$ in Σ_p is in the class Σ_p (α, β, γ) if it satisfies the condition

$$\left| \frac{z^2 f'(z) + 1}{(2\gamma - 1) z^2 f'(z) + (2\alpha\gamma - 1)} \right| < \beta \quad (z \in U^*) \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$, β $(0 < \beta \leq 1)$, and γ $(\frac{1}{2} \leq \gamma \leq 1)$.

For the class $\Sigma_p(\alpha, \beta, \gamma)$, Cho, Lee and Owa [2] proved the following lemma

Lemma 1. A function f(z) defined by (1.2) is in the class $\Sigma_p(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=1}^{\infty} n(1 + 2\beta\gamma - \beta) a_n \leq 2\beta\gamma(1-\alpha) \tag{1.4}$$

for some α $(0 \leq \alpha < 1)$, β $(0 < \beta \leq 1)$ and γ $(\frac{1}{2} \leq \gamma \leq 1)$. The result is sharp.

In view of Lemma 1 all functions belonging to the class $\Sigma_p(\alpha, \beta, \gamma)$ satisfy the coefficient inequality

$$a_n \leq \frac{2\beta\gamma(1-\alpha)}{n(1 + 2\beta\gamma - \beta)} \quad (n \geq 1). \tag{1.5}$$

Making use of (1.5), we now introduce the following class of functions:

Let $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$ denote the subclass of $\Sigma_p(\alpha, \beta, \gamma)$ consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)c_i}{i(1 + 2\beta\gamma - \beta)} z^i + \sum_{n=k+1}^{\infty} a_n z^n \tag{1.6}$$

where

$$a_n \geq 0,\ 0 \leq c_i \leq 1, \text{ and } 0 \leq \sum_{i=1}^{k} c_i \leq 1.$$

For k = 1, the class $\Sigma_{p, c_1}(\alpha, \beta, \gamma) = \Sigma_{p, c}(\alpha, \beta, \gamma)$ was introduced by Aouf and Hossen [1].

In this paper we obtain coefficient estimates and closure theorems for the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$. Further the radius of convexity is obtained for the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$. Techniques used are similar to those of Silverman and Silvia [4], Uralegaddi [5] and Owa and Srivastava [3].

2. COEFFICIENT ESTIMATES

Theorem 1. Let the function f(z) be defined by (1.6). Then f(z) is in the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta)\, a_n \leq 2\beta\gamma\, \beta\gamma\,(1-\alpha)\left(1-\sum_{i=1}^{k} c_i\right), \tag{2.1}$$

where

$$0 \leq c_i \leq 1 \text{ and } 0 \leq \sum_{i=1}^{k} c_i \leq 1.$$

The result (2.1) is sharp.

Proof. Putting

$$a_i = \frac{2\beta\gamma(1-\alpha)\, c_i}{i(1+2\beta\gamma-\beta)} \quad (i = 1, 2, \ldots, k), \tag{2.2}$$

in Lemma 1, we have

$$\sum_{i=1}^{k} 2\beta\gamma(1-\alpha)\, c_i + \sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta)\, a_n \leq 2\beta\gamma(1-\alpha), \tag{2.3}$$

which clearly implies (2.1). Further, by taking the function f(z) of the form

$$f(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)\, c_i}{i(1+2\beta\gamma-\beta)}\, z^i + \frac{2\beta\gamma(1-\alpha)\left(1-\sum_{i=1}^{k} c_i\right)}{n(1+2\beta\gamma-\beta)}\, z^n \tag{2.4}$$

for $n \geq k+1$, we can see that the result (2.1) is sharp.

Corollary 1. Let the function f(z) defined by (1.6) be in the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$. Then

$$a_n \leq \frac{2\beta\gamma(1-\alpha)\left(1-\sum_{i=1}^{k} c_i\right)}{n(1+2\beta\gamma-\beta)} \qquad (n \geq k+1). \tag{2.5}$$

The result (2.5) is sharp for the function f(z) given by (2.4).

3. CLOSURE THEOREMS

Theorem 2. Let the functions

$$f_j(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)c_i}{i(1+2\beta\gamma-\beta)} z^i + \sum_{n=k+1}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0) \qquad (3.1)$$

be in the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$ for every $j = 1, 2, \ldots, m$. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^{m} d_j f_j(z) \quad (d_j \geq 0) \qquad (3.2)$$

is also in the same class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$, where

$$\sum_{j=1}^{m} d_j = 1. \qquad (3.3)$$

Proof: Combining the definitions (3.1) and (3.2) we have

$$F(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)c_i}{i(1+2\beta\gamma-\beta)} z^i + \sum_{n=k+1}^{\infty} \left(\sum_{j=1}^{m} d_j a_{n,j}\right) z^n, \qquad (3.4)$$

where we have also used the relationship (3.3). Since $f_j(z) \in \Sigma_{p, c_k}(\alpha, \beta, \gamma)$ for every $j = 1, 2, \ldots, m$, Theorem 1 yields

$$\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta) a_{n,j} \leq 2\beta\gamma(1-\alpha)\left(1 - \sum_{i=1}^{k} c_i\right) \qquad (3.5)$$

for every $j = 1, 2, \ldots, m$. Thus we obtain

$$\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta)\left(\sum_{j=1}^{m} d_j a_{n,j}\right)$$

$$= \sum_{j=1}^{m} d_j \left(\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta) a_{n,j}\right)$$

$$\leq 2\beta\gamma(1-\alpha)\left(1 - \sum_{i=1}^{k} c_i\right)$$

which (in view of Theorem 1) implies that $F(z) \in \Sigma_{p, c_k}(\alpha, \beta, \gamma)$.

Theorem 3. Let the functions $f_j(z)$ defined by (3.1) be in the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$ for each $j = 1, 2, \ldots, m$, then the function $h(z)$ defined by

$$h(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)c_i}{i(1+2\beta\gamma-\beta)} z^i + \sum_{n=k+1}^{\infty} b_n z^n \quad (b_n \geq 0) \tag{3.6}$$

is also in the same class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$, where

$$b_n = \frac{1}{m} \sum_{j=1}^{m} a_{n, j}. \tag{3.7}$$

Proof: Since $f_j(z) \in \Sigma_{p, c_k}(\alpha, \beta, \gamma)$, it follows from Theorem 1, that

$$\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta) a_{n, j} \leq 2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i). \tag{3.8}$$

Hence

$$\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta) b_n$$

$$= \sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta)\left(\frac{1}{m}\sum_{j=1}^{m} a_{n, j}\right)$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta) a_{n, j}\right)$$

$$\leq 2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i) \tag{3.9}$$

and the result follows.

Theorem 4. The class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$ is closed under convex linear combination.

Proof: Let the function $f_j(z)$ $(j = 1, 2)$ defined by (3.1) be in the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$, it is sufficient to prove that the function $H(z)$ defined by

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \le \lambda \le 1) \tag{3.10}$$

is also in the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$.

Since

$$H(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha) c_i}{i(1+2\beta\gamma)-\beta} z^i + \sum_{n=k+1}^{\infty} \{\lambda a_{n,1} + (1-\lambda) a_{n,2}\} z^n, \tag{3.11}$$

we observe that

$$\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta) \{\lambda a_{n,1} + (1-\lambda) a_{n,2}\}$$

$$\le 2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i), \tag{3.12}$$

with the aid of Theorem 1. Hence $H(z) \in \Sigma_{p, c_k}(\alpha, \beta, \gamma)$. This completes the proof of Theorem 4.

Theorem 5. Let

$$f_k(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha) c_i}{i(1+2\beta\gamma-\beta)} z^i \tag{3.13}$$

and

$$f_n(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha) c_i}{i(1+2\beta\gamma-\beta)} z^i$$

$$+ \frac{2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{n(1+2\beta\gamma-\beta)} z^n \quad (n \ge k+1). \tag{3.14}$$

Then $f(z)$ is the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$ if and only if it can be expressed in the form of

$$f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z) \tag{3.15}$$

where $\lambda_n \ge 0$ $(n \ge k)$ and

$$\sum_{n=k}^{\infty} \lambda_n = 1. \tag{3.16}$$

Proof: We suppose that $f(z)$ can be expressed in the form of (3.15). Then it follows from (3.13), (3.14) and (3.16) that

$$f(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)c_i}{i(1+2\beta\gamma-\beta)} z^i$$

$$+ \sum_{n=k+1}^{\infty} \frac{2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{n(1+2\beta\gamma-\beta)} \lambda_n z^n. \tag{3.17}$$

Note that

$$\sum_{n=k+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i)}{n(1+2\beta\gamma-\beta)} \lambda_n$$

$$= 2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i) \sum_{n=k+1}^{\infty} \lambda_n = 2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i)(1-\lambda_k)$$

$$\leq 2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i), \tag{3.18}$$

which implies that $f(z) \in \Sigma_{p, c_k}(\alpha, \beta, \gamma)$.

For the converse, assume that the function $f(z)$ of the form (1.6) belongs to the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$. Since $f(z)$ satisfies (2.5) for $n \geq k+1$, we may set

$$\lambda_n = \frac{n(1+2\beta\gamma-\beta)a_n}{2\beta\gamma(1-\alpha)(1-\sum_{i=1}^{k} c_i)} \leq 1 \quad (n \geq k+1) \tag{3.19}$$

and

$$\lambda_k = 1 - \sum_{n=k+1}^{\infty} \lambda_n. \tag{3.20}$$

Hence f(z) has the representation (3.15). This evidently completes the proof of Theorem 5.

4. RADIUS OF CONVEXITY

Theorem 6. Let the function f(z) defined by (1.6) be in the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$. Then f(z) is meromorphically convex of order ρ $(0 \leq \rho < 1)$ in the disc $0 < |z| < r = r(\alpha, \beta, \gamma, c_i, \rho)$, where $r(\alpha, \beta, \gamma, c_i, \rho)$ is the largest value for which

$$\sum_{i=1}^{k} \frac{(i+2-\rho)\beta\gamma(1-\alpha)c_i}{(1+2\beta\gamma-\beta)} r^{i+1}$$

$$+\frac{(n+2-\rho)2\beta\gamma(1-\alpha)\left(1-\sum_{i=1}^{k} c_i\right)}{(1+2\beta\gamma-\beta)} r^{n+1} \leq 1-\rho, \tag{4.1}$$

for $n \geq k+1$. The result is sharp for the function f(z) given by (2.4).

Proof: It suffices to show that $\left|\frac{zf''(z)}{f'(z)}+2\right| \leq 1-\rho$ $(0 \leq \rho < 1)$ for $0 < |z| < r(\alpha, \beta, \gamma, c_i, \rho)$. Note that

$$\left|\frac{zf''(z)}{f'(z)}+2\right|$$

$$\leq \frac{\sum_{i=1}^{k} \frac{(i+1)2\beta\gamma(1-\alpha)c_i}{(1+2\beta\gamma-\beta)} r^{i+1} + \sum_{n=k+1}^{\infty} n(n+1)a_n r^{n+1}}{1-\sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)c_i}{(1+2\beta\gamma-\beta)} r^{i+1} - \sum_{n=k+1}^{\infty} na_n r^{n+1}}$$

$$\leq 1-\rho \tag{4.3}$$

for $0 < |z| \leq r$ if and only if

$$\sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)(i+2-\rho)c_i}{1+2\beta\gamma-\beta} r^{i+1} + \sum_{n=k+1}^{\infty} n(n+2-\rho)a_n r^{n+1} \leq 1-\rho. \tag{4.4}$$

Since f (z) is in the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$, from (2.5) we may take

$$a_n = \frac{2\beta\gamma(1-\alpha)\left(1-\sum_{i=1}^{k} c_i\right)}{n(1+2\beta\gamma-\beta)} \lambda_n \quad (n \geq k+1) \tag{4.5}$$

where $\lambda_n \geq 0$ $(n \geq k+1)$ and

$$\sum_{n=k+1}^{\infty} \lambda_n \leq 1. \tag{4.6}$$

For each fixed r, we choose the positive integer $n_0 = n_0(r)$ for which $\frac{(n+2-\rho)}{(1+2\beta\gamma-\beta)} r^{n+1}$ is maximal. Then it follows that

$$\sum_{n=k+1}^{\infty} n(n+2-\rho) a_n r^{n+1} \geq \frac{(n_0+2-\rho)\, 2\beta\gamma(1-\alpha)\left(1-\sum_{i=1}^{k} c_i\right)}{(1+2\beta\gamma-\beta)} r^{n_0+1}. \tag{4.7}$$

Hence f (z) is meromorphically convex of order ρ in $0 < |z| < r(\alpha, \beta, \gamma, c_i, \rho)$ provided that

$$\sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)(i+2-\rho)c_i}{(1+2\beta\gamma-\beta)} r^{i+1} + \frac{(n_0+2-\rho)\, 2\beta\gamma(1-\alpha)\left(1-\sum_{i=1}^{k} c_i\right)}{(a+2\beta\gamma-\beta)} r^{no+1} \leq 1-\rho. \tag{4.8}$$

We find the value $r_0 = r_0(\alpha\ \beta, \gamma, c_i, \rho)$ and the corresponding integer $n_0(r_0)$ so that

$$\sum_{i=1}^{k} \frac{2\beta\gamma(1-\alpha)(i+2-\rho)c_i}{(1+2\beta\gamma-\beta)} r_0^{i+1} + \frac{(n_0+2-\rho)\, 2\beta\gamma(1-\alpha)\left(1-\sum_{i=1}^{k} c_i\right)}{(1+2\beta\gamma-\beta)} r_0^{n+1} = 1-\rho. \tag{4.9}$$

Then this value r_0 is the radius of meromorphically convex of order ρ for functions f (z) belonging to the class $\Sigma_{p, c_k}(\alpha, \beta, \gamma)$.

REFERENCES

[1] M.K. AOUF, and H.M. HOSSEN., A class of meromorphic univalent functions with positive and fixed second coefficients, (submitted).

[2] N.E. CHO., S.H. LEE, and S. OWA., A class of meromorphic univalent functions with positive coefficients, Kobe J. Math. 4 (1987), no. 1, 43–50.

[3] S. OWA, and H.M. SRIVASTAVA., A class of analytic functions with fixed finitely many coefficients, J. Fac. Sci. Tech. Kinki Univ. 31 (1987), 1–10.

[4] H. SILVERMAN, and E.M. SILVIA., Fixed coefficients for subclasses of starlike functions, Houston J. Math. 7 (1981), 129–136.

[5] B.A. URALEGADDI., Meromorphically starlike functions with positive and fixed second second coefficient, Kyungpook Math. J. 29 (1989), no. 1, 64–68.