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F-Absolute Equivalence Of F-Regular Summability Methods

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F-Absolute Equivalence Of F-Regular Summability Methods

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SUMMARY

In this paper, we have defined F-absolute equivalence for F-regular summability methods and also determined necessary and sufficient conditions for F-regular methods to be F-absolutely equivalent for all bounded sequences.

1. INTRODUCTION

Let $A=(a_{nk})$ be an infinite matrix and $x=(x_k)$ be a sequence with complex terms. If the series

$$A_n(x) = \sum_{k=o}^{\infty} a_{nk} x_k$$

is convergent for each n, then the sequence $A(x) = (A_n(x))$ is called the A-transform of the sequence $x = (x_k)$.

Let l_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$, respectively.

The matrix $A = (a_{nk})$ is defined to be regular if the A-transform of x is convergent to the limit of x for each $x \in c$. The regularity conditions of the matrix $A = (a_{nk})$ are known, [1].

A sequence $x = (x_n) \in l_\infty$ is almost convergent if and only if

$$\lim_p \frac{x_n + x_{n+1} + \ldots + x_{n+p}}{p+1} = s$$

uniformly in n, [3], this limit is denoted by f-lim $x = s$. We denote by F and F_0 the linear space of the sequences which are almost convergent and almost convergent to zero, respectively.

Throughout the paper, the sums will be taken from $k = 0$ to $k = \infty$.

Now we recall some known theorems.

Theorem 1.1: A matrix $A = (a_{nk})$ transforms l_∞ into F_0, i.e., $A \in (l_\infty, F_0)$, if and only if

i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$

ii) $\lim_q \sum_{k=o}^{\infty} \left| \frac{1}{q+1} \sum_{i=o}^{q} a_{n+i,k} \right| = 0$

uniformly in n, [2].

Theorem 1.2: A matrix $A = (a_{nk})$ transforms F into F and f—lim $A(x)$ = f—lim x for each $x \in F$ if and only if

i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$

ii) f—$\lim_n \sum_k a_{nk} = 1$

iii) f—$\lim_n a_{nk} = 0$, for each k,

iv) $\lim_q \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} \triangle a_{n+i,k} \right| = 0$,

uniformly in n, where $\triangle a_{n+i,k} = a_{n+i,k} - a_{n+i,k+1}$, [2].

We shall call the matrix $A = (a_{nk})$, F-regular if it transforms F into F and F—lim $A(x)$ = f—lim x for each $x \in F$ (i.e., $A \in (F, F; p)$). Hence Theorem 1.2 gives the necessary and sufficient conditions for a sequence method to be F-regular.

2. F-ABSOLUTE EQUIVALENCE

In [1], it is defined absolute equivalence for regular methods and also given necessary and sufficient conditions for regular methods to be absolutely equivalent for all bounded sequences.

Similary, we shall define F-absolute equivalence for F-regular summability methods and also determine necssary and sufficient conditions for F-regular methods to be F-absolutely equivalent for all bounded sequences.

Let $A = (a_{nk})$ and $B = (b_{nk})$ be two infinite matrices and let $x = (x_k)$ be a sequence such that

$$(1) \qquad z'_n = \sum_k a_{nk} x_k \text{ and } z''_n = \sum_k b_{nk} x_k$$

exits for each n

We now make a definition.

Definition 2.1: Wit the notation of (1) the F-regular methods A and B are said to be F-absolutely equivalent for a given class of sequences (x_k) whenever

$$f\text{—}\lim (z'_n - z''_n) = 0$$

i.e., either (z'_n) and (z''_n) both is almost convergent to the same value, or else neither of them is almost convergent, but their difference is almost convergent to zero.

Let us give some lemmas.

Lemma 2.2: Let the infinite matrix $A = (a_{nk})$ be F-regular. Let us assume that $B = (b_{nk})$ be an infinite matrix such that

$$(2) \qquad \|B\| = \sup_n \sum_k |b_{nk}| < \infty .$$

If the condition

$$(3) \qquad \lim_q \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} (a_{n+i,k} - b_{n+i,k}) \right| = 0, \text{ uniformly in n,}$$

is satisfied then the method B is also F-regular.

Proof: Let condition (3) be hold. If we now set

$$C = (c_{nk}) = (a_{nk} - b_{nk}), \text{ for all n, k,}$$

then $C \in (l_\infty, F_0)$, since the conditions of Theorem 1.1 are satisfied. It is now easy to see that the conditions (i) - (iii) of Theorem 1.2 are satisfied for the matrix C. Therefore, metioned conditions are also satisfied for the matrix B, since A is F-regular. On the other hand, we can write

$$\sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} (\triangle a_{n+i,k} - \triangle b_{n+i,k}) \right|$$

$$\leq \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} (b_{n+i,k} - a_{n+i,k}) \right|$$

$$+ \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} (b_{n+i,k+1} - a_{n+i,k+1}) \right|$$

By considering (3), it is obtained that

$$\lim_{q} \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} (\triangle a_{n+i,k} - \triangle b_{n+i,k}) \right| = 0, \tag{4}$$

uniformly in n. Furthermore, we get

$$\sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} \triangle b_{n+i,k} \right| \leq \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} (\triangle b_{n+i,k} - \triangle a_{n+i,k}) \right| + \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} \triangle a_{n+i,k} \right| .$$

Now, statement (4) and the F-regularity of A together imply that

$$\lim_{q} \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} \triangle b_{n+i,k} \right| = 0, \text{ uniformly in n.}$$

Hence B is F-regular which proves the lemma.

The author wishes to thank E. Öztürk for kindly pointing out that Lemma 2.2 can be proved in a shorter way as follows:

With the notation of (1), for all $x \in l_\infty$, by (3) we get

$$\lim_{q} \left| \frac{1}{q+1} \sum_{v=n}^{n+q} (z'_v - z''_v) \right| = \lim_{q} \left| \sum_{k=o}^{\infty} \frac{1}{q+1} \sum_{i=o}^{q} (a_{n+i,k} - b_{n+i,k}) x_k \right| \leq \|x\| . \lim_{q} \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} (a_{n+i,k} - b_{n+i,k}) \right|$$

$$= 0, \text{ uniformly in n,}$$

where $\|x\| = \sup_k |x_k|$. Hence, we get

$$(5) \qquad f\text{—}\lim (z'_n - z''_n) = 0 .$$

But, since A is F-regular and $F \subset l_\infty$, from (5) we must have B is F-regular.

Lemma: 2.3: Let the matrix $A = (a_{nk})$ be F-regular, and (2) be hold for B. If (3) is valid, then A and B are F-absolutely equivalent for all bounded sequences.

Proof: According to Lemma 2.2, B is also F-regular, since (3) is satisfied. By (1), for all $x \in l_\infty$,

$$z'_n - z''_n = \sum_k (a_{nk} - b_{nk}) x_k$$

$$\lim_q \left| \frac{1}{q+1} \sum_{v=n}^{n+q} (z'_v - z''_v) \right| = \lim_q \left| \sum_{k=o}^{\infty} \frac{1}{q+1} \sum_{i=o}^{q} (a_{n+i,k} - b_{n+i,k}) \right| \leq \|x\| . \lim_q \sum_{k=o}^{\infty} \frac{1}{q+1} \left| \sum_{i=o}^{q} (a_{n+i,k} - b_{n+i,k}) \right| = 0$$

uniformly in n, where $\|x\| = \sup_k |x_k|$. This shows that

$$f\text{—}\lim (z'_n - z''_n) = 0$$

and the lemma is therefore proved.

Lemma 2.4: Let the matrices A and B be given as in Lemma 2.3. If $f\text{—}\lim (z'_n - z''_n) = 0$, then condition (3) holds, where z'_n and z''_n is defined as in (1), for all $x \in l_\infty$.

Proof: Let us define the matrix $C = (c_{nk})$ as follows:

$$C = (c_{nk}) = (a_{nk} - b_{nk})$$

for all n, k. If $f\text{—}\lim (z'_n - z''_n) = 0$, for all $x \in l_\infty$, then we get $C \in (l_\infty, F_o)$. Thus, by Theorem 1.1, condition (3) is obtained. This completes the proof.

Our final result concerns the F-absolute equivalence which gives necessary and sufficient conditions for F-regular methods to be F-absolutely equivalent for all bounded sequences.

Theorem 2.5: Let A and B be two F-regular methods. Then A and B are F-absolutely equivalent for all bounded sequences if and only if condition (3) holds.

Proof: The result follows immediately from Lemma 2.3 and Lemma 2.4.

Remark. In the last theorem, to prove the sufficiency we may take the matrix B as in Lemma 2.2, instead of the F-regularity of B. Because, if (3) holds, then, by Lemma 2.2, B will be necessarily F-regular. But, for the proof of necessity we have take the methods A and B to be F-regular. Because, by Definition 2.3, we know that the methods must be, firstly, F-regular to be F-absolutely equivalent.

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ÖZET

Bu çalışmada, F-regüler toplanabilme metotları için F- mutlak denklik tanımı verilmiş ve bu tip metotların, sınırlı diziler uzayı üzerinde F- mutlak denk olmaları için gerek ve yeter şartlar belirlenmiştir.