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Absolute Summability Factors of Infinite Series

by

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Absolute Summability Factors of Infinite Series

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In this paper we have proved the following theorem which generalizes a theorem of Pati [M.Z. 78 (1962), 293–297] and also a theorem of Prasad and Bhatt [Duke Math. J. 24 (1957), 103–117]

THEOREM *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\Sigma \frac{\lambda_n}{n} < \infty$. *If*

$$\sum_{\upsilon=1}^{n} \frac{|s_\upsilon|}{\upsilon} = O\,(\log n\; \mu_n),$$

where $\{\mu_n\}$ *is a positive non-decreasing sequence such that*

$n \log n\; \mu_n\, \Delta\, (\frac{1}{\mu_n}) = O\,(1),\; n \to \infty,$ *then* $\Sigma\, \frac{a_n \lambda_n}{\mu_n}$ *is summable* $|C, 1|$.

1.1. Let $\Sigma\, a_n$ be any given infinite series with partial sums s_n, and let $t_n = n\; a_n$. By $\{\overset{\alpha}{\sigma}_n\}$ and $\{\overset{\alpha}{t}_n\}$ we denote the n-th Cesàro mean of order α $(\alpha > -1)$ of the sequences $\{s_n\}$ and $\{t_n\}$ respectively.

The following identity is well known [1].

$$\overset{\alpha}{t}_n = n(\overset{\alpha}{\sigma}_n - \overset{\alpha}{\sigma}_{n-1}). \tag{1.1.1}$$

The series $\Sigma\, a_n$ is said to be absolutely summable (C, α), or summable $|C, \alpha|$, $\alpha > -1$, if $\{\overset{\alpha}{\sigma}_n\}$ is a sequence of bounded variation, i. e., if

(1.1.2) $$\sum_n |\overset{\alpha}{\sigma}_n - \overset{\alpha}{\sigma}_{n-1}| < \infty .$$

Let

$$t_n^* = \frac{1}{\log (n+1)} \sum_{\nu=0}^{n} \frac{s_\nu}{n - \nu + 1} .$$

If $t_n^* \in$ B. V., we say that $\Sigma\, a_n$ is absolutely Harmonic summable or simply summable $|N, \frac{1}{n+1}|$.

We write, throughout this paper, for any sequence $\{p_n\}$

$$\Delta\, p_n = p_n - p_{n+1}, \qquad \Delta^2\, p_n = \Delta\, (\Delta\, p_n).$$

A sequence $\{\lambda_n\}$ is said to be convex if $\Delta^2 \lambda_n \geqq 0$ for every positive integer n.

If $\sum_{\nu=1}^{n} \frac{|s_\nu|}{\nu} = O(\log n)$, as $n \to \infty$, then $\Sigma\, a_n$ is said to be strongly bounded [R, log n, 1].

The object of this paper is to prove the following theorem.

1.2 THEOREM 1. *Let* $\{\lambda_n\}$ *be a convex sequence such that*

$$\Sigma \frac{\lambda_n}{n} < \infty . \textit{ If}$$

(1.2.1) $$\sum_{\nu=1}^{n} \frac{|s_\nu|}{\nu} = O(\log n\; \mu_n),$$

where $\{\mu_n\}$ *is a positive non-decreasing sequence such that*

$$n \log n\; \mu_n\; \Delta \left(\frac{1}{\mu_n}\right) = O(1),\; n \to \infty ,$$

then $\Sigma \frac{a_n \lambda_n}{\mu_n}$ *is summable* $|C, 1|$.

It may be remarked that if we consider the special case $\mu_n = 1$, we obtain the following theorem of Pati [5]

THEOREM A. *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\Sigma \frac{\lambda_n}{n} < \infty$. *If* Σa_n *is bounded* $[R, \log n, 1]$, *then* $\Sigma a_n \lambda_n$ *is summable* $|C, 1|$.

On the other hand, if we take $\mu_n = (\log n)^k$ $(k \geqq 0)$, then it is easy to see that our theorem generalizes the following theorem of Prasad and Bhatt [4].

THEOREM B. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma \frac{\lambda_n}{n} < \infty$ *and* $\sum_{\nu=1}^{n} |s_\nu - s| = 0 \{n(\log n)^k\}$, $k \geqq 0$, *as* $n \to \infty$, *then the series* $\Sigma (\log (n+1))^{-k} \lambda_n a_n$ *is summable* $|C, 1|$.

1.3. For the proof of this theorem we require the following lemmas.

LEMMA 1. [2]. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma \frac{\lambda_n}{n} < \infty$, *then* $\{\lambda_n\}$ *is non-negative decreasing sequence and* $\lambda_n \log n = O(1)$, $n \to \infty$.

LEMMA 2. [3]. *If* $\{\lambda_n\}$ *be a convex sequence such that* $\Sigma \frac{\lambda_n}{n} < \infty$, *then* $\sum_n \log (n+1) \Delta \lambda_n < \infty$, *and* $m \log (m+1) \Delta \lambda_m = O(1)$, *as* $m \to \infty$.

LEMMA 3. [6]. *Under the hypothesis of Lemma* 1 *we have as*

$$m \to \infty, \quad \sum_{n=1}^{m} n \log (n+1) \Delta^2 \lambda_n = O(1), \text{ as } m \to \infty.$$

1.4. PROOF OF THE THEOREM. By virtue of the relation (1.1.1) it is sufficient to show that

$$\sum_{n} n^{-1} \mid T_n \mid < \infty$$

where $\{T_n\}$ is the n–th Cesàro mean of order 1 of the sequence $\{\dfrac{n \lambda_n a_n}{\mu_n}\}$ that is

$$T_n = (n+1)^{-1} \sum_{\nu=1}^{n} \frac{\nu \lambda_\nu a_\nu}{\mu_\nu}$$

By Abel's transformation, we get

$$T_n = \frac{1}{n+1} \sum_{\nu=1}^{n-1} s_\nu \Delta \left(\frac{\nu \lambda_\nu}{\mu_\nu}\right) + \frac{1}{n+1} \frac{s_n n \lambda_n}{\mu_n} - \frac{s_0 \lambda_1}{(n+1) \mu_1}$$

$$= \left[\frac{1}{n+1} \left\{ \sum_{\nu=1}^{n-1} s_\nu \left(\frac{\nu \Delta \lambda_\nu}{\mu_\nu} - \frac{\lambda_{\nu+1}}{\mu_{\nu+1}} + \nu \lambda_{\nu+1} \Delta \frac{1}{\mu_\nu} \right) \right\} \right] + \frac{s_n n \lambda_n}{(n+1) \mu_n} - \frac{s_0 \lambda_1}{(n+1) \mu_1}$$

$$= \frac{1}{n+1} \sum_{\nu=1}^{n-1} s_\nu \frac{\nu \Delta \lambda_\nu}{\mu_\nu} - \frac{1}{n+1} \sum_{\nu=1}^{n-1} \frac{s_\nu \lambda_{\nu+1}}{\mu_{\nu+1}}$$

$$+ \frac{1}{n+1} \sum_{\nu=1}^{n-1} s_\nu \,.\, \nu \,.\, \lambda_{\nu+1} \Delta \left(\frac{1}{\mu_\nu}\right)$$

$$+ \frac{n s_n \lambda_n}{(n+1) \mu_n} - \frac{s_0 \lambda_1}{(n+1) \mu_1}$$

$$= K_1 + K_2 + K_3 + K_4 + K_5, \text{ say.}$$

Now

$$\sum_{1}^{m} \frac{|K_1|}{n} = O \left(\sum_{1}^{m} \frac{1}{n(n+1)} \sum_{\nu=1}^{n-1} |s_\nu| \frac{\nu \Delta \lambda_\nu}{\mu_\nu} \right)$$

$$= O\left(\sum_{1}^{m} \frac{|s_\nu|}{\nu} \frac{\nu \Delta \lambda_\nu}{\mu_\nu}\right)$$

$$= O\left[\sum_{1}^{m-1} \Delta\left(\frac{\nu \Delta \lambda_\nu}{\mu_\nu}\right) \sum_{\mu=1}^{\nu} \frac{|s_\mu|}{\mu} + \frac{m \Delta \lambda_m}{\mu_m} \sum_{\mu=1}^{m} \frac{|s_\mu|}{\mu}\right]$$

$$= O\left[\sum_{\nu=1}^{m-1} \left| \Delta\left(\frac{\nu \Delta \lambda_\nu}{\mu_\nu}\right) \right| \log \nu \mu_\nu\right] + O\left[\frac{m \Delta \lambda_m}{\mu_m} \log m . \mu_m\right]$$

$$= O\left[\sum_{\nu=1}^{m-1} \left| \Delta\left(\frac{\nu \Delta \lambda_\nu}{\mu\nu}\right) \right| (\log \nu \mu_\nu)\right] + O\left[m \Delta \lambda_m \log m\right]$$

$$= O\left[\sum_{\nu=1}^{m-1} \left| \Delta\left(\frac{\nu \Delta \lambda_\nu}{\mu_\nu}\right) \log \nu . \mu_\nu .\right] + O(1).$$

$$= O\left[\sum_{\nu=1}^{m-1} \log \nu \mu_\nu \left(\frac{\Delta^2 \lambda_\nu . \nu}{\mu_\nu} + \frac{\Delta \lambda_\nu}{\mu_\nu} + \Delta \lambda_\nu . \nu \Delta \frac{1}{\mu_\nu}\right)\right] + O(1)$$

$$= O\left[\sum_{\nu=1}^{m-1} \nu \Delta^2 \lambda_\nu \log \nu + O\left[\sum_{\nu=1}^{m-1} \log \nu \Delta \lambda_\nu\right] + O\left[\sum_{\nu=1}^{m-1} \nu \log \nu \mu_\nu \Delta \lambda_\nu \Delta \frac{1}{\mu_\nu}\right] + O(1)\right.$$

$$= O(1)$$

by lemmas 1, 2, 3 and the hypothesis of the theorem. Also we have

$$\sum_{n=1}^{m} \frac{|K_2|}{n} = O\left(\sum_{n=1}^{m} \frac{1}{n(n+1)} \sum_{\nu=1}^{n-1} \frac{|s_\nu| \lambda_{\nu+1}}{\mu_{\nu+1}} \right)$$

$$= O\left(\sum_{\nu=1}^{m} \frac{|s_\nu|}{\nu} \frac{\lambda_\nu}{\mu_{\nu+1}} \right)$$

$$= O\left[\sum_{\nu=1}^{m-1} \sum_{\mu=1}^{\nu} \frac{|s_\mu|}{\mu} \Delta \left(\frac{\lambda_\nu}{\mu_{\nu+1}}\right) \right.$$

$$\left. + \sum_{\nu=1}^{m} \frac{|s_\nu|}{\nu} \frac{\lambda_m}{\mu_{m+1}} \right]$$

$$= O\left[\sum_{\nu=1}^{m-1} (\log\nu \, \mu_\nu) \Delta \left(\frac{\lambda_\nu}{\mu_{\nu+1}}\right) \right]$$

$$+ O\left[\log m \, \mu_m \frac{\lambda_m}{\mu_{m+1}} \right]$$

$$= O\left[\sum_{\nu=1}^{m-1} (\log\nu \, \mu_\nu) \Delta \left(\frac{\lambda_\nu}{\mu_{\nu+1}}\right) \right] + O\left[\lambda_m \log m\right]$$

$$= O\left[\sum_{\nu=1}^{m-1} (\log\nu \, \mu_\nu) \Delta \left(\frac{\lambda_\nu}{\mu_{\nu+1}}\right) \right] + O(1)$$

$$= O\left(\sum_{\nu=1}^{m-1} \log\nu \, \Delta \lambda_\nu \right) +$$

$$O\left(\sum_{\nu=1}^{m-1} \log\nu \frac{\lambda_\nu}{\nu} \nu \, \mu_\nu \, \Delta \frac{1}{\mu_{\nu+1}} \right) + O(1)$$

$$= O(1) + O\left(\sum_{\nu=1}^{m-1} \frac{\lambda_\nu}{\nu} \right) + O(1)$$

$$= O(1),$$

by the hypothesis of the theorem and lemmas 1 and 2.

Next

$$\sum_{1}^{m} \frac{|K_3|}{n} = O\left(\sum_{n=1}^{m} \frac{1}{n(n+1)} \sum_{\nu=1}^{n-1} |s_\nu| \nu \lambda_\nu \Delta \frac{1}{\mu_\nu} \right)$$

$$= O\left(\sum_{n=1}^{m} \frac{1}{n(n+1)} \sum_{\nu=1}^{n-1} \frac{|s_\nu| \lambda_\nu}{\log(\nu+1)} \frac{\log(\nu+1)}{\mu_\nu} \nu \mu_\nu \Delta \frac{1}{\mu_\nu} \right)$$

$$= O\left(\sum_{n=1}^{m} \frac{1}{n(n+1)} \sum_{\nu=1}^{n-1} \frac{|s_\nu| \lambda_\nu}{\log(\nu+1) \mu_\nu} \right)$$

$$= O\left(\sum_{\nu=1}^{m} \frac{|s_\nu|}{\nu} \frac{\lambda_\nu}{(\log(\nu+1) . \mu_\nu)} \right)$$

$$= O\left[\sum_{\nu=1}^{m-1} \sum_{\mu=1}^{\nu} \frac{|s_\mu|}{\mu} \Delta \left(\frac{\lambda_\nu}{\log(\nu+1) \mu_\nu} \right) + \sum_{\nu=1}^{m} \frac{|s_\nu|}{\nu} \frac{\lambda_m}{\log(m+1) \mu_m} \right]$$

$$= O\left[\sum_{\nu=1}^{m-1} \log\nu \, \mu_\nu \, \Delta \left(\frac{\lambda_\nu}{\log(\nu+1) \mu_\nu} \right) \right] + O\left[\frac{\log m \, \mu_m \lambda_m}{\log(m+1) \mu_m} \right]$$

$$= O\left[\sum_{\nu=1}^{m-1} \log\nu \frac{\Delta \lambda_\nu . \mu_\nu}{\log(\nu+1) \mu_\nu} \right] +$$

$$O\left[\sum_{\nu=1}^{m-1} \mu_\nu \log \frac{\lambda_\nu}{\mu \, \nu+1} \frac{1}{\nu (\log(\nu+1))^2} \right]$$

$$+ O\left[\sum_{\nu=1}^{m-1} \frac{\mu_\nu \log\nu\ \lambda_\nu}{\log(\nu+1)} \Delta \frac{1}{\mu_\nu}\right] + O(1)$$

$$= O\left(\sum_{\nu=1}^{m-1} \Delta \lambda_\nu\right) + O\left(\sum_{\nu=1}^{m-1} \frac{\lambda_\nu}{\nu}\right)$$

$$+ O\left(\sum_{\nu=1}^{m-1} \lambda_\nu\ \mu_\nu \Delta \frac{1}{\mu_\nu}\right) + O(1)$$

$$= O(1) + O\left(\sum_{1}^{m-1} \frac{\lambda_\nu}{\nu} \nu .\ \mu_\nu\ \Delta \frac{1}{\mu_\nu}\right)$$

$$= O(1),$$

by the hypothesis of the theorem. Also

$$\sum_{1}^{m} \frac{|K_4|}{n} = O\left(\sum_{n=1}^{m} \frac{|s_n|\ \lambda_n}{n\ \mu_n}\right)$$

$$= O\left(\sum_{n=1}^{m-1} \sum_{\nu=1}^{n} \frac{|s_\nu|}{\nu} \Delta\left(\frac{\lambda_n}{\mu_n}\right)\right.$$

$$\left. + \sum_{n=1}^{m} \frac{|s_n|}{n} \frac{\lambda_m}{\mu_m}\right)$$

$$= O\left(\sum_{n=1}^{m} \log n\ \mu_n\ \Delta\left(\frac{\lambda_n}{\mu_n}\right)\right) + O\left(\frac{\log m\ \mu_m\ \lambda_m}{\mu_m}\right)$$

$$= O\left(\sum_{n=1}^{m} \log n\ \mu_n\ \Delta(\lambda_n / \mu_n)\right) + O(1)$$

$$= O\left(\sum_{n=1}^{m} \log n\ \mu_n \frac{\Delta \lambda_n}{\mu_n}\right) +$$

$$O\left(\sum_{n=1}^{m} \log n\ \mu_n\ \lambda_{n+1}\ \Delta \frac{1}{\mu_n}\right) + O(1)$$

$$= O\left(\sum_{n=1}^{m} \log n \, \Delta \lambda_n\right) + O\left(\sum_{n=1}^{m} \frac{\lambda_n}{n}\right) + O(1)$$

$$= O(1),$$

by lemmas 1,2 and hypothesis of the theorem. Lastly

$$\sum_{1}^{m} \frac{|K_5|}{n} = O\left(\sum_{n=1}^{m} \frac{1}{n(n+1)}\right)$$

$$= O(1)$$

Therefore, we have

$$\sum_{1}^{m} \frac{|T_n|}{n} = O(1), \; m \longrightarrow \infty$$

This completes the proof of the theorem.

1.5. Concerning absolute Harmonic summability factors Singh [8] has recently proved the following theorem.

THEOREM C. *If* Σa_n *is summable* $|C, 1|$, *then* $\Sigma \frac{a_n \log n}{n}$ *is summable* $\left|N, \frac{1}{n+1}\right|$.

From the above theorem we deduce the following result for summability $\left|N, \frac{1}{n+1}\right|$.

THEOREM 2. *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\Sigma \frac{\lambda_n}{n} < \infty$. *If*

$$\sum_{1}^{n} \frac{|s_\nu|}{\nu} = O(\log n \, \mu_n), \; n \longrightarrow \infty$$

where $\{\mu_n\}$ *is a positive non-decreasing sequence such that*

$$n \log n \, \mu_n \, \Delta \left(\frac{1}{\mu_{n+1}}\right) = O(1), \; n \longrightarrow \infty,$$

then $\Sigma \dfrac{a_n \lambda_n \log n}{n \mu_n}$ *is summable* $\left| N, \dfrac{1}{n+1} \right|$.

Taking $\mu_n = 1$, we have the following theorem of Lal [7].

THEOREM D. *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\Sigma \dfrac{\lambda_n}{n} < \infty$. *If* Σa_n *is bounded* $[R, \log n, 1]$, *then* $\Sigma \dfrac{\lambda_n a_n \log(n+1)}{n}$ *is absolutely Harmonic summable.*

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ÖZET

Pati'nin bir teoremini [M. Z. 78 (1962), 293–297] ve aynı zamanda Prasad ve Bhatt'ın bir teoremini [Duke Math. J. 24 (1967), 103–117] genelleştiren bir teoremi bu makalede ispatladık.

TEOREM $\{\lambda_n\}$, $\Sigma \dfrac{\lambda_n}{n} < \infty$ *olacak şekilde bir konveks dizi olsun.*

$\{\mu_n\}$, *n log n* $\mu_n \Delta \left(\dfrac{1}{\mu_n}\right) = O(1)$, $n \to \infty$ *olacak şekilde pozitif bir azalmayan dizi olsun. Bu takdirde* $\sum_{\upsilon=1}^{n} \dfrac{|S\nu|}{\upsilon} = O$ $(\log n\ \mu_n)$ *ise,* $\Sigma \dfrac{a_n \lambda_n}{\mu_n}$ $|C, 1|$ *toplanabilir.*

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