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HERMITE-HADAMARD-FEJÉR INEQUALITIES FOR DOUBLE INTEGRALS

Hüseyin BUDAK and Mehmet Zeki SARIKAYA

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, TURKEY

Abstract. In this paper, we first obtain Hermite-Hadamard-Fejer inequalities for co-ordinated convex functions in a rectangle from the plane $\mathbb{R}^2$. Moreover, we give the some refinement of these obtained Hermite-Hadamard-Fejer inequalities utilizing two mapping. The inequalities obtained in this study provide generalizations of some result given in earlier works.

1. Introduction

The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard (see, e.g., [8], [22, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [4], [9]- [11], [14], [21], [25], [29], [30], [32]).

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called

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hsyn.budak@gmail.com-Corresponding author; sarikayamz@gmail.com

0000-0001-8843-955X; 0000-0002-6165-9242.

Hermite-Hadamard-Fejér inequalities, In [13], Fejer gave a weighted generalization of the inequalities (1) as the following:

Theorem 1. *$f : [a,b] \to \mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{2}$$

holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$ (i.e. $g(x) = g(a+b-x)$).

A formal defination for co-ordinated convex function may be stated as follows:

Definition 2. *A function $f : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x,u),(y,v) \in \Delta$ and $t,s \in [0,1]$, if it satifies the following inequality:*

$$\begin{aligned} & f(tx+(1-t)\ y, su+(1-s)\ v) \qquad (3) \\ \le\ & ts\ f(x,u)+t(1-s)f(x,v)+s(1-t)f(y,u)+(1-t)(1-s)f(y,v). \end{aligned}$$

The mapping f is a co-ordinated concave on Δ if the inequality (3) holds in reversed direction for all $t,s \in [0,1]$ and $(x,u),(y,v) \in \Delta$.

In [7], Dragomir proved the following inequalities which is Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 3. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ dydx \qquad (4) \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ &+ \left. \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ &\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp. The inequalities in (4) hold in reverse direction if the mapping f is a co-ordinated concave mapping.

Over the years, many papers are dedicated on the generalizations and new versions of the inequalities (4) using the different type convex functions. For the other Hermite-Hadamard type inequalities for co-ordinated convex functions, please refer to ([1], [3], [5] , [6], [23], [24], [20], [26], [27], [31])

Alomari and Darus proved the following Hermite-Hadamard-Fejér inequalities for double integrals in [2]:

Theorem 4. *Let $p : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ be a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. Let $f : \Delta \to \mathbb{R}$ be a co-ordinated convex on Δ, then we have the following Hermite-Hadamard-Fejer type inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx &\leq \frac{\int_a^b \int_c^d f(x,y)p(x,y)dydx}{\int_a^b \int_c^d p(x,y)dydx} \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

Moreover, Farid et al. established a weighted version of the inequalities (4) in [12]. Please see ([15]- [19], [28]) for other papers focused on Hermite-Hadamard-Fejér inequalities for co-ordinated convex functions.

The aim of this paper is to establish a new weighed generalizations of Hermite-Hadamard type integral inequalities (4). The results presented in this paper provide extensions of those given in [2], [7] and [12].

We will use the following lemma to proof of main result:

Lemma 5. *[2] Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a co-ordinated convex function and let*

$$a \leq y_1 \leq x_1 \leq x_2 \leq y_2 \leq b \text{ with } x_1 + x_2 = y_1 + y_2,$$

$$c \leq w_1 \leq v_1 \leq v_2 \leq w_2 \leq d \text{ with } v_1 + v_2 = w_1 + w_2.$$

Then for the convex partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(x) = f(x, y)$ for all $x \in [a, b]$ and $f_x\, [c, d] \to \mathbb{R}$, $f_x(y) = f(x, y)$ for all $y \in [c, d]$, the following hold:

$$f(x_1, s) + f(x_2, s) \leq f(y_1, s) + f(y_2, s), \quad \forall s \in [c, d] \tag{5}$$

and

$$f(t, v_1) + f(t, v_2) \leq f(t, w_1) + f(t, w_2), \quad \forall t \in [a, b]. \tag{6}$$

2. Hermite-Hadamard-Fejer Inequalities

Lets start the following Hermite-Hadamard-Fejer inequalities:

Theorem 6. *Let $p : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ be a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. Let $f : \Delta \to \mathbb{R}$ be a co-ordinated convex on Δ, then we have* *the following Hermite-Hadamard-Fejer type inequalities*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx \qquad (7)\\
\leq\ & \frac{1}{2}\int_a^b\int_c^d\left[f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},y\right)\right]p(x,y)dydx\\
\leq\ & \int_a^b\int_c^d f(x,y)p(x,y)dydx\\
\leq\ & \frac{1}{4}\int_a^b\int_c^d\left[f(x,c)+f(x,d)+f(a,y)+f(b,y)\right]p(x,y)dydx\\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx.
\end{aligned}$$

Proof. Since f is co-ordinated convex on Δ, if we define the mappings $f_x:[c,d]\to\mathbb{R}$, $f_x(y)=f(x,y)$ and $p_x:[c,d]\to\mathbb{R}$, $p_x(y)=p(x,y)$, then $f_x(y)$ is convex on $[c,d]$ and $p_x(y)$ is positive, integrable and symmetric about $\frac{c+d}{2}$ for all $x\in[a,b]$. If we apply the inequality (2) for the convex function $f_x(y)$, then we have

$$f_x\left(\frac{c+d}{2}\right)\int_c^d p_x(y)dy\leq\int_c^d f_x(y)p_x(y)dy\leq\frac{f_x(c)+f_x(d)}{2}\int_c^d p_x(y)dy. \qquad (8)$$

That is,

$$f\left(x,\frac{c+d}{2}\right)\int_c^d p(x,y)dy\leq\int_c^d f(x,y)p(x,y)dy\leq\frac{f(x,c)+f(x,d)}{2}\int_c^d p(x,y)dy. \qquad (9)$$

Integrating the inequality (9) with respect to x from a to b, we obtain

$$\begin{aligned}
\int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)p(x,y)dydx &\leq \int_a^b\int_c^d f(x,y)p(x,y)dydx \qquad (10)\\
&\leq \frac{1}{2}\int_a^b\int_c^d\left[f(x,c)+f(x,d)\right]p(x,y)dydx.
\end{aligned}$$

Similarly, as f is co-ordinated convex on Δ, if we define the mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(x)=f(x,y)$ and $p_y:[a,b]\to\mathbb{R}$, $p_y(x)=p(x,y)$, then $f_y(x)$ is convex on $[a,b]$

and $p_y(x)$ is positive, integrable and symmetric about $\frac{a+b}{2}$ for all $y \in [c,d]$. Utilizing the inequality (2) for the convex function $f_y(x)$, then we obtain the inequality

$$f_y\left(\frac{a+b}{2}\right)\int_a^b p_y(x)dx \le \int_a^b f_y(x)p_y(x)dx \le \frac{f_y(a)+f_y(b)}{2}\int_a^b p_y(x)dx, \tag{11}$$

i.e.

$$\begin{aligned} f\left(\frac{a+b}{2},y\right)\int_a^b p(x,y)dx &\le \int_a^b f(x,y)p(x,y)dx \\ &\le \frac{f(a,y)+f(b,y)}{2}\int_a^b p(x,y)dx. \end{aligned} \tag{12}$$

Integrating the inequality (12) with respect to y on $[c,d]$, we get

$$\begin{aligned} \int_a^b\int_c^d f\left(\frac{a+b}{2},y\right)p(x,y)dydx &\le \int_a^b\int_c^d f(x,y)p(x,y)dydx \\ &\le \frac{1}{2}\int_a^b\int_c^d \left[f(a,y)+f(b,y)\right]p(x,y)dydx. \end{aligned} \tag{13}$$

Summing the inequalities (10) and (13), we obtain the second and third inequalities in (7).

Since $f\left(\frac{a+b}{2},y\right)$ is convex on $[c,d]$ and $p_x(y)$ is positive, integrable and symmetric about $\frac{c+d}{2}$, using the first inequality in (2), we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d p_x(y)dy \le \int_c^d f\left(\frac{a+b}{2},y\right)p_x(y)dy. \tag{14}$$

Integrating the inequality (12) with respect to x on $[a,b]$, we get

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx \le \int_a^b\int_c^d f\left(\frac{a+b}{2},y\right)p(x,y)dydx. \tag{15}$$

Since $f\left(x,\frac{c+d}{2}\right)$ is convex on $[c,d]$ and $p_y(x)$ is positive, integrable and symmetric about $\frac{a+b}{2}$, utilizing the first inequality in (2), we have the following inequality

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx \le \int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)p(x,y)dydx. \tag{16}$$

From the inequalities (15) and (16), we have the first inequality in (7).

For the proof of last inequality in (7), using the second inequality in (2) for the convex functions $f(x,c)$ and $f(x,d)$ on $[a,b]$ and for the symmetric function $p_y(x)$, we obtain the inequalities

$$\int_a^b f(x,c)p_y(x)dx \leq \frac{f(a,c)+f(b,c)}{2}\int_a^b p_y(x)dx \tag{17}$$

and

$$\int_a^b f(x,d)p_y(x)dx \leq \frac{f(a,d)+f(b,d)}{2}\int_a^b p_y(x)dx. \tag{18}$$

Similarly, applying the second inequality in (2) for the convex functions $f(a,y)$ and $f(b,y)$ on $[c,d]$ and for the symmetric function $p_x(y)$, we have

$$\int_c^d f(a,y)p_x(y)dy \leq \frac{f(a,c)+f(a,d)}{2}\int_c^d p_x(y)dy \tag{19}$$

and

$$\int_c^d f(b,y)p_x(y)dy \leq \frac{f(b,c)+f(b,d)}{2}\int_c^d p_x(y)dy. \tag{20}$$

Integrating the inequalities (17) and (18) on $[c,d]$ and inequalities (19) and (20) on $[a,b]$, then summing the resulting inequality we obtain the last inequality in (7).

This completes the proof. $\square$

Remark 7. *Under assumptions of Theorem 6 with $p(x,y)=1$, the inequalities (7) reduce to inequalities (4) proved by Dragomir in [7].*

Remark 8. *Let $g_1:[a,b]\longrightarrow\mathbb{R}$ and $g_1:[c,d]\longrightarrow\mathbb{R}$ be two positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$, respectively. If we choose $p(x,y)=\frac{g_1(x)g_2(y)}{G_1G_2}$ for all $(x,y)\in\Delta$ in Theorem 6, then we have the following inequalities*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x,\frac{c+d}{2}\right)g_1(x)dx+\frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},y\right)g_2(y)dy\right] \\
\leq\ & \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx
\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{4}\left[\frac{1}{G_1}\int\limits_a^b\left[f(x,c)+f(x,d)\right]g_1(x)dx+\frac{1}{G_2}\int\limits_c^d\left[f(a,y)+f(b,y)\right]g_2(y)dy\right]\\&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\end{aligned}$$

where

$$G_1=\int\limits_a^b g_1(x)dx \text{ and } G_2=\int\limits_c^d g_2(y)dy$$

which is the same result proved by Farid et al. in [12].

3. Refinements of the Hermite-Hadamard-Fejer Inequalities

In this section, using two mappings we establish the refinements of the Hermite-Hadamard-Fejer inequalities:

Theorem 9. *Let $p:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. Let $f:\Delta\to\mathbb{R}$ be a co-ordinated convex on Δ and define the mappings Λ_1 and Λ_2 by*

$$\begin{aligned}\Lambda_1(t,s) &= \frac{1}{2}\int\limits_a^b\int\limits_c^d\left[f\left(tx+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\&\left.+f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx\end{aligned}$$

and

$$\begin{aligned}\Lambda_2(t,s) &= \frac{1}{2}\int\limits_a^b\int\limits_c^d\left[f\left(tx+(1-t)\frac{a+b}{2},y\right)\right.\\&\left.+f\left(x,sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx.\end{aligned}$$

Then the functions Λ_1 and Λ_2 are co-ordinated convex functions on $[0,1]^2$, non-decreasing on $[0,1]^2$ and we have the following refinement of Hermite-Hadamard-Fejer inequalities

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int\limits_a^b\int\limits_c^d p(x,y)dydx \qquad (21)\\&\leq \Lambda_1(t,s)\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\int_a^b\int_c^d \left[f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},y\right)\right]p(x,y)dydx \\
&\le \Lambda_2(t,s) \\
&\le \int_a^b\int_c^d f(x,y)p(x,y)dydx.
\end{aligned}
$$

Moreover we have

$$
\inf_{(t,s)\in[0,1]^2}\Lambda_1(t,s)=\Lambda_1(0,0)=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx, \tag{22}
$$

$$
\begin{aligned}
\sup_{(t,s)\in[0,1]^2}\Lambda_1(t,s) &= \Lambda_1(1,1) \\
&= \frac{1}{2}\int_a^b\int_c^d \left[f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},y\right)\right]p(x,y)dydx,
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\inf_{(t,s)\in[0,1]^2}\Lambda_2(t,s) &= \Lambda_2(0,0) \\
&= \frac{1}{2}\int_a^b\int_c^d \left[f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},y\right)\right]p(x,y)dydx,
\end{aligned} \tag{24}
$$

and

$$
\sup_{(t,s)\in[0,1]^2}\Lambda_2(t,s)=\Lambda_2(1,1)=\int_a^b\int_c^d f(x,y)p(x,y)dydx. \tag{25}
$$

Proof. Fix $s\in[0,1]$ and let $t_1,t_2\in[0,1]$ and $\alpha,\beta\ge 0$ with $\alpha+\beta=1$. Then by using the co-ordinated convexity of f we have

$$
\begin{aligned}
&\Lambda_1(\alpha t_1+\beta t_2,s) \\
&= \frac{1}{2}\int_a^b\int_c^d \left[f\left((\alpha t_1+\beta t_2)x+(1-(\alpha t_1+\beta t_2))\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
&\quad \left. +f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{2}\int_a^b\int_c^d \left[f\left(\alpha\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+\beta\left(t_2x+(1-t_2)\frac{a+b}{2}\right),\frac{c+d}{2}\right)\right.\\
&\quad \left.+f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx\\
&= \frac{1}{2}\int_a^b\int_c^d \left[\alpha f\left(t_1x+(1-t_1)\frac{a+b}{2},\frac{c+d}{2}\right)+\beta f\left(t_2x+(1-t_2)\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\quad \left.+f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx\\
&= \frac{\alpha}{2}\int_a^b\int_c^d \left[f\left(t_1x+(1-t_1)\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\quad \left.+f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx\\
&\quad +\frac{\beta}{2}\int_a^b\int_c^d \left[f\left(t_2x+(1-t_2)\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\quad \left.+f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx\\
&= \alpha\Lambda_1(t_1,s)+\beta\Lambda_1(t_2,s).
\end{aligned}$$

Similarly, if $t\in[0,1]$, then for $s_1,s_2\in[0,1]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, we can also obtain

$$\Lambda_1(t,\alpha s_1+\beta s_2)\leq \alpha\Lambda_1(t,s_1)+\beta\Lambda_1(t,s_2)$$

which gives that Λ_1 is co-ordinated convex function on $[0,1]^2$.

Fix $s\in[0,1]$ and let $0\leq t_1\leq t_2\leq 1$ with $x=\frac{a+b}{2}$. Then, we have

$$\begin{aligned}
t_2x+(1-t_2)\frac{a+b}{2} &\leq t_1x+(1-t_1)\frac{a+b}{2} \\
&\leq t_1(a+b-x)+(1-t_1)\frac{a+b}{2}\\
&\leq t_2(a+b-x)+(1-t_2)\frac{a+b}{2}
\end{aligned}\tag{26}$$

and

$$\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+\left(t_1(a+b-x)+(1-t_1)\frac{a+b}{2}\right)\tag{27}$$

$$= \left(t_2 x + (1-t_2)\frac{a+b}{2}\right) + \left(t_2\left(a+b-x\right) + (1-t_2)\frac{a+b}{2}\right).$$

From the inequality (5) of Lemma 5, since p is positive, integrable and symmetric to $\frac{a+b}{2}$, we obtain

$$\begin{aligned}
&\Lambda_1(t_1, s) \\
= \;& \frac{1}{2}\int_a^b\int_c^d \left[f\left(t_1 x + (1-t_1)\frac{a+b}{2}, \frac{c+d}{2}\right)\right. \\
&\left. + f\left(\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx \\
= \;& \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_1 x + (1-t_1)\frac{a+b}{2}, \frac{c+d}{2}\right)\right. \\
&\left. + f\left(\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx \\
&+ \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_1\left(a+b-x\right) + (1-t_1)\frac{a+b}{2}, \frac{c+d}{2}\right)\right. \\
&\left. + f\left(\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right)\right] p(a+b-x,y)dydx \\
= \;& \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_1 x + (1-t_1)\frac{a+b}{2}, \frac{c+d}{2}\right)\right. \\
&+ f\left(t_1\left(a+b-x\right) + (1-t_1)\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\left. + f\left(\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx \\
\leq \;& \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_2 x + (1-t_2)\frac{a+b}{2}, \frac{c+d}{2}\right)\right. \\
&+ f\left(t_2\left(a+b-x\right) + (1-t_2)\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\left. + f\left(\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_2x+(1-t_2)\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
&\quad \left.+f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx \\
&\quad +\frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_2\left(a+b-x\right)+(1-t_2)\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
&\quad \left.+f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(a+b-x,y)dydx \\
&= \frac{1}{2}\int_a^b\int_c^d \left[f\left(t_2x+(1-t_2)\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
&\quad \left.+f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx \\
&= \Lambda_1(t_2,s),
\end{aligned}$$

wich gives that $\Lambda_1(t,.)$ is non-decreasing on $[0,1]$. Similar way, we can also prove that $\Lambda_1(.,s)$ is non-decreasing on $[0,1]$ by the assumption $t\in[0,1]$ is fixed and by using the (6) of Lemma 5. Therefore Λ_1 is co-ordinated monotonic non-decreasing on $[0,1]^2$.

It can easily shown that Λ_2 is co-ordinated convex function on $[0,1]^2$ similar to proof of co-ordinated convexity of Λ_1. To prove that Λ_2 is co-ordinated monotonic non-decreasing on $[0,1]^2$, consider the assumptions (26) and (27). Using the inequality (5) of Lemma 5, we have

$$\begin{aligned}
&\Lambda_2(t_1,s) \\
&= \frac{1}{2}\int_a^b\int_c^d \left[f\left(t_1x+(1-t_1)\frac{a+b}{2},y\right)+f\left(x,sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx \\
&= \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_1x+(1-t_1)\frac{a+b}{2},y\right)+f\left(x,sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx \\
&\quad +\frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_1\left(a+b-x\right)+(1-t_1)\frac{a+b}{2},y\right)\right. \\
&\quad \left.+f\left(a+b-x,sy+(1-s)\frac{c+d}{2}\right)\right]p(a+b-x,y)dydx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_1 x + (1-t_1)\frac{a+b}{2}, y\right) + f\left(t_1(a+b-x) + (1-t_1)\frac{a+b}{2}, y\right)\right. \\
&\quad \left. + f\left(x, sy + (1-s)\frac{c+d}{2}\right) + f\left(a+b-x, sy+(1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx \\
&\le \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_2 x + (1-t_2)\frac{a+b}{2}, y\right) + f\left(t_2(a+b-x) + (1-t_2)\frac{a+b}{2}, y\right)\right. \\
&\quad \left. + f\left(x, sy + (1-s)\frac{c+d}{2}\right) + f\left(a+b-x, sy+(1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx \\
&= \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_2 x + (1-t_2)\frac{a+b}{2}, y\right) + f\left(x, sy + (1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx \\
&\quad \frac{1}{2}\int_a^{\frac{a+b}{2}}\int_c^d \left[f\left(t_2(a+b-x) + (1-t_2)\frac{a+b}{2}, y\right)\right. \\
&\quad \left. + f\left(a+b-x, sy+(1-s)\frac{c+d}{2}\right)\right] p(a+b-x,y)dydx \\
&= \frac{1}{2}\int_a^b\int_c^d \left[f\left(t_2 x + (1-t_2)\frac{a+b}{2}, y\right) + f\left(x, sy + (1-s)\frac{c+d}{2}\right)\right] p(x,y)dydx \\
&= \Lambda_2(t_2, s).
\end{aligned}
$$

This finishes the proof that $\Lambda_2(t,.)$ is non-decreasing on $[0,1]$. Similarly, we can also obtain that $\Lambda_2(.,s)$ is non-decreasing on $[0,1]$ by the assumption $t \in [0,1]$ is fixed and by using the (6) of Lemma 5. Thus, Therefore Λ_2 is also co-ordinated monotonic non-decreasing on $[0,1]^2$.

The proofs of the equalities (22)-(25) are obvious from that Λ_1 and Λ_2 are co-ordinated monotonic non-decreasing on $[0,1]^2$.

The proof of the Theorem 9 is completely completed. $\square$

Corollary 10. *Under assumptions of Theorem 9 with $p(x,y) = \frac{1}{(b-a)(d-c)}$, then we have the mappings Ω_1 and Ω_2 defined by*

$$
\Omega_1(t,s) = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)dx\right.
$$

$$+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)dy\Bigg]$$

and

$$\begin{aligned}\Omega_2(t,s) &= \frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d\left[f\left(tx+(1-t)\frac{a+b}{2}, y\right)\right. \\ &\quad \left.+f\left(x, sy+(1-s)\frac{c+d}{2}\right)\right]dydx.\end{aligned}$$

Then, the functions Ω_1 *and* Ω_2 *are co-ordinated convex functions on* $[0,1]^2$, *non-decreasing on* $[0,1]^2$ *and we have the following refinement of Hermite-Hadamard-Fejer inequalities*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (28) \\ \leq\ & \Omega_1(t,s) \\ \leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ \leq\ & \Omega_2(t,s) \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx.\end{aligned}$$

Moreover we have

$$\inf_{(t,s)\in[0,1]^2}\Omega_1(t,s) = \Omega_1(0,0) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right),$$

$$\begin{aligned}\sup_{(t,s)\in[0,1]^2}\Omega_1(t,s) &= \Omega_1(1,1) \\ &= \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right],\end{aligned}$$

$$\inf_{(t,s)\in[0,1]^2}\Omega_2(t,s) = \Omega_2(0,0)$$

$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]$$

and

$$\sup_{(t,s)\in[0,1]^2}\Omega_2(t,s)=\Omega_2(1,1)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx.$$

The inequalities (28) was first given by Ali et al. in [3].

The following Corollary give the refinements of the inequalities obtained by Farid et al. in [12].

Corollary 11. *Let $g_1:[a,b]\longrightarrow\mathbb{R}$ and $g_1:[c,d]\longrightarrow\mathbb{R}$ be two positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$, respectively. If we choose $p(x,y)=\frac{g_1(x)g_2(y)}{G_1G_2}$ for all $(x,y)\in\Delta$ in Theorem 9, then we have the mappings Ω_3 and Ω_4 defined by*

$$\begin{aligned}\Omega_3(t,s) &= \frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(tx+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)dx\right.\\ &\left.+\frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dy\right]\end{aligned}$$

and

$$\Omega_4(t,s)=\frac{1}{2G_1G_2}\int_a^b\int_c^d\left[f\left(tx+(1-t)\frac{a+b}{2},y\right)+f\left(x,sy+(1-s)\frac{c+d}{2}\right)\right]dydx.$$

Then the functions Ω_3 and Ω_4 are co-ordinated convex functions on $[0,1]^2$, non-decreasing on $[0,1]^2$ and we have the following refinement of Hermite-Hadamard-Fejer inequalities

$$\begin{aligned}& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (29)\\ \leq\ & \Omega_3(t,s)\\ \leq\ & \frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\ \leq\ & \Omega_4(t,s)\end{aligned}$$

$$\leq \quad \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)dydx.$$

Moreover we have

$$\inf_{(t,s)\in[0,1]^2}\Omega_3(t,s)=\Omega_3(0,0)=f\left(\frac{a+b}{2},\frac{c+d}{2}\right),$$

$$\sup_{(t,s)\in[0,1]^2}\Omega_3(t,s)=\Omega_3(1,1)=\frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right],$$

$$\inf_{(t,s)\in[0,1]^2}\Omega_4(t,s)=\Omega_4(0,0)=\frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]$$

and

$$\sup_{(t,s)\in[0,1]^2}\Omega_4(t,s)=\Omega_4(1,1)=\frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)dydx.$$

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