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MAGNETIC NON-NULL CURVES ACCORDING TO PARALLEL TRANSPORT FRAME IN MINKOWSKI 3-SPACE

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Abstract. In this study, we define the notions of T-magnetic, N_1-magnetic and N_2-magnetic timelike and spacelike curves in Minkowski 3-space. We obtain the magnetic vector field V when the timelike or spacelike curve is a T-magnetic, N_1-magnetic or N_2-magnetic trajectory of V and give some examples for these magnetic curves.

1. Introduction

Recently, magnetic curves that have been proposed for computer graphics purposes are a particle tracing technique that generates a wide variety of curves and spirals under the influence of a magnetic field. In a uniform magnetic field, the motion of a particle of charge q and mass m, travelling with velocity $\vec{v}$ under magnetic induction $\vec{B}$ is the result of Lorentz force, $F = q(\vec{v} \times \vec{B})$, which can be written as $m\frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B})$, where $\times$ represents the cross product operation. It describes the motion of charged particles experiencing Lorentz force. In [15], the authors have obtained the components of magnetic curves and investigated the magnetic curves with constant logarithmic curvature graph (LCG) and logarithmic torsion graph (LTG) gradient.

Also, the magnetic curves on a Riemannian manifold (M, g) are trajectories of charged particles moving on M under the action of a magnetic field F. A *magnetic field* is a closed 2-form F on M and the *Lorentz force* of the magnetic field F on (M, g) is a (1,1)-tensor field Φ given by $g(\Phi(X), Y) = F(X, Y)$, for any vector fields $X, Y \in \chi(M)$. In dimension 3, the magnetic fields may be defined using divergence-free vector fields. As Killing vector fields have zero divergence, one may define a special class of magnetic fields called *Killing magnetic fields*.

Different approaches in the study of magnetic curves for a certain magnetic field and on the fixed energy level have been reviewed by Munteanu in [11]. He has

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emphasized them in the case when the magnetic trajectory corresponds to a Killing vector field associated to a screw motion in the Euclidean 3-space. In [12], the authors have investigated the trajectories of charged particles moving in a space modeled by the homogeneous 3-space $S^2 \times \mathbb{R}$ under the action of the Killing magnetic fields.

In [6], the authors have classified all magnetic curves in the 3-dimensional Minkowski space corresponding to the Killing magnetic field $V = a\partial_x + b\partial_y + c\partial_z$, with $a, b, c \in \mathbb{R}$. They have found that, they are helices in E_1^3 and draw the most relevant of them. In 3D semi-Riemannian manifolds, Özdemir et. al. have determined the notions of T-magnetic, N-magnetic and B-magnetic curves and give some characterizations for them, where T, N an B are the tangent, normal and binormal vectors of a curve α, respectively [14]. Also, in [10], the authors have studied on magnetic pseudo null and magnetic null curves in Minkowski 3-space.

In any 3D Riemannian manifold (M, g), magnetic fields of nonzero constant length are one to one correspondence to almost contact structure compatible to the metric g. From this fact, many authors have motivated to study magnetic curves with closed fundamental 2-form in almost contact metric 3-manifolds, Sasakian manifolds, quasi-para-Sasakian manifolds and etc (see [4], [8], [9], [5]).

On the other hand, a lot of characterizations of the space curves has been studied by many mathematicians by using Frenet-Serret theorem. The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. But, if the second derivative of the curve is zero, then the curvature may vanish at some points on the curve. For this reason, we need an alternative frame in E^3. Hence, an alternative moving frame along a curve is defined by Bishop in 1975 and he called it Bishop frame or parallel transport frame which is well defined as well the curve has vanishing second curvature [2]. The Bishop frame have many applications in Biology and Computer Graphics. For example, it may be possible to compute information about the shape of sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animations [3]. Also, after defining this useful alternative frame, the parallel transport frame has been defined for non-null curves in Minkowski 3-space [13].

In this paper, firstly we define the notions of T-magnetic, N_1-magnetic and N_2-magnetic timelike and spacelike curves in Minkowski 3-space. Also, we obtain the magnetic vector field V when the timelike or spacelike curve is a T-magnetic, N_1-magnetic or N_2-magnetic trajectory of V and give some examples for these magnetic curves.

2. Preliminaries

We know that, an alternative moving frame along a curve in an Euclidean 3-space is defined by Bishop in 1975 [2]. For defining an alternative moving frame which is called *Bishop frame* or *parallel transport frame* in E^3, one can parallel transport each component of an orthonormal frame along the curve. Moreover, this frame is well defined as well the curve has vanishing second curvature. The Bishop frame is written as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.1}$$

where T is the tangent vector of the curve and $\{N_1, N_2\}$ are any convenient arbitrary basis for the remainder of the frame in an Euclidean 3-space. Here, $\{T, N_1, N_2\}$ is called *Bishop trihedra* and k_1 and k_2 are called *Bishop curvatures* of the curve α. The relation between Frenet frame and Bishop frame is given by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & \sin\theta(t) \\ 0 & -\sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.2}$$

where $\theta(t) = \arctan \frac{k_2}{k_1}$, the torsion and curvature of the curve according to Frenet frame are $\tau(t) = \theta'(t)$ and $\kappa(t) = \sqrt{(k_1)^2 + (k_2)^2}$, respectively. Also, the Bishop curvatures are defined by $k_1 = \kappa \cos\theta(t)$ and $k_2 = \kappa \sin\theta(t)$.

Now, we will recall the parallel transport frame of a non-null curve in Minkowski 3-space.

Let E_1^3 be a 3-dimensional Minkowski space defined as a space to be usual 3-dimensional vector space consisting of vectors $\{(x^0, x^1, x^2) : x^0, x^1, x^2 \in \mathbb{R}\}$, but with a linear connection ∇ corresponding to its Minkowski metric g given by

$$g(x, y) = -x^0y^0 + x^1y^1 + x^2y^2.$$

Here, there are three categories of vector fields, namely,

spacelike if $g(X, X) > 0$ or $X = 0$,

timelike if $g(X, X) < 0$,

lightlike if $g(X, X) = 0$, $X \neq 0$. In general, the type into which a given vector field X falls is called the *causal character* of X [7].

In three dimensional Minkowski space, the parallel transport frames for timelike and spacelike curves can be defined as follows:

If the curve is timelike, then the parallel transport frame is written as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.3}$$

where T is the timelike tangent vector of the curve and $\{N_1, N_2\}$ are any convenient arbitrary basis for the remainder of the frame in Minkowski 3-space. Here, both of the vectors of $\{N_1, N_2\}$ are spacelike. Now, $\{T, N_1, N_2\}$ is called *parallel transport trihedra* and k_1 and k_2 are called *parallel transport curvatures* of the curve α. The relation between Frenet frame and parallel transport frame is given by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & \sin\theta(t) \\ 0 & -\sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.4}$$

where $\theta(t) = \arctan\frac{k_2}{k_1}$, the torsion and curvature of the curve according to Frenet frame are $\tau(t) = \theta'(t)$ and $\kappa(t) = \sqrt{(k_1)^2 + (k_2)^2}$, respectively. Also, the parallel transport curvatures are defined by $k_1 = g(T', N_1) = \kappa\cos\theta(t)$ and $k_2 = g(T', N_2) = \kappa\sin\theta(t)$.

If the curve is spacelike, then the parallel transport frame is written as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -\epsilon_{N_1}k_1 & 0 & 0 \\ -\epsilon_{N_2}k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.5}$$

where T is the spacelike tangent vector of the curve and $\{N_1, N_2\}$ are any convenient arbitrary basis for the remainder of the frame in Minkowski 3-space such that one them is spacelike and the other one is timelike and $\epsilon_X = g(X, X)$. Here, $\{T, N_1, N_2\}$ is called *parallel transport trihedra* and k_1 and k_2 are called *parallel transport curvatures* of the curve α. The relation between Frenet frame and parallel transport frame is given by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta(t) & \sinh\theta(t) \\ 0 & \sinh\theta(t) & \cosh\theta(t) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.6}$$

where $\theta(t) = \operatorname{arctanh}\frac{k_2}{k_1}$, the torsion and curvature of the curve according to Frenet frame are $\tau(t) = \epsilon_{N_1}\theta'(t)$ and $\kappa(t) = \sqrt{\epsilon_{N_1}(k_1)^2 + \epsilon_{N_2}(k_2)^2}$, respectively. Also, the parallel transport curvatures are defined by $k_1 = \epsilon_{N_1}g(T', N_1) = \kappa\cosh\theta(t)$ and $k_2 = \epsilon_{N_2}g(T', N_2) = \kappa\sinh\theta(t)$.

Moreover, we assume that $\{T, N_1, N_2\}$ is positively oriented and the vector products of these vectors are defined as follows:

$$T \times N_1 = \epsilon_{N_2}N_2,\ N_1 \times N_2 = \epsilon_T T,\ N_2 \times T = \epsilon_{N_1}N_1.$$

(for detail, see [13]).

Now, we will give some informations about the magnetic curves in 3-dimensional semi-Riemannian manifolds.

A divergence-free vector field defines a magnetic field in a three-dimensional semi-Riemannian manifold M. It is known that, $V \in \chi(M^n)$ is a Killing vector field if and only if $L_V g = 0$ or, equivalently, $\nabla V(p)$ is a skew-symmetric operator in $T_p(M^n)$, at each point $p \in M^n$. It is clear that, any Killing vector field on (M^n, g) is divergence-free. In particular, if $n = 3$, then every Killing vector field defines a magnetic field that will be called a *Killing magnetic field* [1].

Let (M, g) be an n-dimensional semi-Riemannian manifold. A *magnetic field* is a closed 2-form F on M and the *Lorentz force* Φ of the magnetic field F on (M, g) is defined to be a skew-symmetric operator given by

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \chi(M). \tag{2.7}$$

The *magnetic trajectories* of F are curves α on M that satisfy the *Lorentz equation* (sometimes called the *Newton equation*)

$$\nabla_{\alpha'}\alpha' = \Phi(\alpha'). \tag{2.8}$$

The Lorentz equation generalizes the equation satisfied by the geodesics of M, namely $\nabla_{\alpha'}\alpha' = 0$.

Note that, one can define on M the cross product of two vectors $X, Y \in \chi(M)$ as follows

$$g(X \times Y, Z) = dv_g(X, Y, Z), \quad \forall Z \in \chi(M).$$

If V is a Killing vector field on M, let $F_V = \imath_V dv_g$ be the corresponding Killing magnetic field. By $\imath$ we denote the inner product. Then, the Lorentz force of F_V is

$$\Phi(X) = V \times X.$$

Consequently, the Lorentz force equation (2.8) can be written as

$$\nabla_{\alpha'}\alpha' = V \times \alpha' \tag{2.9}$$

(for detail see [11], [14]).

3. Magnetic Non-Null Curves According to Parallel Transport Frame in Minkowski 3-Space

In this section, we will investigate the T-magnetic, N_1-magnetic and N_2-magnetic timelike and spacelike curves in Minkowski 3-space. Also, we obtain the magnetic vector field V when the timelike or spacelike curve is a T-magnetic, N_1-magnetic or N_2-magnetic trajectory of V and give some examples for these magnetic curves.

3.1. Magnetic Timelike Curves According to Parallel Transport Frame in Minkowski 3-Space.

Definition 1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E_1^3$ be a timelike curve in Minkowski 3-space and F_V be a magnetic field in E_1^3. Then,*

i) *if the tangent vector field of the curve satisfies the Lorentz force equation $\nabla_{\alpha'}T = \Phi(T) = V \times T$, then the curve α is called a **T-magnetic timelike curve according to parallel transport frame**.*

ii) *If the vector field N_1 of the parallel transport frame satisfies the Lorentz force equation $\nabla_{\alpha'}N_1 = \Phi(N_1) = V \times N_1$, then the curve α is called an **N_1-magnetic timelike curve according to parallel transport frame**.*

iii) *If the vector field N_2 of the parallel transport frame satisfies the Lorentz force equation $\nabla_{\alpha'}N_2 = \Phi(N_2) = V \times N_2$, then the curve α is called an **N_2-magnetic timelike curve according to parallel transport frame.***

Proposition 1. ***i)*** *Let α be a T-magnetic timelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the Lorentz force according to the parallel transport frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & \rho \\ k_2 & -\rho & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.1}$$

where ρ is a certain function defined by $\rho = g(\Phi(N_1), N_2)$.

ii) *Let α be an N_1-magnetic timelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the Lorentz force according to the parallel transport frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & \mu \\ k_1 & 0 & 0 \\ \mu & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.2}$$

where μ is a certain function defined by $\mu = g(\Phi(T), N_2)$.

iii) *Let α be an N_2-magnetic timelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the Lorentz force according to the parallel transport frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & \gamma & k_2 \\ \gamma & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.3}$$

where γ is a certain function defined by $\gamma = g(\Phi(T), N_1)$.

Proof. Let α be a T-magnetic timelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. From the definition of the T-magnetic timelike curve according to parallel transport frame, we know that $\Phi(T) = k_1 N_1 + k_2 N_2$. Since $\Phi(N_1) \in Sp\{T, N_1, N_2\}$, we have $\Phi(N_1) = a_1 T + a_2 N_1 + a_3 N_2$. So, we get

$$\begin{aligned} a_1 &= -g(\Phi N_1, T) = g(N_1, \Phi T) = g(N_1, k_1 N_1 + k_2 N_2) = k_1, \\ a_2 &= g(\Phi N_1, N_1) = 0, \\ a_3 &= g(\Phi N_1, N_2) = \rho \end{aligned}$$

and hence we obtain that, $\Phi(N_1) = k_1 T + \rho N_2$.

Furthermore, from $\Phi(N_2) = b_1 T + b_2 N_1 + b_3 N_2$, we have

$$\begin{aligned} b_1 &= -g(\Phi N_2, T) = g(N_2, \Phi T) = g(N_2, k_1 N_1 + k_2 N_2) = k_2, \\ b_2 &= g(\Phi N_2, N_1) = -g(N_2, \Phi N_1) = -\rho, \\ b_3 &= g(\Phi N_2, N_2) = 0 \end{aligned}$$

and so, we can write $\Phi(N_2) = k_2 T - \rho N_1$.

ii) and iii) can be proven with the similar procedure in i). □

Proposition 2. ***i)*** *Let α be a unit speed T-magnetic timelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the timelike curve α is a T-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = \rho T - k_2 N_1 + k_1 N_2 \tag{3.4}$$

along the curve α.

ii) *Let α be a unit speed N_1-magnetic timelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the timelike curve α is an N_1-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\mu N_1 + k_1 N_2 \tag{3.5}$$

along the curve α.

iii) *Let α be a unit speed N_2-magnetic timelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the timelike curve α is an N_2-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -k_2 N_1 + \gamma N_2 \tag{3.6}$$

along the curve α.

Proof. Let α be a T-magnetic timelike trajectory of a Killing magnetic vector field V. Using Proposition 1 and taking $V = aT + bN_1 + cN_2$; from $\Phi(T) = V \times T$, we get

$$b = -k_2, \ c = k_1;$$

from $\Phi(N_1) = V \times N_1$, we get

$$a = \rho, \ c = k_1$$

and from $\Phi(N_2) = V \times N_2$, we get

$$a = \rho, \ b = -k_2$$

and so the Killing magnetic vector field V can be written by (3.4). Conversely, if the Killing magnetic vector field V is the form of (3.4), then one can easily see that $V \times T = \Phi(T)$ holds. So, the timelike curve α is a T-magnetic projectory of the Killing magnetic vector field V.

ii) and iii) can be proven with the similar procedure in i). □

Corollary 1. ***i)*** *If the unit speed timelike curve α with parallel transport frame $\{T, N_1, N_2\}$ is a T-magnetic trajectory of a Killing magnetic vector field V in Minkowski 3-space, then the Killing magnetic vector field V can be a spacelike, timelike or null vector.*

ii) *If the unit speed timelike curve α with parallel transport frame $\{T, N_1, N_2\}$ is an N_1-magnetic trajectory of a Killing magnetic vector field V in Minkowski 3-space, then the Killing magnetic vector field V is a spacelike vector.*

iii) *If the unit speed timelike curve α with parallel transport frame $\{T, N_1, N_2\}$ is an N_2-magnetic trajectory of a Killing magnetic vector field V in Minkowski 3-space, then the Killing magnetic vector field V is a spacelike vector.*

Example 1. *Let us consider the unit speed timelike curve*

$$\alpha(t) = \frac{1}{\sqrt{3}}(2t, \cos t, \sin t) \tag{3.7}$$

in Minkowski 3-space. Here, one can easily calculate its Frenet-Serret trihedra and curvatures as

$$\begin{aligned} T &= \frac{1}{\sqrt{3}}(2, -\sin t, \cos t), \\ N &= (0, -\cos t, -\sin t), \\ B &= \frac{1}{\sqrt{3}}(-1, 2\sin t, -2\cos t), \\ \kappa &= \frac{1}{\sqrt{3}},\ \tau = \frac{2}{\sqrt{3}}, \end{aligned} \tag{3.8}$$

respectively. Now, we will obtain its parallel transport frame and curvatures. For this, we find the $\theta(t)$ with the aid of $\tau(t) = \theta'(t)$ as

$$\theta(t) = \int_0^t \frac{2}{\sqrt{3}} dt = \frac{2t}{\sqrt{3}}. \tag{3.9}$$

So, the transformation matrix can be expressed as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2t}{\sqrt{3}} & \sin\frac{2t}{\sqrt{3}} \\ 0 & -\sin\frac{2t}{\sqrt{3}} & \cos\frac{2t}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}. \tag{3.10}$$

Using the method of Cramer, we can obtain the parallel transport trihedra of the timelike curve α as follows

$$\begin{aligned} T &= \frac{1}{\sqrt{3}}(2, -\sin t, \cos t), \qquad (3.11) \\ N_1 &= \begin{pmatrix} \frac{1}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\sin t\sin\frac{2t}{\sqrt{3}} - \cos t\cos\frac{2t}{\sqrt{3}}, \\ \frac{2}{\sqrt{3}}\cos t\sin\frac{2t}{\sqrt{3}} - \sin t\cos\frac{2t}{\sqrt{3}} \end{pmatrix}, \\ N_2 &= \begin{pmatrix} -\frac{1}{\sqrt{3}}\cos\frac{2t}{\sqrt{3}}, \frac{2}{\sqrt{3}}\sin t\cos\frac{2t}{\sqrt{3}} - \cos t\sin\frac{2t}{\sqrt{3}}, \\ -\frac{2}{\sqrt{3}}\cos t\cos\frac{2t}{\sqrt{3}} - \sin t\sin\frac{2t}{\sqrt{3}} \end{pmatrix} \end{aligned}$$

and the parallel transport curvatures can be obtained as

$$\begin{aligned} k_1 &= g(T', N_1) = \kappa\cos\theta = \frac{1}{\sqrt{3}}\cos\frac{2t}{\sqrt{3}}, \\ k_2 &= g(T', N_2) = \kappa\sin\theta = \frac{1}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}}. \end{aligned} \tag{3.12}$$

Now, for example, let us find the Killing magnetic vector field V when the timelike curve (3.7) is an N_2-magnetic trajectory of the Killing magnetic vector field V according to parallel transport frame (3.11):

If the timelike curve α is N_2-magnetic according to parallel transport frame, then from (3.6), we obtain the Killing magnetic vector field V as

$$\begin{aligned} V &= -\frac{1}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}}\left(\begin{array}{c} \frac{1}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\sin t\sin\frac{2t}{\sqrt{3}} - \cos t\cos\frac{2t}{\sqrt{3}}, \\ \frac{2}{\sqrt{3}}\cos t\sin\frac{2t}{\sqrt{3}} - \sin t\cos\frac{2t}{\sqrt{3}} \end{array}\right) \qquad (3.13) \\ &\quad +\gamma\left(\begin{array}{c} -\frac{1}{\sqrt{3}}\cos\frac{2t}{\sqrt{3}}, \frac{2}{\sqrt{3}}\sin t\cos\frac{2t}{\sqrt{3}} - \cos t\sin\frac{2t}{\sqrt{3}}, \\ -\frac{2}{\sqrt{3}}\cos t\cos\frac{2t}{\sqrt{3}} - \sin t\sin\frac{2t}{\sqrt{3}} \end{array}\right). \end{aligned}$$

When the timelike curve α is N_2-magnetic according to parallel transport frame, the figure of α and V can be drawn as following. Similarly, the Killing magnetic

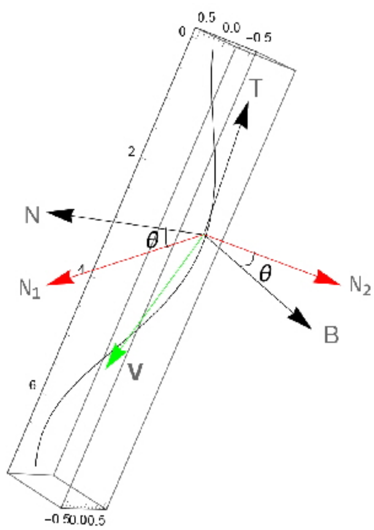

FIGURE 1. .

vector field V when the curve (3.7) is a T-magnetic or N_1-magnetic trajectory of the Killing magnetic vector field V according to parallel transport frame (3.11) can be found as the above procedure.

3.2. **Magnetic Spacelike Curves According to Parallel Transport Frame in Minkowski 3-Space.**

Definition 2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E_1^3$ be a spacelike curve in Minkowski 3-space and F_V be a magnetic field in E_1^3. Then,*

i) *if the tangent vector field of the curve satisfies the Lorentz force equation* $\nabla_{\alpha'} T = \Phi(T) = V \times T$, *then the curve* α *is called a* T***-magnetic spacelike curve according to parallel transport frame****.*

ii) *If the vector field* N_1 *of the parallel transport frame satisfies the Lorentz force equation* $\nabla_{\alpha'} N_1 = \Phi(N_1) = V \times N_1$, *then the curve* α *is called an* N_1***-magnetic spacelike curve according to parallel transport frame****.*

iii) *If the vector field* N_2 *of the parallel transport frame satisfies the Lorentz force equation* $\nabla_{\alpha'} N_2 = \Phi(N_2) = V \times N_2$, *then the curve* α *is called an* N_2***-magnetic spacelike curve according to parallel transport frame.***

Proposition 3. ***i)*** *Let* α *be a* T*-magnetic spacelike curve in Minkowski 3-space with the parallel transport frame* $\{T, N_1, N_2\}$ *and the parallel transport curvatures* $\{k_1, k_2\}$*. Then, the Lorentz force according to the parallel transport frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -\epsilon_{N_1} k_1 & 0 & \epsilon_{N_2}\rho \\ -\epsilon_{N_2} k_2 & -\epsilon_{N_1}\rho & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.14}$$

where ρ *is a certain function defined by* $\rho = g(\Phi(N_1), N_2)$*.*

ii) *Let* α *be an* N_1*-magnetic spacelike curve in Minkowski 3-space with the parallel transport frame* $\{T, N_1, N_2\}$ *and the parallel transport curvatures* $\{k_1, k_2\}$*. Then, the Lorentz force according to the parallel transport frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & \epsilon_{N_2}\mu \\ -\epsilon_{N_1} k_1 & 0 & 0 \\ -\mu & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.15}$$

where μ *is a certain function defined by* $\mu = g(\Phi(T), N_2)$*.*

iii) *Let* α *be an* N_2*-magnetic spacelike curve in Minkowski 3-space with the parallel transport frame* $\{T, N_1, N_2\}$ *and the parallel transport curvatures* $\{k_1, k_2\}$*. Then, the Lorentz force according to the parallel transport frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_{N_1}\gamma & k_2 \\ -\gamma & 0 & 0 \\ -\epsilon_{N_2} k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.16}$$

where γ *is a certain function defined by* $\gamma = g(\Phi(T), N_1)$*.*

Proof. Let α be a T-magnetic spacelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. From the definition of the T-magnetic spacelike curve according to parallel transport frame, we know that $\Phi(T) = k_1 N_1 + k_2 N_2$. Since $\Phi(N_1) \in Sp\{T, N_1, N_2\}$, we have $\Phi(N_1) = a_1 T + a_2 N_1 + a_3 N_2$. So, we get

$$\begin{aligned} a_1 &= g(\Phi N_1, T) = -g(N_1, \Phi T) = -g(N_1, k_1 N_1 + k_2 N_2) = -\epsilon_{N_1} k_1, \\ a_2 &= \epsilon_{N_1} g(\Phi N_1, N_1) = 0, \\ a_3 &= \epsilon_{N_2} g(\Phi N_1, N_2) = \epsilon_{N_2}\rho \end{aligned}$$

and hence we obtain that, $\Phi(N_1) = -\epsilon_{N_1} k_1 T + \epsilon_{N_2} \rho N_2$.

Furthermore, from $\Phi(N_2) = b_1 T + b_2 N_1 + b_3 N_2$, we have

$$\begin{aligned} b_1 &= g(\Phi N_2, T) = -g(N_2, \Phi T) = -g(N_2, k_1 N_1 + k_2 N_2) = -\epsilon_{N_2} k_2, \\ b_2 &= \epsilon_{N_1} g(\Phi N_2, N_1) = -\epsilon_{N_1} g(N_2, \Phi N_1) = -\epsilon_{N_1} \rho, \\ b_3 &= \epsilon_{N_2} g(\Phi N_2, N_2) = 0 \end{aligned}$$

and so, we can write $\Phi(N_2) = -\epsilon_{N_2} k_2 T - \epsilon_{N_1} \rho N_1$.

ii) and iii) can be proven with the similar procedure in i). □

Proposition 4. ***i)*** *Let α be a unit speed T-magnetic spacelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the spacelike curve α is a T-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = \rho T - \epsilon_{N_2} k_2 N_1 + \epsilon_{N_1} k_1 N_2 \tag{3.17}$$

along the curve α.

ii) *Let α be a unit speed N_1-magnetic spacelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the spacelike curve α is an N_1-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\mu N_1 + \epsilon_{N_1} k_1 N_2 \tag{3.18}$$

along the curve α.

iii) *Let α be a unit speed N_2-magnetic spacelike curve in Minkowski 3-space with the parallel transport frame $\{T, N_1, N_2\}$ and the parallel transport curvatures $\{k_1, k_2\}$. Then, the spacelike curve α is an N_2-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\epsilon_{N_2} k_2 N_1 + \gamma N_2 \tag{3.19}$$

along the curve α.

Proof. Let α be a T-magnetic spacelike trajectory of a Killing magnetic vector field V. Using Proposition 3 and taking $V = aT + bN_1 + cN_2$; from $\Phi(T) = V \times T$, we get

$$b = -\epsilon_{N_2} k_2, \ c = \epsilon_{N_1} k_1;$$

from $\Phi(N_1) = V \times N_1$, we get

$$a = \rho, \ c = \epsilon_{N_1} k_1$$

and from $\Phi(N_2) = V \times N_2$, we get

$$a = \rho, \ b = -\epsilon_{N_2} k_2$$

and so the Killing magnetic vector field V can be written by (3.17). Conversely, if the Killing magnetic vector field V is the form of (3.17), then one can easily see that $V \times T = \Phi(T)$ holds. So, the spacelike curve α is a T-magnetic projectory of the Killing magnetic vector field V.

ii) and iii) can be proven with the similar procedure in i). □

Corollary 2. *If the unit speed spacelike curve α with parallel transport frame $\{T, N_1, N_2\}$ is a T-magnetic, N_1-magnetic or N_2-magnetic trajectory of a Killing magnetic vector field V in Minkowski 3-space, then the Killing magnetic vector field V can be a spacelike, timelike or null vector.*

Example 2. *Let us consider the unit speed spacelike curve*

$$\alpha(t) = \frac{1}{\sqrt{2}}(\cosh t, \sinh t, t) \tag{3.20}$$

in Minkowski 3-space. Here, one can easily calculate its Frenet-Serret trihedra and curvatures as

$$\begin{aligned} T &= \frac{1}{\sqrt{2}}(\sinh t, \cosh t, 1), \\ N &= (\cosh t, \sinh t, 0), \\ B &= \frac{1}{\sqrt{2}}(\sinh t, \cosh t, -1), \\ \kappa &= \tau = \frac{1}{\sqrt{2}}, \end{aligned} \tag{3.21}$$

respectively. Now, we will obtain its parallel transport frame and curvatures. For this, we find the $\theta(t)$ with the aid of $\tau(t) = \epsilon_{N_1}\theta'(t)$ as

$$\theta(t) = -\int_0^t \frac{1}{\sqrt{2}}dt = -\frac{t}{\sqrt{2}}. \tag{3.22}$$

So, the transformation matrix can be expressed as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\frac{t}{\sqrt{2}} & -\sinh\frac{t}{\sqrt{2}} \\ 0 & -\sinh\frac{t}{\sqrt{2}} & \cosh\frac{t}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}. \tag{3.23}$$

Using the method of Cramer, we can obtain the parallel transport trihedra of the spacelike curve α as follows

$$\begin{aligned} T &= \frac{1}{\sqrt{2}}(\sinh t, \cosh t, 1), \qquad (3.24) \\ N_1 &= \left(\begin{array}{c} \frac{1}{\sqrt{2}}\sinh t \sinh\frac{t}{\sqrt{2}} + \cosh t \cosh\frac{t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh t \sinh\frac{t}{\sqrt{2}} + \sinh t \cosh\frac{t}{\sqrt{2}}, \\ -\frac{1}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}} \end{array} \right), \\ N_2 &= \left(\begin{array}{c} \frac{1}{\sqrt{2}}\sinh t \cosh\frac{t}{\sqrt{2}} + \cosh t \sinh\frac{t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh t \cosh\frac{t}{\sqrt{2}} + \sinh t \sinh\frac{t}{\sqrt{2}}, \\ -\frac{1}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}} \end{array} \right) \end{aligned}$$

and the parallel transport curvatures can be obtained as

$$\begin{aligned} k_1 &= \epsilon_{N_1} g(T', N_1) = \kappa \cosh\theta = \frac{1}{\sqrt{2}} \cosh\frac{t}{\sqrt{2}}, \\ k_2 &= \epsilon_{N_2} g(T', N_2) = \kappa \sinh\theta = -\frac{1}{\sqrt{2}} \sinh\frac{t}{\sqrt{2}}. \end{aligned} \tag{3.25}$$

Here, N_1 is a timelike and N_2 is a spacelike vector. Now, for example, let us find the Killing magnetic vector field V when the timelike curve (3.20) is an N_1-magnetic trajectory of the Killing magnetic vector field V according to parallel transport frame (3.24):

If the spacelike curve α is N_1-magnetic according to parallel transport frame, then from (3.18), we obtain the Killing magnetic vector field V as

$$\begin{aligned} V &= -\mu \begin{pmatrix} \frac{1}{\sqrt{2}} \sinh t \sinh\frac{t}{\sqrt{2}} + \cosh t \cosh\frac{t}{\sqrt{2}}, \\ \frac{1}{\sqrt{2}} \cosh t \sinh\frac{t}{\sqrt{2}} + \sinh t \cosh\frac{t}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \sinh\frac{t}{\sqrt{2}} \end{pmatrix} \\ &\quad -\frac{1}{\sqrt{2}} \cosh\frac{t}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \sinh t \cosh\frac{t}{\sqrt{2}} + \cosh t \sinh\frac{t}{\sqrt{2}}, \\ \frac{1}{\sqrt{2}} \cosh t \cosh\frac{t}{\sqrt{2}} + \sinh t \sinh\frac{t}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \cosh\frac{t}{\sqrt{2}} \end{pmatrix}. \end{aligned} \tag{3.26}$$

When the spacelike curve α is N_1-magnetic according to parallel transport frame, the figure of α and V can be drawn as following:

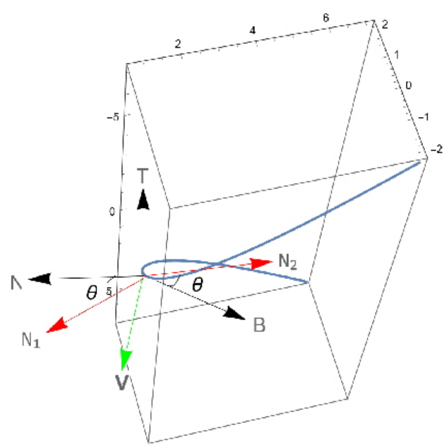

FIGURE 2. .

Similarly, the Killing magnetic vector field V when the curve (3.20) is a T-magnetic or N_2-magnetic trajectory of the Killing magnetic vector field V according to parallel transport frame (3.24) can be found as the above procedure.

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