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FIXED POINT RESULTS FOR PATA CONTRACTION ON A METRIC SPACE WITH A GRAPH AND ITS APPLICATION

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Abstract. Let (X, d) be a metric space endowed with a graph G such that the set $V(G)$ of vertices of G coincides with X. We define the notion of Pata-G-contraction type maps and obtain some fixed point theorems for such mappings. This extends and subsumes many recent results which were obtained for other contractive type mappings on a partially ordered metric space. As an application, we present theorem on the convergence of successive approximations for some linear operators on a Banach space.

1. Introduction

Let f be a selfmap of a metric space (X, d). Following Petrusel and Rus [14], we say that f is a Picard operator (abbr., PO) if f has a unique fixed point x^* and $\lim_{n\to\infty} f^n x = x^*$ for all $x \in X$ and is called a weakly Picard operator (abbr. WPO) if the sequence $(f^n x)_{n\in\mathbb{N}}$ converges, for all $x \in X$ and the limit (which may depends on x) is a fixed point of f. Let (X, d) be a metric space. Let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [5, 11]) by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \in \ X \times X \ : \ (y, x) \in E(G)\}.$$

The letter $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph

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for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \ \cup \ E(G^{-1}). \tag{1}$$

We call $(V^{'}, E^{'})$ a subgraph of G if $V^{'} \subseteq V(G)$, $E^{'} \subseteq E(G)$ and for any edge $(x, y) \in E^{'}$, $x, y \in V^{'}$. Now we recall a few basic notions concerning the connectivity of graph. All of them can be found, e.g., in [5]. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N} \cup \{\not\vdash\}$) is a sequence $(x_i)_{i=0}^{N}$ of $N + 1$ vertices such that

$$x_0 = x, x_N = y \text{ and } (x_{i-1}, x_i) \in E(G) \text{ for } i = 1, \ldots, N.$$

A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Clearly, G_x is connected.

Definition 1. [2] *We say that a mapping $f : X \to X$ is a Banach G-contraction or simply a G-contraction if f preserves edges of G, i.e.,*

$$\forall x, y \in X \quad ((x, y) \in E(G) \text{ implies } (fx, fy) \in E(G)),$$

and f decreases weights of edges of G in the following way:

$$\exists \alpha \in (0, 1) \ , \ \forall x, y \in X \qquad ((x, y) \in E(G) \text{ implies } d(fx, fy) \leq \alpha d(x, y)).$$

For more details, we refer the reader to the papers [1, 3, 12].

2. Iterations and fixed points of Pata-G-contractions

Throughout this section we assume that (X, d) is a metric space, and G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. The set of all fixed point of a mapping f is denoted by $Fixf$.
Recently Pata in [13] introduced a fixed point theorem with weaker hypotheses than those of the Banach contraction principle with an explicit estimate of the convergence rate. This idea was developed by [11, 9, 7, 6, 4].

The aim of this paper is to introduce Pata-G-contractions and obtain results on the existence of a fixed point for single-valued mappings in metric spaces (X, d) by following the technique of Pata [13].

Selecting an arbitrary $x_0 \in X$ we denote

$$\| x \| = d(x, x_0) \text{ for all } x \in X.$$

Let $\psi : [0,1] \to [0,\infty)$ is an increasing function vanishing with continuity at zero. Also consider the vanishing sequence depending on $\alpha \geq 1$, $w_n(\alpha) = (\frac{\alpha}{n})^\alpha \sum_{k=1}^{n} \psi(\frac{\alpha}{k})$.

Definition 2. *We say that a mapping $f : X \to X$ is a Pata-G-contraction if f preserves edges of G, i.e.,*

$$\forall x, y \in X((x,y) \in E(G) \text{ implies } (fx, fy) \in E(G)), \tag{2}$$

and f decreases weights of edges of G in the following way:

$$d(fx, fy) \leq (1-\epsilon)d(x,y) + \Lambda\epsilon^\alpha \psi(\epsilon)[1 + \|x\| + \|y\|]^\beta. \tag{3}$$

That inequality is satisfied for every $\epsilon \in [0,1]$ and every $x, y \in X$. also let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0,\alpha]$ be fixed constants.

Example 3. *Any constant function $f : X \to X$ is a Pata-G-contraction since $E(G)$ contains all loops. (In fact, $E(G)$ must contain all loops if we wish any constant function to be Pata-G-contraction.)*

Example 4. *Let $\preceq$ be a partial order in X. Define the graph G_1 by*

$$E(G_1) := \{(x,y) \in X \times X : x \preceq y\}.$$

Example 5. *Let $X = \{0,1,2,3\}$ and the Euclidean metric $d(x,y) = |x-y|$, $\forall x, y \in X$. The mapping $f : X \to X$, $fx = 0$, for $x \in \{0,1\}$ and $fx = 1$, for $x \in \{2,3\}$ is a Pata-G-contraction where $G = \{(0,1); (0,2); (2,3); (0,0); (1,1); (2,2); (3,3)\}$.*

Proposition 6. *If a mapping $f : X \to X$ is such that (2) (resp., (3)) holds, then (2)(resp. (3)) is also satisfied for graphs G^{-1} and $\tilde{G}$. Hence, if f is a Pata-G-contraction, then f is both a Pata-G^{-1}-contraction and a Pata-$\tilde{G}$-contraction.*

Proof. This is an obvious consequence of symmetry of d and (1).

Example 7. *Let $\preceq$ be a partial order in X. Set*

$$E(G_2) := \{(x,y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

In particular, for this graph (1.2) holds if f is monotone with respect to the order. Moreover, if f satisfies (3) with $G := G_1$ from Example 4, then by proposition 6, (3) holds with $G := G_2$ since $G_2 = \tilde{G_1}$.

Our first result shows that the convergence of successive approximations for Pata G-contractions is closely related to the connectivity of a graph. Also, we say sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ elements of X, are Cauchy equivalent if each of them is a Cauchy sequence and $d(x_n, y_n) \to 0$.

Lemma 8. *Let $f : X \to X$ be a Pata-G-contraction. Then given $x \in X$ and $y \in [x]_{\tilde{G}}$, there exist constants $N(x,y) \in \mathbb{N}$ and $C(x,y) \in \mathbb{R}$ that $N(x,y)$ is number edges that there is from x to y, such that*

$$d(f^n x, f^n y) \leq N(x,y)C(x,y)w_n(\alpha).$$

Proof. Step1: Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then there is a path $(z_i^0)_{i=0}^{N(x,y)}$ in $\tilde{G}$ from x to y, i.e., $z_0^0 = x$, $z_{N(x,y)}^0 = y$. we introduce the sequences

$$z_i^n = f^n z_i^0 \text{ and } c_i^n = \|f^n z_i^0\| = (f^n z_i^0, z_i^0) \text{ for all } i = 1, \ldots, N(x,y).$$

For all $i \in \{1, 2, \ldots, N(x,y)\}$, the sequence $\|f^n z_i^0\| = c_i^n$ is bounded.
Starting from x, Exploiting the inequalities

$$d(f^{n+1}z_i^0, f^n z_i^0) \le (1-\epsilon)d(f^n z_i^0, f^{n-1}z_i^0) + \Lambda\epsilon^\alpha\psi(\epsilon)[1 + \|f^n z_i^0\| + \|f^{n-1}z_i^0\|]^\beta.$$

Since (2) is true for all $\epsilon \in [0,1]$, we put $\epsilon = 0$. Then we have the following relations

$$d(f^{n+1}z_i^0, f^n z_i^0) \le d(f^n z_i^0, f^{n-1}z_i^0) \le \ldots \le d(f z_i^0, z_i^0) = c_i^1.$$

By triangle inequality, we have

$$d(f^n z_i^0, z_i^0) \le d(f^n z_i^0, z_i^1) + d(z_i^1, z_i^0),$$

$$d(f^n z_i^0, z_i^1) \le d(f^n z_i^0, f^{n+1}z_i^0) + d(f^{n+1}z_i^0, z_i^1).$$

We deduce the bound

$$c_i^n = d(f^n z_i^0, z_i^0) \le d(f^{n+1}z_i^0, z_i^1) + 2c_i^1 \text{ for } i = 1, \ldots, N(x,y),$$

therefore, as $\beta \le \alpha$, we infer from (2) that

$$\begin{aligned} c_i^n &\le (1-\epsilon)c_i^n + \Lambda\epsilon^\alpha\psi(\epsilon)[1 + c_i^n + c_i^0]^\beta + 2c_i^1 \\ &\le (1-\epsilon)c_i^n + a\epsilon^\alpha\psi(\epsilon)(c_i^n)^\alpha + b, \end{aligned}$$

for some $a, b > 0$. Accordingly,

$$\epsilon c_i^n \le a\epsilon^\alpha\psi(\epsilon)(c_i^n)^\alpha + b.$$

If there is a subsequence $c_i^{n_\iota} \to \infty$, the choice $\epsilon = \epsilon_\iota = \frac{(1+b)}{c_i^{n_\iota}}$ leads to the contradiction

$$1 \le a(1+b)^\alpha\psi(\epsilon_\iota) \to 0.$$

Step2: put $C(x,y) = \sup_{n\in\mathbb{N}} \Lambda[1 + \|c_1^n\| + \|c_2^n\| + \ldots + \|c_{N(x,y)}^n\|]^\beta < \infty$. We prove following

$$d(f^n x, f^n y) \le N(x,y)C(x,y)w_n(\alpha) \text{ for all } n \in \mathbb{N}.$$

By induction on n, we show the sequence $p_n^i = n^\alpha d(f^n z_i^0, f^n z_{i-1}^0) \le C(x,y)\alpha^\alpha\Sigma_{k=1}^n\psi(\frac{\alpha}{k})$ where $i \in \{1, 2, \ldots, N(x,y)\}$. for $n = 1$,

$$p_1^i = d(f z_i^0, f z_{i-1}^0) \le (1-\epsilon)d(z_{i-1}^0, z_i^0) + \Lambda\epsilon^\alpha\psi(\epsilon)[1 + \|z_{i-1}^0\| + \|z_i^0\|]^\beta,$$

for all $\epsilon \in [0,1]$. By putting $\epsilon = 1$, we have

$$p_1^i \leq \Lambda \epsilon^\alpha \psi(\epsilon)[1 + \|z_{i-1}^0\| + \|z_i^0\|]^\beta$$

which implies

$$p_1^i \leq C(x,y)\alpha^\alpha \psi(\alpha).$$

So, we have

$$p_n^i = n^\alpha d(f^n z_i^0, f^n z_{i-1}^0) \leq n^\alpha (1-\epsilon) d(f^{n-1} z_i^0, f^{n-1} z_{i-1}^0) + C(x,y)\epsilon^\alpha \psi(\epsilon),$$

choosing at each n

$$\epsilon = 1 - (\frac{n}{n+1})^\alpha \leq \frac{\alpha}{n+1},$$

$$p_n^i \leq (n-1)^\alpha d(f^{n-1} z_i^0, f^{n-1} z_{i-1}^0) + C(x,y)\alpha^\alpha \psi(\frac{\alpha}{n}),$$

we end up with

$$p_{n+1}^i \leq p_n^i + C(x,y)\alpha^\alpha \psi(\frac{\alpha}{n+1}).$$

Since $p_0^i = 0$, this gives

$$p_n^i \leq C(x,y)\alpha^\alpha \sum_{k=1}^{n} \psi(\frac{\alpha}{k}),$$

and a final division by n^α will do. Now, since $i \in \{1, 2, \ldots, N(x,y)\}$, by triangle inequality, we have

$$d(f^n x, f^n y) \leq N(x,y)C(x,y)w_n(\alpha).$$

Theorem 9. *Let (X,d) be a metric space endowed with a graph G and $f : X \to X$ be a Pata-G-contraction such that the graph G is weakly connected. For all $x, y \in X$, the sequences $(f^n x)_{n\in\mathbb{N}}$ and $(f^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent.*

Proof. Let f be a Pata-G-contraction, m be fixed and $x, y \in X$. By hypothesis $[x]_{\tilde{G}} = X$, so $f^m x = x_m \in [x]_{\tilde{G}}$. By Lemma 8, we get

$$d(x_n, x_{n+m}) = d(f^n x, f_m^n x) \leq N(x, x_m)C(x, x_m)\ w_n(\alpha),$$

as $n \to \infty$, $d(f^n x, f^n x_m) \to 0$. This show that sequence $(f^n x)_{n\in\mathbb{N}}$ is Cauchy. So since $y \in [x]_{\tilde{G}}$, Lemma 8 yields

$$d(f^n x, f^n y) \leq N(x,y)C(x,y)w_n(\alpha).$$

As $n \to \infty$, $d(f^n x, f^n y) \to 0$. Thus sequence $(f^n y)_{n\in\mathbb{N}}$ is Cauchy and $(f^n x)_{n\in\mathbb{N}}$, $(f^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent.

Corollary 10. *Let (X, d) be complete. The following statement are equivalent:*
(i) *G is weakly connected;*
(ii) *for any Pata-G-contraction $f : X \to X$, there is $x_* \in X$ such that $\lim_{n\to\infty} f^n x = x_*$ for all $x \in X$.*

Proposition 11. *Assume that $f : X \to X$ is a Pata-G-contraction such that for some $x_0 \in X, fx_0 \in [x_0]_{\tilde{G}}$. Let $\tilde{G_{x_0}}$ be the component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is f-invariant and $f|_{[x_0]_{\tilde{G}}}$ is a Pata-$\tilde{G}_{x_0}$-contraction. Moreover, if $x, y \in [x_0]_{\tilde{G}}$ then $(f^n x)_{n\in\mathbb{N}}$ and $(f^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent.*

Proof. Let $x \in [x_0]_{\tilde{G}}$. Then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to x, i.e., $x_N = x$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1, \ldots, N$. By Proposition 6, f is a Pata-$\tilde{G}$-contraction which yields $(fx_{i-1}, fx_i) \in E(\tilde{G})$ for $i = 1, \ldots, N$, i.e., $(fx_i)_{i=0}^N$ is a path in $\tilde{G}$ from fx_0 to fx. Thus $fx \in [fx_0]_{\tilde{G}}$. Since, by hypothesis, $fx_0 \in [x_0]_{\tilde{G}}$, i.e., $[fx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$, we infer $fx \in [x_0]_{\tilde{G}}$. Thus $[x_0]_{\tilde{G}}$ is f-invariant.
Now let $(x, y) \in E(\tilde{G}_{x_0})$. This means there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to y such that $x_{N-1} = x$. Let $(y_i)_{i=0}^M$ be a path in $\tilde{G}$ from x_0 to fx_0. Repeating the argument from the first part of the proof, we infer $(y_0, y_1, \ldots, y_M, fx_1, \ldots, fx_N)$ is a path in $\tilde{G}$ from x_0 to fy; in particular, $(fx_{N-1}, fx_N) \in E(\tilde{G}_{x_0})$, i.e., $(fx, fy) \in E(\tilde{G}_{x_0})$. Moreover, since $E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$ and f is a Pata-$\tilde{G}$-contraction, we infer (3) holds for the graph $\tilde{G}_{x_0}$. Thus $f|_{[x_0]_{\tilde{G}}}$ is a Pata-$\tilde{G}_{x_0}$-contraction.
Finally, in view of Theorem 9, the second statement follows immediately from the first one since $\tilde{G}_{x_0}$ is connected.

Theorem 12. *Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a Pata-G-contraction. Let $X_f := \{x \in X : (x, fx) \in E(G)\}$. We have the following property:*

$$\begin{aligned} &for\ any\ (x_n)_{n\in\mathbb{N}}\ in\ X,\ if\ x_n \to x\ and\ (x_n, x_{n+1}) \in E(G)\ for\ n \in \mathbb{N}, \\ &Then\ there\ is\ a\ subsequence\ (x_{k_n})_{n\in\mathbb{N}}\ with\ (x_{k_n}, x) \in E(G)\ for\ n \in \mathbb{N}. \end{aligned} \tag{4}$$

Then the following statements hold.
1^o *$cardFixf = card\{[x]_{\tilde{G}} : x \in X_f\}$.*
2^o *For any $x \in X_f, f|_{[x]_{\tilde{G}}}$ is a PO.*
3^o *If $X' := \cup\{[x]_{\tilde{G}} : x \in X_f\}$, then $f|_{X'}$ is a WPO.*
4^o *If $f \subseteq E(G)$, then f is a WPO.*

Proof. We begin with point 2^0. Let $x \in X_f$, then $fx \in [x]_{\tilde{G}}$, so by Proposition 11, if $y \in [x]_{\tilde{G}}$, then $(f^n x)_{n\in\mathbb{N}}$ and $(f^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent. By completeness, $(f^n x)_{n\in\mathbb{N}}$ converges to some $x_* \in X$. Clearly, also $\lim_{n\to\infty} f^n y = x_*$. Since $(x, fx) \in E(G)$, (2) yields

$$(f^n x, f^{n+1} x) \in E(G)\ for\ n \in \mathbb{N}. \tag{5}$$

By (4), there is a subsequence $(f^{k_n} x)_{n\in\mathbb{N}}$ such that $(f^{k_n} x, x_*) \in E(G)$ for $n \in \mathbb{N}$. Hence and by (5), we infer $(x, fx, f^2 x, \ldots, f^{k_1} x, x_*)$ is a path in G (hence also in

$\tilde{G}$) from x to x_*, i.e., $x_* \in [x]_{\tilde{G}}$. Moreover, by (3), we have

$$d(f^{k_{n+1}}x, fx_*) \leq (1-\epsilon)d(f^{k_n}x, x_*) + C\epsilon^{\alpha}\psi(\epsilon) \ for \ n \in \mathbb{N},$$

holds for all $\epsilon \in [0,1]$, put $\epsilon = 0$, so

$$d(f^{k_{n+1}}x, fx_*) \leq d(f^{k_n}x, x_*).$$

Hence, letting n tend to ∞ we conclude $x_* = fx_*$. Thus $f|_{[x]_{\tilde{G}}}$ is a PO.
Now 3^o is an easy consequence of 2^o. To show 4^o observe that $f \subseteq E(G)$ means $X_f = X$. This yields $X^{'} = X$, so f is a WPO in view of 3^o.
To prove 1^o, consider a mapping π define by

$$\pi(x) := [x]_{\tilde{G}} \ for \ all \ x \in Fix \ f.$$

It suffices to show π is a bijection of $Fixf$ onto $C := \{[x]_{\tilde{G}} : x \in X_f\}$. since $E(G) \supseteq \Delta$, we infer $Fixf \subseteq X_f$ which yields $\pi(Fixf) \subseteq C$. On the other hand, if $x \in X_f$, then by 2^o, $\lim_{n\to\infty} f^n x \in [x]_{\tilde{G}} \cap Fixf$ which implies $\pi(\lim_{n\to\infty} f^n x) = [x]_{\tilde{G}}$. Thus f is a surjection of $Fixf$ onto C. Now, if $x_1, x_2 \in Fixf$ are such that $\pi(x_1) = \pi(x_2)$, i.e., $[x_1]_{\tilde{G}} = [x_2]_{\tilde{G}}$, then $x_2 \in [x_1]_{\tilde{G}}$, so by 2^0,

$$\lim_{n\to\infty} f^n x_2 \in [x_1]_{\tilde{G}} \cap Fixf = \{x_1\},$$

i.e., $x_1 = x_2$ since $f^n x_2 = x_1$. Consequently, f is injective. Thus 1^o is proved.

Remark 13. *If we assume that a graph G is such that $E(G)$ is a quasi-order (i.e., it is transitive), then (4) is equivalent to the following:*

$$\begin{aligned} &for \ any \ (x_n)_{n\in\mathbb{N}} \ in \ X, \ if \ x_n \to x \ and \ (x_n, x_{n+1}) \in E(G) \ for \ n \in \mathbb{N}, \\ &Then \ (x_n, x) \in E(G) \ for \ n \in \mathbb{N}. \end{aligned} \tag{6}$$

Proposition 14. *If $E(G)$ is a quasi-order and given $x \in X$, the set $\{y \in X : (x,y) \in E(G)\}$ is closed, then (X,d,G) has property (6).*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be such that $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ and $x_n \to x$. By transitivity, given $n \in \mathbb{N}$,

$$x_m \in \{y \in X : (x_n, y) \in E(G)\} \ for \ m \geq n.$$

Letting m tend to ∞, in view of the hypothesis we get $(x_n, x) \in E(G)$.

3. Application: A generalization of the Kelisky-Rivlin theorem

In 1967, Kelisky and Rivlin defined the Bernstein operator $B_n (n \in \mathbb{N})$ on the space $C[0,1]$ by

$$(B_n\varphi)(t) := \sum_{k=0}^{n} \varphi(\frac{k}{n}) \binom{n}{k} t^k (1-t)^{n-k},$$

for all $\varphi \in C[0,1]$, $t \in [0,1]$ (see [5]). They proved that each Bernstein operator B_n is a WPO. Moreover,

$$\lim_{j\to\infty}(B_n^j\varphi)(t) = \varphi(0) + (\varphi(1) - \varphi(0))t,$$

for all $\varphi \in C[0,1]$, $t \in [0,1]$ and $n \geq 1$, where $\{B_n^j\}_{j\geq 1}$ is the sequence of the iterates of B_n. In 2008, a simple proof of the Kelisky-Rivlin theorem was given by Rus with the help of some trick with the Contraction Principle (see[6]). For more details about Kelisky-Rivlin theorem, we refer the reader to the paper [11].

Our purpose here is to show that the Bernstein operator B_n is a Pata-G-contraction for some graph G such that $B_n \subseteq E(G)$, and hence, in view of theorem 12, B_n is a WPO.

Theorem 15. *Let X be a Banach space and X_0 a closed subspace of X. Let $T : X \to X$ be a linear operator(not necessarily continuous on X) such that $\|T|_{X_0}\| < 1$. If the corresponding field $I - T$ is such that $(I-T)(X) \subseteq X_0$, then T is a WPO. Moreover, $CardFixT = CardX \setminus X_0$. and*

$$(x + X_0) \cap FixT = \{\lim_{n\to\infty} T^n x\} \ for\ x \in X.$$

Proof. Define the following graph $G : V(G) := X$ and for $x, y \in X$,

$$(x, y) \in E(G) \ if\ x - y \in X_0.$$

Clearly, $E(G)$ is an equivalence relation; in particular, $E(G) \supseteq \Delta$ and by symmetry, $\tilde{G} = G$. We show Theorem 12 as an application here. First we prove T is a Pata-G-contraction. Let $x, y \in E(G)$, i.e., $x - y \in X_0$. Then we have

$$Tx - Ty = (y - Ty) - (x - Tx) + (x - y) \in X_0,$$

since, by hypothesis, $y - Ty, x - Tx \in X_0$. Thus $(Tx, Ty) \in E(G)$ and moreover,

$$\|Tx - Ty\| = \|T(x - y)\| \leq \|T|_{X_0}\|\|x - y\|.$$

Since $\|T|_{X_0}\| < 1$, we infer T is a G-contraction.

By using [section 3 of [13]], we have

$$\|Tx - Ty\| \leq (1 - \epsilon)\|x - y\| + \Lambda\epsilon^{1+\gamma}[1 + \|x\| + \|y\|], \quad \forall\gamma > 0,$$

where $\alpha = \beta = 1, \psi(\epsilon) = \epsilon^\gamma$ and

$$\Lambda = \Lambda(\gamma, \lambda) = \frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}}\frac{1}{(1-\lambda)^\gamma}.$$

So, we infer T is a Pata-G-contraction.

Observe that given $x \in X$,

$$\{y \in X : (x, y) \in E(G)\} = x + X_0.$$

Since X_0 is closed, so is $x + X_0$. Thus Proposition 14 implies (X, d, G) has property (6) since, in particular, $E(G)$ is a quasi-order. Now condition $(I - T)(X) \subseteq X_0$

means $(x, Tx) \in E(G)$ for $x \in X$, i.e., $T \subseteq E(G)$. So Theorem 12 imply T is a WPO. Moreover, since $E(\tilde{G})(= E(G))$ is transitive, we infer that given $x \in X$,

$$[x]_{\tilde{G}} = \{y \in X : (x, y) \in E(G)\} = x + X_0.$$

Hence and by Theorem 12 (1^o),

$$cardFixT = card\{x + X_0 : x \in X_T\} = cardX \setminus X_0,$$

since $X_T = X$. Finally, Theorem 12 (2^o) yields the last statement of the thesis.

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