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A SIMULATION STUDY OF THE BAYES ESTIMATOR FOR PARAMETERS IN WEIBULL DISTRIBUTION

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Abstract. The Weibull distribution is one of the most popular distributions in analyzing the lifetime data. In this study, we consider the Bayes estimators of the scale and shape parameters of Weibull distribution under the assumptions of Gamma priors and squared error loss function. While computing the Bayes estimates for a Weibull distribution, the continuous conjugate joint prior distribution of the shape and scale parameters does not exist and the closed form expressions of the Bayes estimators cannot be obtained.

In this study first we will consider the Bayesian inference of the scale parameter under the assumption that the shape parameter is known. We will assume that the scale parameter has a Gamma prior. Under these assumptions Bayes estimate can be obtained in explicit form. When both the parameters are unknown, the Bayes estimates cannot be obtained in closed form. In this case, we will assume that the scale parameter has the Gamma prior, and the shape parameter also has the Gamma prior and they are independently distributed. We will use the Lindley approximation to obtain the approximate Bayes estimators.

Under these assumptions, we will compute approximate Bayes estimators and compare with the maximum likelihood estimators by Monte Carlo simulations.

1. INTRODUCTION

The Weibull distribution has been widely studied since its introduction in 1951 [1]. The distribution is frequently used to model survival, reliability, wind speed and other data. The Weibull distribution is characterized by two parameters, one is the shape parameter (β) and the other is the scale parameter (γ).

If $X \sim$ Weibull(β, γ) then its density function is defined as [2,3]:

$$f(x|\gamma, \beta) = \begin{cases} \beta\gamma x^{\beta-1} e^{-\gamma x^{\beta}} & , \quad x > 0 \\ 0 & , \quad x \leq 0 \end{cases} . \tag{1}$$

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The distribution function can also be derived and is defined as:

$$F(x|\gamma,\beta) = P(X \le x) \ = \left\{ \begin{array}{ll} 0 \quad\quad\quad , & x < 0 \\ 1 - e^{-\gamma x^{\beta}}, & x \ge 0 \end{array} \right. \tag{2}$$

β and γ are non-negative.

For different values of the scale parameter (γ), the graphs of the probability density function of Weibull distribution are shown in Figure 1.

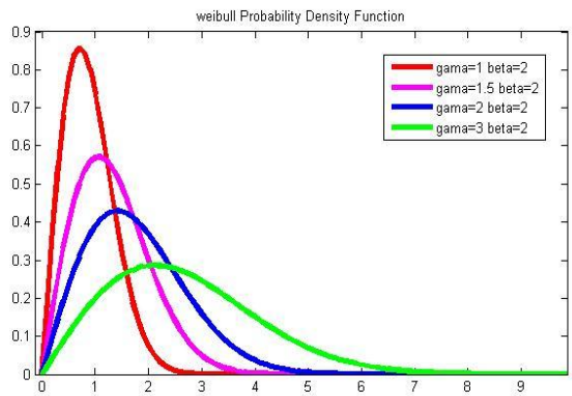

FIGURE 1. Weibull density curve with various values of γ ,$\beta = 2$

For different values of the shape parameter (β), the graphs of the probability density function of Weibull distribution are shown in Figure 2.

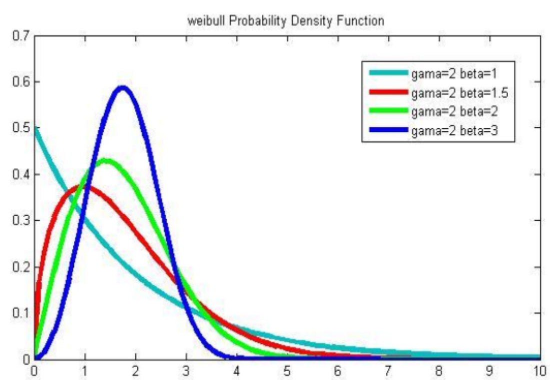

FIGURE 2. Weibull density curve with various values of β, $\gamma = 2$

2. PARAMETER ESTIMATION

2.1. **Maximum Likelihood Estimation.** Maximum-likelihood estimation (MLE) is one of the most common parameter estimation methods for statistical models.

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed Weibull(β, γ) random variables, where the parameters are assumed unknown. To estimate the parameters β and γ the maximum likelihood method is employed. The likelihood function of $X_1, X_2, \ldots, X_n$ can be constructed from Equation (1) as

$$L(\gamma, \beta \,| x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \beta\gamma {x_i}^{\beta-1} e^{-\gamma {x_i}^{\beta}}$$

$$= \beta^n \gamma^n \prod_{i=1}^{n} x_i^{\beta-1} exp\left\{-\gamma \sum_{i=1}^{n} x_i^{\beta}\right\} \tag{3}$$

Taking natural logarithm for both sides yields

$$ln(L) = nln(\beta) + nln(\gamma) + (\beta-1)\sum_{i=1}^{n} lnx_i - \gamma \sum_{i=1}^{n} {x_i}^{\beta} \tag{4}$$

and differentiating ln $L(\gamma, \beta)$ with respect to β and γ respectively and equating to zero we obtain the estimating equations as follows

$$\frac{\partial lnL}{\partial \gamma} = \left(\frac{n}{\gamma}\right) - \sum_{i=1}^{n} {x_i}^{\beta} = 0 \tag{5}$$

$$\frac{\partial lnL}{\partial \beta} = \left(\frac{n}{\beta}\right) + \sum_{i=1}^{n} lnx_i - \gamma \sum_{i=1}^{n} {x_i}^{\beta} lnx_i = 0. \tag{6}$$

From (5) we obtain estimator of γ as,

$$\widehat{\gamma} = \frac{n}{\sum_{i=1}^{n} {x_i}^{\widehat{\beta}}} \tag{7}$$

and on substitution of (7) in (6) we obtain

$$n + \widehat{\beta} \sum_{i=1}^{n} lnx_i = \frac{n\widehat{\beta} \sum_{i=1}^{n} {x_i}^{\widehat{\beta}} lnx_i}{\sum_{i=1}^{n} {x_i}^{\widehat{\beta}}} \ . \tag{8}$$

Equation (8) can be solved numerically for $\widehat{\beta}$. When $\widehat{\beta}$ is obtained, the value of $\widehat{\gamma}$ follows from (7). In this study we have used Newton Raphson Method to obtain $\widehat{\beta}$.

3. Bayes Estimators

3.1. **Shape Parameter Known.** First we consider the shape parameter of Weibull distribution is known and we want to estimate scale parameter. In this case, the prior distribution of the scale parameter may be taken Gamma distribution with probability density function,

$$\pi\left(\gamma|a,b\right)=\frac{b^{a}}{\Gamma\ (\mathrm{a})}\gamma^{a-1}e^{-b\gamma}\quad,\qquad\gamma>0\tag{9}$$

with the hyperparameters $a>0$ and $b>0$.

Posterior distribution of γ, given the likelihood in (3) and the hyperparameters a and b,

$$\begin{aligned}
P(x,\beta,\gamma) &= L(f\,(x|\beta,\gamma))\times\ \pi\left(\gamma|a,b\right)=\beta^{n}\gamma^{n}\ \left(\prod x^{\beta-1}\right)\ e^{-\gamma\sum x^{\beta}}\ \ \times\frac{b^{a}}{\Gamma\ (\mathrm{a})}\gamma^{a-1}e^{-b\gamma}\\
&= \frac{b^{a}}{\Gamma\ (\mathrm{a})}\beta^{n}\ \gamma^{n+a-1}\ \ \left(\prod x^{\beta-1}\right)\ e^{-\gamma(\sum x^{\beta}+b)} \qquad (10)\\
P(x) &= \int_{o}^{\infty}\mathrm{P}(x,\beta,\gamma)d\gamma=\frac{b^{a}}{\Gamma\ (\mathrm{a})}\beta^{n}\left(\prod x^{\beta-1}\right)\int_{0}^{\infty}\ \gamma^{n+a-1}\ e^{-\gamma(\sum x^{\beta}+b)}d\gamma\\
;t &= \gamma(\sum x^{\beta}+b)\ \rightarrow dt=\ d\ \gamma(\sum x^{\beta}+b)\\
&= \frac{b^{a}}{\Gamma\ (\mathrm{a})}\beta^{n}\left(\prod x^{\beta-1}\right)\int_{0}^{\infty}\ (t/\sum x^{\beta}+b)^{n+a-1}\ e^{-t}(1/\sum x^{\beta}+b)dt\\
&= \frac{b^{a}}{\Gamma\ (\mathrm{a})}\beta^{n}\left(\prod x^{\beta-1}\right)(1/\sum x^{\beta}+b)^{n+a}\int_{0}^{\infty}\ t^{n+a-1}\ e^{-t}dt\ \leftarrow\ \Gamma(n+a)\\
&= \frac{b^{a}}{\Gamma\ (\mathrm{a})}\beta^{n}\left(\prod x^{\beta-1}\right)(1/\sum x^{\beta}+b)^{n+a}\Gamma(n+a)
\end{aligned}$$

$$P\left(\gamma|\mathrm{x}\right)=\frac{\mathrm{P}(x,\beta,\gamma)}{\mathrm{P}(x)}=\frac{\frac{b^{a}}{\Gamma\ (\mathrm{a})}\beta^{n}\ \gamma^{n+a-1}\ \left(\prod x^{\beta-1}\right)\ e^{-\gamma(\sum x^{\beta}+b)}}{\frac{b^{a}}{\Gamma\ (\mathrm{a})}\beta^{n}\left(\prod x^{\beta-1}\right)\ \ (1/\sum x^{\beta}+b)^{n+a}\Gamma\ (\mathrm{n+a})}$$

$\frac{\gamma^{n+a-1}}{\Gamma\ (\mathrm{n+a})}\ e^{-\gamma(\sum x^{\beta}+b)}\ (\sum x^{\beta}+b)^{n+a}\sim$ Gamma(n+a, $\sum x^{\beta}+b$)

is Gamma $(n+a,\ \ \sum_{i=1}^{n}{x_i}^{\beta}+b)$. Therefore, the Bayes estimate of γ under the squared error loss function becomes

$$\widehat{\gamma}=\ E(\gamma)=\frac{\mathrm{n+a}}{\sum_{i=1}^{n}x_{i}^{\beta}+b}.$$

3.2. **Shape Parameter Unknown.** When both parameters of the Weibull distribution are considered as random variables, Soland (1969) states that the Weibull distribution does not have a conjugate continuous joint prior distribution [2]. He has suggested use of mixed prior distributions, discrete for the shape parameter, continious for scale parameter. In [4], many different prior distributions have been proposed (Inverted Gamma- Compound Inverted Gamma, Discrete mass function -

Compound Inverted Gamma, Uniform distribution - Compound Inverted Gamma, respectively for the shape and scale parameter). In [3] a Gamma prior on scale parameter and no specific prior on shape parameter is assumed. In [5], the Gamma prior on both the scale and shape parameters have been considered for censored data from Weibull distribution.

In this study we assume that, both the shape and scale parameters are unknown and they are independent of each other. Independent priors for parameters are taken as following,

$$\pi_1(\beta) = \pi(\beta|q_1, p_1) = \frac{{p_1}^{q_1}}{\Gamma(q_1)}\beta^{q_1-1}e^{-p_1\beta}, \qquad \beta > 0 \tag{11}$$

$$\pi(\gamma) = \pi(\gamma|q_2, p_2) = \frac{{p_2}^{q_2}}{\Gamma(q_2)}\gamma^{q_2-1}e^{-p_2\gamma}, \qquad \gamma > 0 \tag{12}$$

respectively. Joint prior distribution for γ and β are being,

$$\pi(\gamma, \beta) = \frac{{p_1}^{q_1}{p_2}^{q_2}}{\Gamma(q_1)\Gamma(q_2)}\gamma^{q_2-1}\beta^{q_1-1}exp - \{(p_1\beta + p_2\gamma)\}. \tag{13}$$

Here, the hyper parameters q_1, p_1 and q_2, p_2 are assumed to be known real numbers. The joint posterior density function of γ and β can be written as

$$\pi(\gamma, \beta \mid x_1, x_2, \ldots, x_n) = \frac{\pi(\gamma, \beta)\, L(\gamma, \beta\, |x_1, x_2, \ldots, x_n)}{\iint \pi(\gamma, \beta)\, L(\gamma, \beta\, |x_1, x_2, \ldots, x_n)\, d\gamma d\beta} \tag{14}$$

$$\pi(\gamma, \beta \mid x_1, x_2, \ldots, x_n) = \frac{\pi(x,\beta,\gamma)}{\mathrm{P}(x)}$$

$$= \frac{\frac{{p_1}^{q_1}{p_2}^{q_2}}{\Gamma(q_1)\Gamma(q_2)}\gamma^{n+q_2-1}\beta^{n+q_1-1}exp - \{(p_1\beta)\}\prod_{i=1}^{n} x_i^{\beta-1} exp\left\{-\gamma\ \sum_{i=1}^{n} x_i^{\beta} + p_2\right\}}{\iint \frac{{p_1}^{q_1}{p_2}^{q_2}}{\Gamma(q_1)\Gamma(q_2)}\gamma^{q_2-1}\beta^{q_1-1}exp - \{(p_1\beta + p_2\gamma)\}\beta^n\gamma^n\prod_{i=1}^{n} x_i^{\beta-1} exp\left\{-\gamma\ \sum_{i=1}^{n} x_i^{\beta}\right\} d\gamma d\beta} \tag{15}$$

$$\mathrm{P}(x) = \iint \pi(x,\beta,\gamma)\, d\gamma d\beta$$

$$= \frac{{p_1}^{q_1}{p_2}^{q_2}}{\Gamma(q_1)\Gamma(q_2)}\Gamma\,(\mathrm{n}+q_2)\int_0^{\infty} \mathrm{e}^{-p_1\beta}\beta^{n+q_1-1}\prod_{i=1}^{n} x_i^{\beta-1}\left(\frac{1}{\sum_{i=1}^{n} x_i^{\beta} + p_2}\right)^{n+q_2} d\beta \tag{16}$$

$$\pi(\gamma, \beta \mid x_1, x_2, \ldots, x_n) = \frac{\gamma^{n+q_2-1}\beta^{n+q_1-1}\prod_{i=1}^{n} x_i^{\beta-1} e^{-\gamma\ \left(\sum_{i=1}^{n} x_i^{\beta} + p_2\right)}\mathrm{e}^{-p_1\beta}}{\Gamma\,(\mathrm{n}+q_2)\int_0^{\infty} \mathrm{e}^{-p_1\beta}\beta^{n+q_1-1}\prod_{i=1}^{n} x_i^{\beta-1}\left(\frac{1}{\sum_{i=1}^{n} x_i^{\beta} + p_2}\right)^{n+q_2} d\beta}. \tag{17}$$

And marginal posterior distribution for γ and β,

$$\pi(\beta \mid x_1, x_2, \ldots, x_n) = \int \pi(\gamma, \beta \mid x_1, x_2, \ldots, x_n)\, d\gamma$$

$$\pi(\beta \mid x_1, x_2, \ldots, x_n) = \frac{\beta^{n+q_1-1} \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-p_1\beta} \left((1/(\sum_{i=1}^n x_i^{\beta} + p_2))^{n+q_2}\right)}{\int \beta^{n+q_1-1} \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-p_1\beta} (1/\ (\sum_{i=1}^n x_i^{\beta} + p_2))^{n+q_2}\ d\beta} \tag{18}$$

$$\pi(\gamma \mid x_1, x_2, \ldots, x_n) = \int \pi(\gamma, \beta \mid x_1, x_2, \ldots, x_n)\, d\beta$$

$$= \frac{\int_0^\infty \gamma^{n+q_2-1} \beta^{n+q_1-1} exp - \{(p_1\beta)\} \prod_{i=1}^n x_i^{\beta-1} exp \left\{-\gamma\ \sum_{i=1}^n x_i^{\beta} + p_2\right\} d\beta}{\Gamma\ (\mathrm{n}+q_2) \int_0^\infty \mathrm{e}^{-p_1\beta} \beta^{n+q_1-1} \prod_{i=1}^n x_i^{\beta-1} \left(\frac{1}{\sum_{i=1}^n x_i^{\beta} + p_2}\right)^{n+q_2} d\beta}. \tag{19}$$

Under the squared error loss function the Bayes estimator of γ and β,

$$\widehat{\beta} = E(\beta) = \frac{\int \beta^{n+q_1} \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-p_1\beta} \left(\frac{1}{\sum_{i=1}^n x_i^{\beta} + p_2}\right)^{n+q_2} d\beta}{\int \beta^{n+q_1-1} \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-p_1\beta} \left(\frac{1}{\sum_{i=1}^n x_i^{\beta} + p_2}\right)^{n+q_2} d\beta} \tag{20}$$

and

$$\widehat{\gamma} = E(\gamma) \frac{\iint_0^\infty \beta^{n+q_1-1} \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-p_1\beta}\ \gamma^{n+q_2}\ e^{-\gamma\left(\sum_{i=1}^n x_i^{\beta} + p_2\right)} d\beta d\gamma}{\Gamma\ (\mathrm{n}+q_2)\ \int \beta^{n+q_1-1} \left(\prod_{i=1}^n x_i^{\beta-1}\right) e^{-p_1\beta} (1/\ (\sum_{i=1}^n x_i^{\beta} + p_2))^{n+q_2}\ d\beta}. \tag{21}$$

It can be seen that (17) cannot be reduced to a closed form and numerical approximations are needed. There exist many tecniques to produce such approximations. In this study, we have used the Lindley's approximation to obtain the approximate Bayes estimators.

3.3. **Lindley's approximation.** Lindley (1980) considered an approximation for the ratio of integrals of the form [6],

$$I = \frac{\int w(\theta)\, exp\{L(\theta)\}\, d\theta}{\int \pi(\theta)\, exp\{L(\theta)\}\, d\theta} \tag{22}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ is the parameter, $L(\theta)$ is the logarithm of the likelihood function, $\pi(\theta)$ is joint prior distribution of θ. $w(\theta)$ is function of θ and let $w(\theta) = u(\theta)\,\pi(\theta)$ we have the posterior expectation,

$$I = E(\ u(\theta) \mid x_1, x_2, \ldots, x_n) = \frac{\int u(\theta)\, exp\{L(\theta) + G(\theta)\}\, d\theta}{\int exp\{L(\theta) + G(\theta)\}\, d\theta} \tag{23}$$

where $G(\theta) = log\pi(\theta)$.

Lindley's expansion of (23) leads to

$$I = E(\ u(\theta) \mid x_1, x_2, \ldots, x_n)$$

$$\approx \left\{ u + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\left(u_{ij}+2u_i g_j\right)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} l_{ijk}\sigma_{ij}\sigma_{kl}u_l \right\}$$

where

$$l_{ijk} = \frac{d^3 l}{d\theta_i d\theta_j d\theta_k}, i = 1,2,\ldots,p, j = 1,2,\ldots,p, k = 1,2,\ldots,p$$

$$l_{ij} = \frac{d^2 l}{d\theta_i d\theta_j}, i = 1,2,\ldots,p, j = 1,2,\ldots,p$$

$$u_i = \frac{du(\theta)}{d\theta_i}, i = 1,2,\ldots,p$$

$$u_{ij} = \frac{d^2 u(\theta)}{d\theta_i d\theta_j}, i = 1,2,\ldots,p, j = 1,2,\ldots,p$$

$$\sigma_{ij} = \left[-l_{ij}\right]^{-1},\ i = 1,2,\ldots,p, j = 1,2,\ldots,p.$$

For two parameters $p = 2$ Lindley's approximation can be written as follows;

$$u\left(\widehat{\theta}\right)_{Bayes} = E\left(\ u\left(\theta\right) \mid x_1, x_2, \ldots, x_n\right)$$

$$\approx u\left(\hat{u}_1, \hat{u}_2\right) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\left(u_{ij}+2u_i g_j\right)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} l_{ijk}\sigma_{ij}\sigma_{kl}u_l \tag{24}$$

where $\hat{u}_1$ and $\hat{u}_2$ are MLE of u_1 and u_2. Lindley's approximation of (17) can be written as follows,

$$\begin{aligned} g(\beta,\gamma) &= \frac{{p_1}^{q_1}}{\Gamma\ (q_1)}\beta^{q_1-1}e^{-p_1\beta} \times \frac{{p_2}^{q_2}}{\Gamma\ (q_2)}\gamma^{q_2-1}e^{-p_2\gamma} \\ &= \frac{{p_1}^{q_1}}{\Gamma\ (q_1)}\frac{{p_2}^{q_2}}{\Gamma\ (q_2)}\beta^{q_1-1}\gamma^{q_2-1}e^{-p_1\beta-p_2\gamma} \quad ,\beta>0\ \gamma>0 \end{aligned}$$

$$\begin{aligned} G(\beta,\gamma) &= logg(\beta,\gamma) = q_1 logp_1 + q_2 \log p_2\ - log\Gamma\ \ (q_1) - \log\Gamma\ \ (q_1) \\ &\quad +(q_1-1\ )\log\beta +(q_2-1\ )\log\gamma\ \ - p_1\beta - p_2\gamma \end{aligned}$$

$$\begin{aligned} g_1 &= \frac{dG(\beta,\gamma)}{d\beta} = \frac{q_1-1}{\beta} - p_1 \\ g_2 &= \frac{dG(\beta,\gamma)}{d\gamma} = \frac{q_2-1}{\gamma} - p_2 \\ L\left(\beta,\gamma\right) &= \beta^n\gamma^n\ \left(\prod x^{\beta-1}\right)\ e^{-\gamma\sum x^{\beta}} \end{aligned}$$

$$\log L\left(\beta,\gamma\right)\ = n\log\beta + n\log\gamma + (\beta+1)\log(\prod x)\ \ - \gamma\sum x^{\beta}$$

$$L_1 = \frac{d\log L\left(\beta,\gamma\right)}{d\beta} = \frac{n}{\beta} + \sum\log x\ \ - \gamma\sum x^{\beta}\ln x$$

$$L_{12} = \frac{d^2\log L\left(\beta,\gamma\right)}{d\beta d\gamma} = \frac{d}{d\gamma}\left(\frac{n}{\beta} + \sum\log x\ \ - \gamma\sum x^{\beta}\ln x\ \right) = -\sum x^{\beta}\ln x$$

$$L_{112} = \frac{d^3 \log L(\beta,\gamma)}{d\beta^2 d\gamma} = \frac{d}{d\beta}\left(-\sum x^\beta \ln x\right) = -\sum x^\beta (\ln x)^{\,2}$$

$$L_{121} = \frac{d^3 \log L(\beta,\gamma)}{d\beta d\gamma d\beta} = \frac{d}{d\beta}\left(-\sum x^\beta \ln x\right) = -\sum x^\beta (\ln x)^{\,2}$$

$$L_{122} = \frac{d^3 \log L(\beta,\gamma)}{d\beta d\gamma^2} = \frac{d^2}{d\beta d\gamma}\left(\frac{d}{d\gamma}\left(\frac{n}{\gamma} - \sum x^\beta\right)\right) = 0$$

$$L_2 = \frac{d \log L(\beta,\gamma)}{d\gamma} = \frac{n}{\gamma} - \sum x^\beta$$

$$L_{21} = \frac{d^2 \log L(\beta,\gamma)}{d\gamma d\beta} = -\sum x^\beta \ln x$$

$$L_{221} = \frac{d^3 \log L(\beta,\gamma)}{d\gamma^2 d\beta} = 0$$

$$L_{212} = \frac{d^3 \log L(\beta,\gamma)}{d\gamma d\beta d\gamma} = 0$$

$$L_{211} = \frac{d^3 \log L(\beta,\gamma)}{d\gamma d\beta^2} = \frac{d}{d\gamma}\left(-\frac{n}{\beta^2} - \gamma \sum x^\beta (\ln x)^{\,2}\right) = -\sum x^\beta (\ln x)^{\,2}$$

$$L_{222} = \frac{d^3 \log L(\beta,\gamma)}{d\gamma^3} = \frac{2n}{\gamma^3}$$

$$L_{11} = \frac{d^2 \log L(\beta,\gamma)}{d\beta^2} = -\frac{n}{\beta^2} - \gamma \sum x^\beta (\ln x)^{\,2}$$

$$L_{111} = \frac{d^3 \log L(\beta,\gamma)}{d\beta^3} = \frac{2n}{\beta^3} - \gamma \sum x^\beta (\ln x)^{\,3}$$

$$L_{21} = \frac{d^2 \log L(\beta,\gamma)}{d\gamma d\beta} = -\sum x^\beta \ln x$$

$$L_{22} = \frac{d^2 \log L(\beta,\gamma)}{d\gamma^2} = -\frac{n}{\gamma^2}$$

$$G_{ij} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}^{-1}$$

$$G_{ij} = \frac{1}{\det(G_{ij})}\begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{bmatrix}$$

$$= \frac{1}{\left(-\frac{n}{\gamma^2}\right) \times \left(-\frac{n}{\beta^2} - \gamma \sum x^\beta (\ln x)^{\,2}\right) - \left(-\sum x^\beta \ln x\right)^2}$$

$$\times \begin{vmatrix} -\frac{n}{\gamma^2} & -\sum x^\beta \ln x \\ -\sum x^\beta \ln x & -\frac{n}{\beta^2} - \gamma \sum x^\beta (\ln x)^{\,2} \end{vmatrix}$$

$$T = \left(-\frac{n}{\gamma^2}\right) \times \left(-\frac{n}{\beta^2} - \gamma \sum x^\beta (\ln x)^{\,2}\right) - \left(-\sum x^\beta \ln x\right)^2$$

$$U = -\ \frac{n}{\gamma^2}$$
$$V = -\sum x^{\beta} \ln x$$
$$W = -\frac{n}{\beta^2} - \gamma \sum x^{\beta} (\ln x)\ ^2$$
$$G_{ij} = \begin{bmatrix} \frac{U}{T} & \frac{V}{T} \\ \frac{V}{T} & \frac{W}{T} \end{bmatrix}$$

Let $U(\beta, \gamma) = \beta$ then 1, $U_2 = U_{12} = U_{21} = U_{11} = U_{22} = 0$

$$\widehat{\beta}_{Bayes} = \ \widehat{\beta}_{Mle} + \ G_{11}g_1 + G_{12}g_2 + \frac{1}{2}(L_{111}{G_{11}}^2 + 3L_{112}G_{11}G_{12} + L_{222}G_{12}G_{22}).$$

$$\begin{aligned} \widehat{\beta}_{Bayes} \quad = \quad & \widehat{\beta} + \frac{U}{T}\left(\frac{q_1 - 1}{\beta} - p_1\right) + \frac{V}{T}\left(\frac{q_2 - 1}{\gamma} - p_2\right) \\ & + \frac{1}{2}\left(\frac{2n}{\beta^3} - \gamma \sum x^{\beta} (\ln x)\ ^3 \times \left(\frac{U}{T}\right)^2 + 3\left(-\sum x^{\beta} (\ln x)\ ^2\right)\right. \\ & \left. \times \frac{U}{T} \times \frac{V}{T} \ + \ \frac{2n}{\gamma^3} \times \frac{V}{T} \times \frac{W}{T}\right) \end{aligned} \tag{25}$$

Let $U(\beta, \gamma) = \gamma$ then 1, $U_2 = U_{12} = U_{21} = U_{11} = U_{22} = 0$

$$\widehat{\gamma}_{Bayes} = \ \widehat{\gamma}_{MLE} + G_{21}g_1 + G_{22}g_2 + \frac{1}{2}(L_{222}{G_{22}}^2 + L_{112}(G_{11}G_{22} + 2{G_{12}}^2) + L_{111}G_{11}G_{12})$$

$$\widehat{\gamma}_{Bayes} = \widehat{\gamma} + \frac{V}{T}\left(\frac{q_1 - 1}{\beta} - p_1\right) + \frac{W}{T}\left(\frac{q_2 - 1}{\gamma} - p_2\right) + \frac{1}{2}$$
$$+ \left(-\sum x^{\beta} (\ln x)\ ^2\right) \times \left(\frac{U}{T} \times \frac{W}{T} + 2\left(\frac{V}{T}\right)^2\right) + \left(\frac{2n}{\beta^3} - \gamma \sum x^{\beta} (\ln x)\ ^3 \times \frac{U}{T} \times \frac{V}{T}\right)$$

4. Simulation Study

In this section, the performances of the maximum likelihood and Bayes estimators of the shape and the scale parameters of Weibull distribution are compared with respect to the the mean square error (MSE) criteria.

We assume that both parameters are unknown and both the shape and scale parameters have independent Gamma priors. We compute approximated Bayes estimates using Lindley's approximation. We also compute the maximum likelihood estimates and we compare the Bayes estimates with the maximum likelihood estimates.

In simulation study, for comparing the performances of the estimators, we have generated random data from Weibull distribution based on Monte Carlo simulation study of 1000 replications for different sample sizes *n=50,100,150* and different parameter values. We used mean of the squared error for comparing criteria calculated as,

$$\boldsymbol{MSE} = \frac{\sum_{i=1}^{1000}\left(\beta_i - \widehat{\beta}_i\right)^2 + (\gamma_i - \widehat{\gamma}_i)^2}{1000}.$$

The resuts are summarized in Table 1.

Table 1. Simulated mean, *MSE* values for the estimators of β=1 and γ=1,1.5,2.

Parameter Values		MLE			LINDLEY		
	n	$\hat{\beta}$	$\hat{\gamma}$	MSE	$\hat{\beta}$	$\hat{\gamma}$	MSE
β=1 γ=1	50	1.0053	1.0264	0.0368	0.9993	1.1094	0.5531
	100	1.0014	1.0102	0.0186	1.0005	1.0458	0.3538
	150	1.0012	1.0082	0.0114	1.00046	1.0197	0.0115
β=1 γ=1.5	50	0.9953	1.5453	0.0434	1.0113	1.5709	0.0459
	100	1.0011	1.5256	0.0189	1.0091	1.5382	0.0197
	150	1.0001	1.5138	0.0128	1.0055	1.5221	0.0131
β=1 γ=2	50	0.9958	2.0440	0.0586	1.0137	2.0704	0.0619
	100	0.9984	2.0299	0.0305	1.0074	2.0429	0.0316
	150	0.9999	2.0186	0.0186	1.0059	2.0272	0.0190

5. CONCLUSION

In this study, we consider the estimating the parameters of Weibull distribution. We want to estimate the parameters by Bayesian method. It is observed that the Bayesian estimator of parameters cannot be obtained under explicit form and numerical integration is required. We used Lindley's approximation and compared the approximate Bayesian estimators under Gamma priors to the maximum likelihood estimators through a simulation study.

In simulation study we see that as sample size increases, both maximum likelihood estimation and Bayes estimation have a decrease in MSE. The results show that Lindley's approximation works well. We can say that these two methods give similar results for parameter estimates.

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