Communications in Advanced Mathematical Sciences
Vol. IV, No. 4, 180-189, 2021
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.947259

Results on the Behavior of the Solutions for Linear Impulsive Neutral Delay Differential Equations with Constant Coefficients

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Abstract
We have given some results regarding the behavior of solutions for first order linear impulsive neutral delay differential equations with constant coefficients. These results were obtained using two different real roots of the corresponding characteristic equation. Finally, two examples are given for solutions of impulsive neutral delay differential equations.

Keywords: Behavior of solutions, characteristic equation, neutral delay differential equation
2010 AMS: Primary 34K06, 34K38, 34K40, 34K45

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Received: 2 June 2021, **Accepted:** 12 October 2021, **Available online:** 27 December 2021

1. Introduction and Preliminaries

The author [1] has recently obtained some results regarding asymptotic behavior and stability for solutions of first order linear impulsive neutral delay differential equations with constant coefficient and constant delay. These results are obtained using a real root of the corresponding characteristic equations. Our aim in this article is to obtain different results from the article in [1] by using two different real roots of the corresponding characteristic equation.

Consider the linear impulsive neutral delay differential equation

$$[x(t)+cx(t-\sigma)]' = ax(t)+bx(t-\tau)\,, \quad t\neq t_k\,, \quad t\geq 0, \tag{1.1}$$

$$\triangle x(t_k) = \ell_k\,, \quad k\in\mathbb{Z}^+ = \{1,2,\cdots\}, \tag{1.2}$$

where σ and τ are positive constants, a,b,c and ℓ_k are real constants, $x(t)\in\mathbb{R}$ and $\triangle x(t_k) = x(t_k^+)-x(t_k^-)$. The impulse points t_k satisfy $0<t_1<\cdots<t_k<t_{k+1}<\cdots$ and $\lim_{k\to\infty} t_k = \infty$ and also $t_k-\sigma$ be not impulsive points for all $k\in\mathbb{Z}^+$.

Let's introduce the positive constant h defined by $h=\max\{\sigma,\tau\}$. Together with (1.1), an initial condition is indicated, i.e.

$$x(t) = \phi(t)\,, \quad -h\leq t\leq 0, \tag{1.3}$$

where *the initial function* ϕ is any given continuous real-valued function on the interval $[-h,0]$.

With the equation (1.1) we associate its *characteristic equation*

$$\lambda\left(1+ce^{-\lambda\sigma}\right) = a+be^{-\lambda\tau}. \tag{1.4}$$

Equation (1.4) is obtained from (1.1) by looking for solutions of the form $x(t) = e^{\lambda t}$ for $t \in \mathbb{R}$.

The authors in [2]-[6] obtained interesting results for the solutions of linear impulsive neutral delay differential equations in the form of (1.1). The authors in [2] examined some classes of integro-functional inequalities of the Gronwall type for piecewise continuous functions, and through the results obtained from them, they made estimates for the solutions of impulsive functional differential equations. As an application, they have proven the existence of solutions of certain nonlinear equations with arbitrarily long lifespan for sufficiently small initial functions. Later on, in the article [3] made by the same authors, the problem of stability under persistent disturbances of an impulsive systems of differential-difference equations of neutral type is investigated. An as application the existence of a global solutions of a systems with quadratic nonlinearities is proved for sufficiently small initial data. In [4], by means of Lyapunov's direct method sufficient conditions for uniform asymptotic stability of the zero solution of impulsive systems of differential-difference equations of neutral type are found. In [5], the authors examined the asymptotic behavior of positive solutions of first-order neutral impulsive differential equations with constant coefficients and constant delays, and established the necessary and sufficient conditions for the existence of such solutions. Finally, the authors in [6] established some criteria for the asymptotic stability of a neutral delay control system by applying the Lyapunov functions and Razumikhin technique, which combine with impulsive feedback control. They also showed that the stability behavior of the system can be controlled by appropriate impulsive perturbations.

In this paper, we construct estimates for (1.1)-(1.3) solutions using two different real roots of the corresponding characteristic equation. We obtained the results using the methods in [1, 7, 8]. Sufficient information about the delay or neutral impulsive differential equations and initial value problem (1.1)-(1.3) is given in [1]. For more results regarding delay or neutral impulsive differential equations, we refer the reader to [9]-[16] and references therein.

2. The Main Result

In this section, before going to the main result, we will give an lemma about two different real roots of the characteristic equation (1.4) by Philos and Purnaras [8]. In the following lemma, only the first part of the lemma in [8] is considered.

Lemma 2.1. *([8], Lemma 3.1) Suppose that* $c \le 0 \quad \text{and} \quad b < 0$. *Let* λ_0 *be a nonpositive real root of the characteristic equation* (1.4) *and let* $\beta(\lambda_0) = b\tau e^{-\lambda_0\tau} + ce^{-\lambda_0\sigma}(1-\lambda_0\sigma)$. *Then*

$$1+\beta(\lambda_0) > 0$$

if (1.4) *has another real root less than* λ_0, *and*

$$1+\beta(\lambda_0) < 0$$

if (1.4) *has another nonpositive real root greater than* λ_0.

Now, our main conclusion in this article is that we can give the following theorem.

Theorem 2.2. *Suppose that*

$$c \le 0 \quad \text{and} \quad b < 0.$$

Let λ_0 *be a nonpositive real root of the characteristic equation* (1.4) *with* $1+\beta(\lambda_0) \neq 0$ *where* $\beta(\lambda_0)$ *is defined as in Lemma 2.1, and let*

$$L(\lambda_0;\phi) = \phi(0) + c\phi(-\sigma) + be^{-\lambda_0\tau}\int_{-\tau}^{0} e^{-\lambda_0 s}\phi(s)ds - c\lambda_0 e^{-\lambda_0\sigma}\int_{-\sigma}^{0} e^{-\lambda_0 s}\phi(s)ds.$$

Let also λ_1 *be a nonpositive real root of* (1.4) *with* $\lambda_0 \neq \lambda_1$.

(I) Assume that $\lambda_0 > \lambda_1$ *and* $\ell_i > 0$ *for* $i \in \mathbb{Z}^+$. *Also let there be a number* $d_1 > 0$ *such that it is provided*

$$1+\beta(\lambda_0) \ge \frac{1}{d_1}\sum_{i=1}^{\infty} \ell_i e^{-\lambda_0 t_i}, \tag{2.1}$$

then, for any $\phi \in C\big([-h,0],\mathbb{R}\big)$ *such that*

$$\phi(t) \le e^{\lambda_0 t}\left[d_1 + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right] \quad for \quad t \in [-h,0], \tag{2.2}$$

the solution x of (1.1)-(1.3) *satisfies*

$$D_1(\lambda_0,\lambda_1;\phi) \le e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for all } t \ge 0, \tag{2.3}$$

where

$$D_1(\lambda_0,\lambda_1;\phi) = \min_{-h\le t\le 0}\left\{e^{-\lambda_1 t}\left[\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right]\right\}.$$

Note: Since $\lambda_0 > \lambda_1$, according to the Lemma 2.1 is $1+\beta(\lambda_0) > 0$.

(II) Assume that $\lambda_0 < \lambda_1$ and $\ell_i < 0$ for $i \in \mathbb{Z}^+$. Also let there be a number $d_2 > 0$ such that it is provided

$$1+\beta(\lambda_0) \le \frac{1}{d_2}\sum_{i=1}^{\infty}\ell_i e^{-\lambda_0 t_i}, \tag{2.4}$$

then, for any $\phi \in C([-h,0],\mathbb{R})$ such that

$$e^{\lambda_0 t}\left[d_2 + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right] \le \phi(t) \quad \text{for} \quad t \in [-h,0], \tag{2.5}$$

the solution x of (1.1)-(1.3) *satisfies*

$$d_2 e^{(\lambda_0-\lambda_1)t} \le e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \le D_2(\lambda_0,\lambda_1;\phi) \quad \text{for all } t \ge 0, \tag{2.6}$$

where

$$D_2(\lambda_0,\lambda_1;\phi) = \max_{-h\le t\le 0}\left\{e^{-\lambda_1 t}\left[\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right]\right\}.$$

Note: Since $\lambda_0 < \lambda_1$, according to the Lemma 2.1 is $1+\beta(\lambda_0) < 0$.

Proof. (Proof of Part (I) of the Theorem 2.2): We will show that the double inequality (2.3) is first

$$D_1(\lambda_0,\lambda_1;\phi) \le e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \quad \text{for all } t \ge 0,$$

and

$$e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for all } t \ge 0,$$

respectively. Let $\phi \in C([-h,0],\mathbb{R})$ such that satisfies (2.2) and x be the solution of (1.1)-(1.3). Furthermore, let $y(t) = e^{-\lambda_0 t}x(t)$ for $t \ge -h$. As it has been shown ([1], Lemma 1.1), the fact that x satisfies (1.1)-(1.3) for $t \ge 0$ is equivalent to the fact that y satisfies

$$\begin{aligned} y(t)+ce^{-\lambda_0\sigma}y(t-\sigma) &= L(\lambda_0;\phi) + \sum_{i=1}^{n(t)}\ell_i e^{-\lambda_0 t_i} \\ &\quad + (a-\lambda_0)\int_0^t y(s)ds + be^{-\lambda_0\tau}\int_0^{t-\tau}y(s)ds - c\lambda_0 e^{-\lambda_0\sigma}\int_0^{t-\sigma}y(s)ds \end{aligned} \tag{2.7}$$

where

$$n(t) = max\{k \in \mathbb{Z}^+ : \quad t_k \le t\} \quad \text{and} \quad n(t) = 0 \quad \text{if } t < t_1.$$

In addition, the initial condition (1.3) can be made equivalent to

$$y(t) = e^{-\lambda_0 t}\phi(t) \quad \text{for } t \in [-h,0].$$

Later on, by using the fact that λ_0 is root of (1.4) and by using $z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}$ for $t \ge -h$, then (2.7) becomes

$$z(t) + ce^{-\lambda_0\sigma}z(t-\sigma) = \sum_{i=1}^{n(t)}\ell_i e^{-\lambda_0 t_i} - be^{-\lambda_0\tau}\int_{t-\tau}^{t}z(s)ds + c\lambda_0 e^{-\lambda_0\sigma}\int_{t-\sigma}^{t}z(s)ds \quad \text{for } t \ge 0 \tag{2.8}$$

and we immediately see that the initial condition (1.3) becomes

$$z(t) = e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for } t \in [-h,0]. \tag{2.9}$$

Next, let us define

$$w(t) = e^{(\lambda_0-\lambda_1)t} z(t) \quad \text{for } t \geq -h.$$

By the use of the function w, (2.8) becomes

$$\begin{aligned} w(t) + ce^{-\lambda_1\sigma} w(t-\sigma) &= e^{(\lambda_0-\lambda_1)t} \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\ &- be^{-\lambda_0\tau} \int_{t-\tau}^{t} e^{(\lambda_0-\lambda_1)(t-s)} w(s)ds + c\lambda_0 e^{-\lambda_0\sigma} \int_{t-\sigma}^{t} e^{(\lambda_0-\lambda_1)(t-s)} w(s)ds \quad \text{for } t \geq 0. \end{aligned} \tag{2.10}$$

Also, (2.9) takes the following equivalent form

$$w(t) = e^{-\lambda_1 t}\left[\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t}\right] \quad \text{for } t \in [-h,0].$$

By way of the definitions of y, z and w, we have

$$w(t) = e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t}\right] \quad \text{for } t \geq -h. \tag{2.11}$$

Thus, from the definition of the constant $D_1(\lambda_0,\lambda_1;\phi)$, it follows that the double inequality (2.3) in the conclusion of our theorem can equivalently be written as follows

$$\min_{-h\leq s\leq 0} w(s) \leq w(t) \leq d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for all } t \geq 0. \tag{2.12}$$

The proof of the theorem will be accomplished by proving the double inequality (2.12). First, let's prove the following inequality of the double inequality (2.12)

$$\min_{-h\leq s\leq 0} w(s) \leq w(t) \quad \text{for all } t \geq 0. \tag{2.13}$$

To prove (2.13), we consider an arbitrary real number A such that $A < \min_{-h\leq s\leq 0} w(s)$. Clearly,

$$A < w(t) \quad \text{for } -h \leq t \leq 0. \tag{2.14}$$

We will show that

$$A < w(t) \quad \text{for all } t \geq 0. \tag{2.15}$$

To this end, let us assume that (2.15) fails to hold. Then, because of (2.14), there exists a point $t^* > 0$ so that

$$A < w(t) \quad \text{for } -h \leq t < t^*, \quad \text{and} \quad w(t^*) = A.$$

Thus, by using the hypothesis that $c \leq 0$, $b < 0$, $\ell_i > 0 \quad$ for $i \in \mathbb{Z}^+$ and taking into account the fact that $\lambda_0 \leq 0$, from (2.10) we obtain

$$
\begin{aligned}
A = w(t^*) &= -ce^{-\lambda_1\sigma}w(t^*-\sigma) + e^{(\lambda_0-\lambda_1)t^*}\sum_{i=1}^{n(t^*)}\ell_i e^{-\lambda_0 t_i} \\
&\quad - be^{-\lambda_0\tau}\int_{t^*-\tau}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}w(s)ds + c\lambda_0 e^{-\lambda_0\sigma}\int_{t^*-\sigma}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}w(s)ds \\
&> A\left\{-ce^{-\lambda_1\sigma} - be^{-\lambda_0\tau}\int_{t^*-\tau}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}ds + c\lambda_0 e^{-\lambda_0\sigma}\int_{t^*-\sigma}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}ds\right\} + e^{(\lambda_0-\lambda_1)t^*}\sum_{i=1}^{n(t^*)}\ell_i e^{-\lambda_0 t_i} \\
&> A\left\{-ce^{-\lambda_1\sigma} - be^{-\lambda_0\tau}\int_{t^*-\tau}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}ds + c\lambda_0 e^{-\lambda_0\sigma}\int_{t^*-\sigma}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}ds\right\} \\
&= A\left\{-ce^{-\lambda_1\sigma} - be^{-\lambda_0\tau}\left(\frac{1}{\lambda_1-\lambda_0}\right)\left[1-e^{(\lambda_0-\lambda_1)\tau}\right] + c\lambda_0 e^{-\lambda_0\sigma}\left(\frac{1}{\lambda_1-\lambda_0}\right)\left[1-e^{(\lambda_0-\lambda_1)\sigma}\right]\right\} \\
&= \frac{A}{\lambda_1-\lambda_0}\left\{-c\lambda_1 e^{-\lambda_1\sigma} + be^{-\lambda_1\tau} + c\lambda_0 e^{-\lambda_0\sigma} - be^{-\lambda_0\tau}\right\} \\
&= \frac{A}{\lambda_1-\lambda_0}\left\{(\lambda_1-a)+(a-\lambda_0)\right\} = A.
\end{aligned}
$$

This is a contradiction and hence (2.15) is always satisfied. We have thus proved that (2.15) holds true for all real numbers A with $A < \min_{-h\le s\le 0} w(s)$. This guarantees that (2.13) is fulfilled and so, the first part of the double inequality (2.12) (or, (2.3)) is proved.

Now, let's prove the second part of the double inequality (2.3). Property (2.2) implies $\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t} \le d_1 e^{\lambda_0 t}$. So, if both sides of this inequality are multiplied by $e^{-\lambda_1 t}$, using the definition (2.11), it follows that

$$
e^{-\lambda_1 t}\left[\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for } t\in[-h,0]
$$

or by way of the definition of w, we have

$$
w(t) \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for } t\in[-h,0]. \tag{2.16}
$$

We will show that $d_1 e^{(\lambda_0-\lambda_1)t}$ is a bound of w on the whole interval $[-h,\infty]$, namely that

$$
w(t) \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for all } t\ge -h. \tag{2.17}
$$

For the sake of contradiction suppose that there exists a $\bar{t} > 0$ such that $w(\bar{t}) > d_1 e^{(\lambda_0-\lambda_1)\bar{t}}$. Let

$$
t_* = \inf\left\{\bar{t}: \quad w(\bar{t}) > d_1 e^{(\lambda_0-\lambda_1)\bar{t}}\right\}.
$$

Now, by right continuity, either $w(t_*) = d_1 e^{(\lambda_0-\lambda_1)t_*}$ if there is no impulsive point at t_*, or $w(t_*) \ge d_1 e^{(\lambda_0-\lambda_1)t_*}$ as a consequence of a t_*. Whatever the case, using right continuity, we thus have $w(t) \le d_1 e^{(\lambda_0-\lambda_1)t}$ for $t\in[-h,t_*)$, where $w(t_*) = d_1 e^{(\lambda_0-\lambda_1)t_*}$ if this occors at a non-impulsive point. Then, by using the hypothesis that $c\le 0$, $b<0$, $\ell_i>0$ for $i\in\mathbb{Z}^+$ and taking into account the fact that $\lambda_0\le 0$, and also using (2.1), from (2.10) we have that

$$\begin{aligned}
d_1 e^{(\lambda_0-\lambda_1)t_*} &= w(t_*) = -ce^{-\lambda_1\sigma} w(t_*-\sigma) + e^{(\lambda_0-\lambda_1)t_*} \sum_{i=1}^{n(t_*)} \ell_i e^{-\lambda_0 t_i} \\
&\quad - be^{-\lambda_0\tau} \int_{t_*-\tau}^{t_*} e^{(\lambda_0-\lambda_1)(t_*-s)} w(s)ds + c\lambda_0 e^{-\lambda_0\sigma} \int_{t_*-\sigma}^{t_*} e^{(\lambda_0-\lambda_1)(t_*-s)} w(s)ds \\
&\le -cd_1 e^{-\lambda_1\sigma} e^{(\lambda_0-\lambda_1)(t_*-\sigma)} + e^{(\lambda_0-\lambda_1)t_*} \sum_{i=1}^{n(t_*)} \ell_i e^{-\lambda_0 t_i} \\
&\quad - bd_1 e^{-\lambda_0\tau} \int_{t_*-\tau}^{t_*} e^{(\lambda_0-\lambda_1)(t_*-s)} e^{(\lambda_0-\lambda_1)s} ds + cd_1\lambda_0 e^{-\lambda_0\sigma} \int_{t_*-\sigma}^{t_*} e^{(\lambda_0-\lambda_1)(t_*-s)} e^{(\lambda_0-\lambda_1)s} ds \\
&< d_1 e^{(\lambda_0-\lambda_1)t_*} \left\{ -ce^{-\lambda_0\sigma} - be^{-\lambda_0\tau}\tau + c\lambda_0 e^{-\lambda_0\sigma}\sigma + \frac{1}{d_1}\sum_{i=1}^{\infty} \ell_i e^{-\lambda_0 t_i} \right\} \\
&= d_1 e^{(\lambda_0-\lambda_1)t_*} \left\{ -\beta(\lambda_0) + \frac{1}{d_1}\sum_{i=1}^{\infty} \ell_i e^{-\lambda_0 t_i} \right\} \le d_1 e^{(\lambda_0-\lambda_1)t_*}.
\end{aligned}$$

This gives us the desired contradiction, since we proved $w(t_*) < d_1 e^{(\lambda_0-\lambda_1)t_*}$, and we assumed $w(t_*) = d_1 e^{(\lambda_0-\lambda_1)t_*}$ if t_* is a continuity point, or $w(t_*) \ge d_1 e^{(\lambda_0-\lambda_1)t_*}$ if t_* is a discontinuity point. So (2.17) is true and the second part of the double inequality (2.12) (or, (2.3)) is proved. As a result, the Part (I) of Theorem 2.2 has been proven.

(Proof of Part (II) of the Theorem 2.2): As in Part (I), the double inequality (2.6) can be shown to be

$$e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t}\right] \le D_2(\lambda_0,\lambda_1;\phi) \quad \text{for all } t \ge 0,$$

and

$$d_2 e^{(\lambda_0-\lambda_1)t} \le e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t}\right] \quad \text{for all } t \ge 0$$

respectively. So, from the definition (2.11), it follows that the double inequality (2.6) in the conclusion of our theorem can equivalently be written as follows

$$w(t) \le \max_{-h\le t\le 0} w(t) \quad \text{for all } t \ge 0,$$

and

$$d_2 e^{(\lambda_0-\lambda_1)t} \le w(t) \quad \text{for all } t \ge 0,$$

respectively. Thus, using the hypothesis in the Part (II), it can be proved similarly as in the Part (I). As a result, the proof of the Part (II) of Theorem 2.2 here is omitted.

□

It is immediately clear that the following corollary of double inequalities ((2.3) and (2.6)) in Theorem 2.2 can be written as equivalent.

Corollary 2.3. *Assume that the conditions in Theorem 2.2 are provided. Then the solution of* (1.1)-(1.3) *satisfies*
(I) for $\lambda_1 < \lambda_0$

$$D_1(\lambda_0,\lambda_1;\phi)e^{\lambda_1 t} + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t} \le x(t) \le e^{\lambda_0 t}\left(d_1 + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right) \quad \textit{for all } t \ge 0,$$

(II) for $\lambda_0 < \lambda_1$

$$e^{\lambda_0 t}\left(d_2 + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right) \le x(t) \le D_2(\lambda_0,\lambda_1;\phi)e^{\lambda_1 t} + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t} \quad \textit{for all } t \ge 0.$$

Also, if $\lambda_0, \lambda_1 < 0$*, then from (I) and (II) the solution of* (1.1)-(1.3) *satisfies*

$$\lim_{t\to\infty} x(t) = 0.$$

Example 2.4. Consider

$$[x(t)-\frac{1}{3}x(t-\frac{1}{4})]' = \frac{1}{2}x(t)-\frac{1}{2}x(t-\frac{1}{2})\,, \quad t \neq t_k\,, \quad t \geq 0, \tag{2.18}$$

$$\triangle x(t_k) = \left(\frac{1}{4}\right)^k, \quad k \in \mathbb{Z}^+, \tag{2.19}$$

$$x(t) = \phi(t)\,, \quad -\frac{1}{2} \leq t \leq 0,$$

where $\phi \in C\left([-\frac{1}{2},0],\mathbb{R}\right)$ and t_k are arbitrary impulsive points, such that $t_k - \frac{1}{4}$ are not impulsive points for all $k \in \mathbb{Z}^+$.

The characteristic equation of (2.18) is

$$2\lambda\left(3-e^{-\frac{\lambda}{4}}\right) = 3\left(1-e^{-\frac{\lambda}{2}}\right). \tag{2.20}$$

We see that $\lambda = 0$ and $\lambda \approx -2.08$ are real roots of (2.20). Let $\lambda_0 = 0$ and $\lambda_1 = -2.08$. Let's choose the number $d_1 = 1$. We have $\lambda_0 > \lambda_1$, $\ell_i = \left(\frac{1}{4}\right)^i > 0 \quad i \in \mathbb{Z}^+$ and from (2.1)

$$1+\beta(0) = 1-\frac{1}{4}-\frac{1}{3} = \frac{5}{12} > \sum_{i=1}^{\infty}\left(\frac{1}{4}\right)^i = \frac{1}{3}.$$

Thus, by applying Theorem 2.2-(I) and Corollary 2.3-(I), we obtain the following results:

According to (2.2), for any $\phi \in C\big([-\frac{1}{2},0],\mathbb{R}\big)$ such that

$$\phi(t) \leq \left[1+\frac{L(0;\phi)}{5/12}\right], \quad \text{for } t \in \left[-\frac{1}{2},0\right], \tag{2.21}$$

the solution x of (2.18)-(2.19) satisfies

$$D_1\left(0,-2.08;\phi\right) \leq e^{2.08t}\left[x(t)-\frac{L(0;\phi)}{5/12}\right] \leq e^{2.08t} \quad \text{for all } \ t \geq 0,$$

or equivalent

$$D_1\left(0,-2.08;\phi\right)e^{-2.08t}+\frac{L(0;\phi)}{5/12} \leq x(t) \leq 1+\frac{L(0;\phi)}{5/12} \quad \text{for all } t \geq 0,$$

where

$$L(0;\phi) = \phi(0)-\frac{1}{3}\phi\left(-\frac{1}{4}\right)-\frac{1}{2}\int_{-\frac{1}{2}}^{0}\phi(s)ds$$

and

$$D_1\left(0,-2.08;\phi\right) = \min_{-\frac{1}{2}\leq t\leq 0}\left\{e^{2.08t}\left[\phi(t)-\frac{L(0;\phi)}{5/12}\right]\right\}.$$

Now let's take the special case of $\phi(t) = 1$. Then

$$L(0;1) = 1-\frac{1}{3}-\frac{1}{2}\int_{-\frac{1}{2}}^{0}ds = \frac{5}{12} \quad \text{and} \quad D_1\left(0,-2.08;1\right) = \min_{-\frac{1}{2}\leq t\leq 0}\left\{e^{2.08t}\left[1-\frac{L(0;1)}{5/12}\right]\right\} = 0.$$

Thus, for $\phi(t) = 1$ the inequality (2.21) is provided and the solution x of (2.18)-(2.19) satisfies

$$0 \leq e^{2.08t}\left[x(t)-1\right] \leq e^{2.08t} \quad \text{for all } \ t \geq 0,$$

or equivalent

$$1 \leq x(t) \leq 2 \quad \text{for all } t \geq 0.$$

Example 2.5. Consider

$$[x(t)-e^{-\frac{1}{4}}x(t-\frac{1}{2})]' = -x(t)+e^{-\frac{1}{4}}x(t-\frac{1}{2}), \quad t\neq t_k = k, \quad t\geq 0, \tag{2.22}$$

$$\triangle x(t_k) = -\left(\frac{1}{e^2}\right)^k, \quad k\in\mathbb{Z}^+, \tag{2.23}$$

$$x(t)=\phi(t), \quad -\frac{1}{2}\leq t\leq 0,$$

where $\phi\in C\left([-\frac{1}{2},0],\mathbb{R}\right)$.

The characteristic equation of (2.22) is

$$\lambda\left(1-e^{-\frac{1}{4}}e^{-\frac{\lambda}{2}}\right) = -1+e^{-\frac{1}{4}}e^{-\frac{\lambda}{2}}$$

or

$$\lambda-(\lambda+1)e^{-\frac{1}{4}(2\lambda+1)}+1=0. \tag{2.24}$$

We see that $\lambda=-1$ and $\lambda=-\frac{1}{2}$ are real roots of (2.24). Let $\lambda_0=-1$ and $\lambda_1=-\frac{1}{2}$. Let's choose the number $d_2=\frac{5}{4}$. We have $\lambda_0<\lambda_1$, $\ell_i=-\frac{1}{2}\left(\frac{1}{e^2}\right)^i<0 \quad i\in\mathbb{Z}^+$ and from (2.4)

$$\begin{aligned}1+\beta(-1) &= 1-e^{\frac{1}{4}}\approx -0.284\\ &< -\frac{4}{5}\sum_{i=1}^{\infty}\frac{1}{2}\left(\frac{1}{e^2}\right)^i e^i = -\frac{2}{5}\sum_{i=1}^{\infty}\frac{1}{e^i}\approx -0.232.\end{aligned}$$

Thus, by applying Theorem 2.2-(II) and Corollary 2.3-(II), we obtain the following results:

According to (2.5), for any $\phi\in C\left([-\frac{1}{2},0],\mathbb{R}\right)$ such that

$$e^{-t}\left[\frac{5}{4}+\frac{L(-1;\phi)}{1-e^{\frac{1}{4}}}\right]\leq\phi(s) \quad \text{for } t\in\left[-\frac{1}{2},0\right], \tag{2.25}$$

the solution x of (2.22)-(2.23) satisfies

$$\frac{5}{4}e^{-\frac{t}{2}}\leq e^{\frac{t}{2}}\left[x(t)+\frac{L(-1;\phi)}{1-e^{\frac{1}{4}}}e^{-t}\right]\leq D_2\left(-1,-\frac{1}{2};\phi\right)$$

for all $t\geq 0$, where

$$\begin{aligned}L(-1;\phi) &= \phi(0)-e^{-\frac{1}{4}}\phi\left(-\frac{1}{2}\right)+e^{\frac{1}{4}}\int_{-\frac{1}{2}}^{0}e^s\phi(s)ds-e^{\frac{1}{4}}\int_{-\frac{1}{2}}^{0}e^s\phi(s)ds\\ &= \phi(0)-e^{-\frac{1}{4}}\phi\left(-\frac{1}{2}\right)\end{aligned}$$

and

$$D_2\left(-1,-\frac{1}{2};\phi\right) = \max_{-\frac{1}{2}\leq t\leq 0}\left\{e^{\frac{t}{2}}\left[\phi(t)+\frac{L(-1;\phi)}{1-e^{\frac{1}{4}}}e^{-t}\right]\right\}.$$

Now let's take the special case of $\phi(t)=1$. Then $L(-1;1)=1-e^{-\frac{1}{4}}$ and

$$D_2\left(-1,-\frac{1}{2};\phi\right) = \max_{-\frac{1}{2}\leq t\leq 0}\left\{e^{\frac{t}{2}}\left[1+\frac{1-e^{-\frac{1}{4}}}{1-e^{\frac{1}{4}}}e^{-t}\right]\right\} = \max_{-\frac{1}{2}\leq t\leq 0}\left\{e^{\frac{t}{2}}\left[1+e^{-\left(t+\frac{1}{4}\right)}\right]\right\} = 1+e^{-\frac{1}{4}}.$$

Thus, for $\phi(t)=1$ the inequality (2.25) is provided, i.e.

$$e^{-t}\left[\frac{5}{4}-e^{-\frac{1}{4}}\right]\leq 1 \quad \text{for} \quad t\in\left[-\frac{1}{2},0\right],$$

and for $\phi(t) = 1$ the solution x of (2.22)-(2.23) satisfies

$$\frac{5}{4}e^{-\frac{t}{2}} \leq e^{\frac{t}{2}} \left[x(t) + e^{-\left(t+\frac{1}{4}\right)}\right] \leq 1 + e^{-\frac{1}{4}} \quad \text{for all } \ t \geq 0$$

or from Corollary 2.3-(II) it follows that

$$e^{-t}\left(d_2 + \frac{L(-1;\phi)}{1+\beta(-1)}\right) \leq x(t) \leq D_2\left(-1, -\frac{1}{2}; \phi\right) e^{-\frac{t}{2}} + \frac{L(-1;\phi)}{1+\beta(-1)} e^{-t},$$

$$e^{-t}\left(\frac{5}{4} + \frac{1-e^{-\frac{1}{4}}}{1-e^{\frac{1}{4}}}\right) \leq x(t) \leq \left(1+e^{-\frac{1}{4}}\right) e^{-\frac{t}{2}} + \frac{1-e^{-\frac{1}{4}}}{1-e^{\frac{1}{4}}} e^{-t},$$

$$e^{-t}\left(\frac{5}{4} - e^{-\frac{1}{4}}\right) \leq x(t) \leq \left(1+e^{-\frac{1}{4}}\right) e^{-\frac{t}{2}} - e^{-\left(t+\frac{1}{4}\right)} \quad \text{for all } \ t \geq 0.$$

Also, from the last double inequality we get

$$\lim_{t\to\infty} x(t) = 0.$$

3. Conclusion

In this paper, an important result is obtained for the behavior of the solutions by making use of two appropriate real roots of the characteristic equation and two examples were given. The real roots used in this paper play an important role in determining the results.

Acknowledgement

The author thank the authors which listed in References for many useful support.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] A.F. Yeniçerioğlu, *Stability of linear impulsive neutral delay differential equations with constant coefficients*, J. Math. Anal. Appl., **479** (2019), 2196–2213.

[2] D. D. Bainov, V. Covachev, I.M. Stamova, *Estimates of the solutions of impulsive quasilinear functional differential equations*, Annales de la Facultb des Sciences de Toulouse, **2** (1991), 149-161.

[3] D. D. Bainov, V. Covachev, I. M. Stamova, *Stability under persistent disturbances of impulsive differential- difference equations of neutral type*, J. Math. Anal. Appl., **187** (1994), 790-808.

[4] D. D. Bainov, I. M. Stamova, *Uniform asymptotic stability of impulsive differential-difference equations of neutral type by Lyapunov's direct method*, J. Comput. Appl. Math., **62** (1995), 359-369.

[5] M. B. Dimitrova, V. I. Donev, *Existence and asymptotic behavior of positive solutions of neutral impulsive differential equations*, Nonlinear Oscillations, **8**(3) (2005), 302-317.

[6] I. M. Stamova, T. G. Stamov, *Asymptotic stability of impulsive control neutral-type systems*, Inter. J. Control, **87**(1) (2014), 25-31.

[7] I. G. E. Kordonis, N. T. Niyianni, Ch. G. Philos, *On the behavior of the solutions of scalar first order linear autonomous neutral delay differential equations*, Arch. Math., (Basel) **71** (1998), 454-464.

[8] Ch. G. Philos, I. K. Purnaras, *On the behavior of the solutions for certain first order linear autonomous functional differential equations*, Rocky Mountain J. Math., **36** (2006), 1999-2019.

[9] R. P. Agarwal, F. Karakoç, *A survey on oscillation of impulsive delay differential equations*, Comput. Math. Appl., **60** (2010), 1648-1685.

[10] G.H. Ballinger, *Qualitative Theory of impulsive delay differential Equations*, National Library of Canada, 1999.

[11] X. Liu, J. Shen, *Asymptotic behavior of solutions of impulsive neutral differential equations*, Appl. Math. Letters, **12** (1999), 51-58.

[12] X. Liu, Y. M. Zeng, *Linear multistep methods for impulsive delay differential equations*, Appl. Math. Comput., **321** (2018), 555-563.

[13] F. Jiang, J. Shen,*Asymptotic behavior of solutions for a nonlinear differential equation with constant impulsive jumps*, Acta Math. Hungar., **138**(1–2) (2013), 1–14.

[14] I. M. Stamova, *Stability analysis of impulsive functional differential equations*, Walter de Gruyter, Berlin, 2009.

[15] Z. You, J. Wang, *Stability of impulsive delay differential equations*, J. Appl. Math. Comput., **56** (2018), 253-268.

[16] G.L. Zhang, M.H. Song, M. Z. Liu, *Exponential stability of the exact solutions and the numerical solutions for a class of linear impulsive delay differential equations*, J. Comput. Appl. Math., **285** (2015), 32-44.