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GBS Operators of Bivariate Durrmeyer Operators on Simplex

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Abstract
We define GBS operators of Durrmeyer operators for bivariate functions on simplex and we give their approximations and rate of their approximations for B-continuous and B-differentiable functions. We show that the GBS type the operators of new Durrmeyer have better approximation than the new operators.

Keywords: Bivariate functions, Bögel continuous, Bögel differentiable, Durrmeyer operators on simplex, GBS-operators, Rate of approximation.
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1. Preliminaries

Polynomial approach and the classical approximation theory constitute a basic research area in applied mathematics. The development of the approximation theory played an important role in the numerical solution of partial differential equations, data processing sciences, and many other disciplines. For example, it is widely used in geometric modeling in the aerospace and automotive industries to calculate approximate values with basic functions. Work in this field goes back to the 18th century and still continues as a powerful tool in scientific calculations. Furthermore, it is used in civil engineering projects to analyze the energy efficiency and earthquake resistance data of different types of buildings in thermography calculations and earthquake engineering. The purpose of the approximation theory is to provide an approach between function spaces. In this context, the best approximation uses a linear positive operator. An operator that brings a function of positive value in one function space to another function of positive value in another function space is called a positive operator; whereas the operators that are both positive and linear are called linear positive operators. We will introduce a generalization of Bernstein operators that form the basis of linear positive operators. This new generalization to be defined will be a better version of Bernstein operators that contribute to all of the above mentioned fields of study. In this way, it is aimed to have a better approach. Before introducing the operator, if we need to talk about previous studies. Weierstrass, who laid the foundations of the approach with a linear positive operator, said in 1885, that each continuous function as an element of C [a, b] was a sequence that could be approached with a polynomial in the same closed range, but he did not specify the properties of these sequences. In 1912 Bernstein, proved that the sequences in the Weierstrass theorem were the polynomials referred to by his name and exposed them as follows:

$$B_n(h;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}h\left(\frac{k}{n}\right).$$

The modified Bernstein polynomials,

$$D_n(f;x) = (n+1)\sum_{k=0}^{n}\varphi_n^k(x)\left(\int_0^1 \varphi_n^k(t)f(t)dt\right), (n \geq 1),$$

where $\varphi_n^k(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}$, $(0 \leq x \leq 1)$, were introduced by Durrmeyer [1] and Deriennic [2] gave some results on approximation of function f on [0, 1] by (1)

In [2], it is shown that,for $m \in \mathbb{N}$

$$D_n(t^m;x) = \frac{(n+1)!}{(n+m+1)!}\sum_{r=0}^{m}\begin{pmatrix} m \\ r \end{pmatrix}\frac{m!}{r!}\frac{n!}{(n-r)!}x^r.$$

Denoting by $\Delta = \{(x,y) : x \geq 0, y \geq 0 \text{ and } x+y \leq 1\}$. Singh [3]defined new class of positive linear operators of order n on Δ by

$$S_n(f;x,y) = \frac{(n+2)!}{n!}\sum_{k=0}^{n}\sum_{j=0}^{n-k}P_{n,k,j}(x,y)\int\int_{\Delta}P_{n,k,j}(u,v)f(u,v)dudv \tag{1.1}$$

where $P_{n,k,j}(x,y) = \begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} n-k \\ j \end{pmatrix} x^k y^j (1-x-y)^{n-k-j}$. Singh proved some results on approximation of function f on Δ by (1.1).

Define e_i :=$e_i(x)$=x^i, $E_i := E_i(u,x) = (u-x)^i$, e_{ij} :=$e_{ij}(x,y)$=x^iy^j and $E_{ij} := E_{ij}(u,v;x,y) = (u-x)^i(v-y)^j$.

Lemma 1.1: ([3])

$$S_n(u^pv^q;x,y) = \frac{(n+2)!}{(n+p+q+2)!}\sum_{r=0}^{p}\sum_{l=0}^{q}\begin{pmatrix} p \\ r \end{pmatrix}\begin{pmatrix} q \\ l \end{pmatrix}\frac{p!q!}{r!l!}x^ry^l$$

In particular,

$$S_n(e_{00};x,y) = 1,$$

$$S_n(e_{10};x,y) = \frac{nx+1}{n+3},$$

$$S_n(e_{01};x,y) = \frac{ny+1}{n+3}, \tag{1.2}$$

$$S_n(e_{20};x,y) = \frac{n(n-1)x^2+4nx+2}{(n+3)(n+4)}, \tag{1.3}$$

$$S_n(e_{02};x,y) = \frac{n(n-1)x^2+4nx+2}{(n+3)(n+4)}, \tag{1.4}$$

$$S_n(e_{30};x,y) = \frac{n(n-1)(n-2)x^3+9n(n-1)x^2+18nx+6}{(n+3)(n+4)(n+5)},$$

$$S_n(e_{03};x,y) = \frac{n(n-1)(n-2)y^3+9n(n-1)y^2+18ny+6}{(n+3)(n+4)(n+5)},$$

$$S_n(e_{40};x,y) = \frac{\alpha_n(3)x^4+16\alpha_n(2)x^3+72\alpha_n(1)x+24}{(n+3)(n+4)(n+5)(n+6)},$$

$$S_n(e_{04};x,y)=\frac{\alpha_n(3)y^4+16\alpha_n(2)y^3+72\alpha_n(1)y+24}{(n+3)(n+4)(n+5)(n+6)}.$$

$$S_n(E_{20};x,y) = \frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)},$$

$$S_n(E_{02};x,y) = \frac{2[1+(n-4)y-2(n-6)y^2]}{(n+3)(n+4)},$$

$$S_n(E_{40};x,y)=\frac{12[a_nx^4-2b_nx^3+a_nx^2+6(n-2)x+2]}{(n+3)(n+4)(n+5)(n+6)},$$

and

$$S_n(E_{04};x,y)=\frac{12[a_ny^4-2b_ny^3+a_ny^2+6(n-2)y+2]}{(n+3)(n+4)(n+5)(n+6)}.$$

where $\alpha_n(p)=\dfrac{n!}{(n-p-1)!}$, $a_n=(n^2-31n+30)$, and $b_n=(n^2-28n+20)$. For all $x\in[0,1]$ we have,

$$S_n(E_{20};x,y)\leq\begin{cases}\dfrac{1}{15} & , \quad n=6\\[2ex] \dfrac{4}{5} & , \quad n<6\\[2ex] \dfrac{n+6}{(n+3)(n+4)} & , \quad n>6\end{cases}$$

The situation for $S_n(E_{02};x,y)$ is the same (11). It is easy to see that

$$\frac{24}{(n+3)(n+4)} < \frac{12}{5(n+3)}$$

for all $n>6$.

And also for all $x\in[0,1]$ we have,

$$S_n(E_{40};x,y)\leq\begin{cases}\dfrac{15}{32} & , \quad n\leq 30\\[2ex] \dfrac{24}{(n+3)(n+4)} & , \quad n>30\end{cases}$$

The situation for $S_n(E_{04};x,y)$ is the same (12). It is easy to see that

$$\frac{24}{(n+3)(n+4)} < \frac{12}{17(n+3)}$$

for all $n>30$.

Our aim is to extend the operator (1.1) to case B-continuous (Bögel continuous) functions. The term ”B-continous” first was introduced by K. Bögel ([4], [5]). And then we shall present a GBS (Generalized Bögel Sum) operator of (1.1) and some approximation of properties of this operator. The term GBS(Generalized Boolean Sum) operators were introduced by Dobrescu and Matei [7]. The analogous of the well-known Korovkin theorem for approximation of B-continuous functions using GBS operators was given by C.Badea, I. Badea and H. Gonska [8]. The analogous of first modulus of continuity for bivariate B-continuous functions which is named ”mixed modulus of smoothness” was introduced by I. Badea[9]. (see Also H. H. Gonska[10] , C.Badea and C. Cottin [11]).

We show that the operators (2.1) (GBS type the operators of (1.1)) have better approximation than the operators (1.1) in fügures and numerical values.

Definition 1.1:

a) ([4], [5]) A function f is called a B-Continuous function in $(x_0,y_0)\in X\times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f\left[(x,y),(x_0,y_0)\right]=0.$$

where $\Delta f\left[(x,y),(x_0,y_0)\right]=f(x,y)-f(x_0,y)-f(x,y_0)+f(x_0,y_0)$ represents the mixed difference of f.

b) ([9]).Let $f\in B_b(X\times Y)$. For any $(\delta_1,\delta_2)\in\mathbb{R}^2_{0,+}$, the mixed modulus of smoothness is the function $\omega_{mixed}(f;\delta_1,\delta_2):\mathbb{R}^2_{0,+}\to\mathbb{R}$ defined by

$$\omega_{mixed}(f;\delta_1,\delta_2)=\sup\left\{\left|\Delta f\left[(x,y),(u,v)\right]\right| : |u-x|\le\delta_1, |v-y|\le\delta_2\right\} \tag{1.5}$$

where $\mathbb{R}_{0,+}:=[0,\infty)$.

c) ([6]) A function f is called a B-Differentiable function in $(x_0,y_0)\in X\times Y$ if the following limit is exist and finite,

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f\left[(x,y),(x_0,y_0)\right]}{(x-y)(x_0-y_0)}.$$

This B-Differentiable of f in (x_0,y_0) is denoted by $D_Bf(x_0,y_0)$.

Let F be the class of all functions $f:X\times Y\to\mathbb{R}$. Then we use subsets of F which are given in the following:

$$B(X\times Y)=\{f\in F\ :\ f \text{ bounded on } X\times Y\}$$

with usual sup-norm $\|\|_\infty$.

$$B_b(X\times Y)=\{f\in F : \left|\Delta f\left[(x,y),(x_0,y_0)\right]\right|\le K, \text{on } X\times Y, K>0\}$$

is called $B-$bounded functions class with the norm

$$\|\|_b=\sup_{(x,y),(x_0,y_0)\in X\times Y}\left|\Delta f\left[(x,y),(x_0,y_0)\right]\right|.$$

$$\begin{aligned} C_b(X\times Y) &= \{f\in F : f \text{ is } B-\text{Continuous on } X\times Y\},\\ D_b(X\times Y) &= \{f\in F : f \text{ is } B-\text{Differentable on } X\times Y\}\end{aligned}$$

If $f:X\times Y\to\mathbb{R}$ is a continuous function in (x_0,y_0), it is also B-Continuous function in (x_0,y_0). A B-continuous function is not necessarily continuous(in usual sense), but the converse is true.

The approximation theorems for bivariate functions were first given by Volkov in [12] and approximation of the GBS operators of associate with operators of two variables were established by [8].

The term GBS(Generalized Boolean Sum) operators were introduced by Badea and Kottin as the following [11]

Definition 1.2. Let $L:C_b(X\times Y)\to B(X\times Y)$ be a linear positive operator. The operator $UL:C_b(X\times Y)\to B(X\times Y)$ is defined by

$$(ULf)(x,y)=\left(L\left(f(\bullet,y)+f(x,*)-f(\bullet,*)\right)\right)(x,y)$$

is called the GBS operator associated to the operator L, where "$\bullet$" and "$*$" stand for the first and second variable respectively.

From now on, we write $L(f(u,v);x,y)$ and

$$UL(f(u,v);x,y)=L\left(\left(f(u,y)+f(x,v)-f(u,v)\right);x,y\right)$$

instead of (1.2).

Theorem 1.1. ([8].) Let a,b,c,d be real numbers satisfying the inequalities $a<b,\ c<d$ and let $(T_{n,m})$ $(n,m\in\mathbb{Z}^+)$ be a sequence of bivariate linear positive operators, applying $C([a,b]\times[c,d])$ into itself. Suppose the following relations hold for any $(x,y)\in[a,b]\times[c,d]$.

i) $T_{n,m}(e_{00};x,y)=1$,
ii) $T_{n,m}(e_{10};x,y)=x+u_{n,m}(x,y)$,
iii) $T_{n,m}(e_{01};x,y)=y+v_{n,m}(x,y)$,
iv) $T_{n,m}(e_{20}+e_{02};x,y)=x^2+y^2+w_{n,m}(x,y)$

If each of the sequence of $u_{n,m}(x,y), v_{n,m}(x,y)$ and $w_{n,m}(x,y)$ converges to zero uniformly as $n \to \infty$, $m \to \infty$, then the sequence $(T^*_{n,m})$ $(n,m \in \mathbb{Z}^+)$ converges to f uniformly on $[a,b] \times [c,d]$, where $T^*_{n,m}$ represent the GBS operator associate with $T_{n,m}$

Theorem 1.2 ([12]). Let $T : C([a,b] \times [c,d]) \to C([a,b] \times [c,d])$ be a linear positive operator and T^* the GBS operator associate with T.

Then, for any $f \in C([a,b] \times [c,d])$, $(x,y) \in [a,b] \times [c,d]$ and $\delta_1, \delta_2 > 0$, the following holds

$$\begin{aligned} |T^*(f(u,v);x,y) - f(x,y)| \leq\ & |f(x,y)|\,|1 - T(e_{00};x,y)| + \left\{ |T(e_{00};x,y)| + \frac{1}{\delta_1}\sqrt{T(E_{20};x,y)} + \frac{1}{\delta_2}\sqrt{T(E_{02};x,y)} \right. \\ & \left. + \frac{1}{\delta_1\delta_2}\sqrt{T(E_{20};x,y)\,T(E_{02};x,y)} \right\} \omega_{mixed}(\delta_1,\delta_2). \end{aligned} \tag{1.6}$$

Theorem 1.3 ([4], [5] , [6]). Let $f : [u,v] \times [x,y] \to \mathbb{R}$ be a function. If f is $B-$differentiable on $[u,v] \times [x,y]$, there exist $(x_0,y_0) \in (u,v) \times (x,y)$ such that

$$\Delta f[(u,v),(x,y)] = (u-x)(v-y) D_B f(x_0,y_0).$$

Theorem 1.4([13]).Let $T : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UT : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in D_b(X \times Y)$ with $D_B f \in B(X \times Y)$, any $(x,y) \in X \times Y$ and $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned} |UT(f(u,v);x,y) - f(x,y)| \leq\ & |f(x,y)|\,|1 - T(e_{00};x,y)| + 3\,\|D_B f\|_\infty \sqrt{T(E_{20};x,y)\,T(E_{02};x,y)} \\ & + \left[\sqrt{T(E_{20};x,y)\,T(E_{02};x,y)} + \frac{1}{\delta_1}\sqrt{T(E_{40};x,y)\,T(E_{02};x,y)} \right. \\ & \left. + \frac{1}{\delta_2}\sqrt{T(E_{20};x,y)\,T(E_{04};x,y)} + \frac{1}{\delta_1\delta_2} T(E_{20};x,y)\,T(E_{02};x,y) \right] \omega_{mixed}(D_B f,\, \delta_1, \delta_2). \end{aligned}$$

2. Representation of bivariate GBS operator of Durrmeyer operator

For any $f \in C_b(\Delta)$,($C_b(\Delta)$ is the class of all B-continuous functions on Δ) ,representation of bivariate GBS operator of Durrmeyer operator is

$$F_n(f;x,y) = \frac{(n+2)!}{n!} \sum_{k=0}^{n} \sum_{j=0}^{n-k} P_{n,k,j}(x,y) \int\int_\Delta P_{n,k,j}(u,v)\,[f(u,y) + f(x,v) - f(u,v)]\,du dv. \tag{2.1}$$

It is easy to see $F_n(f;x,y)$ is linear positive operator. Taking into account the relations (1.3), (1.4) and applying Theorem 1.1 we obtain the following theorem.

Theorem 2. 1: The sequence $(F_n)_{n\in\mathbb{N}}$ converges to any $f \in C_b(\Delta)$ uniformly.

It easy tı see the following relation:

$$F_n(u^i v^j;x,y) = x^i y^j \text{ for all } i,\ j = 0, 1, 2, \ldots.$$

That means there is no approximation for any usual continuous function f on $[0,1]$. Mean $F_n(f;x,y) = f$ for all $f \in C(\Delta)$.

Theorem 2. 2 : If $f \in C_b(\Delta)$, then for any $(x,y) \in \Delta$, the following relation holds for all $n > 6$:

$$|F_n(f(u,v);x,y) - f(x,y)| \leq \frac{11}{2}\omega_{mixed}(f;\sqrt{\frac{1}{n+3}},\sqrt{\frac{1}{n+3}}).$$

Proof : Applying Theorem 1.2 , Lemma 1.1 and (11^*) we have

$$\begin{aligned} |F_n(f(u,v);x,y)-f(x,y)| &\leq \left(\frac{1}{\delta_1}\sqrt{S_n(E_{20};x,y)}+\frac{1}{\delta_2}\sqrt{S_n(E_{02};x,y)}+\frac{1}{\delta_1\delta_2}\sqrt{S_n(E_{20};x,y)S_n(E_{02};x,y)}\right)\omega_{mixed}(f;\delta_1,\delta_2) \\ &\leq \left(\frac{1}{\delta_1}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}+\frac{1}{\delta_2}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}\right. \\ &\quad \left.+\frac{1}{\delta_1\delta_2}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}.\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}\right)\omega_{mixed}(f;\delta_1,\delta_2) \\ &\leq \left(\frac{1}{\delta_1}\sqrt{\frac{12}{5}}\sqrt{\frac{1}{n+3}}+\frac{1}{\delta_2}\sqrt{\frac{12}{5}}\sqrt{\frac{1}{n+3}}+\frac{1}{\delta_1\delta_2}\frac{12}{5}\sqrt{\frac{1}{n+3}}\sqrt{\frac{1}{n+3}}\right)\omega_{mixed}(f;\delta_1,\delta_2) \end{aligned}$$

If we choose $\delta_1=\sqrt{\frac{1}{n+3}}$ and $\delta_2=\sqrt{\frac{1}{n+3}}$, we get

$$|F_n(f(u,v);x,y)-f(x,y)| \leq \left(2\sqrt{\frac{12}{5}}+\frac{12}{5}\right).\omega_{mixed}(f;\delta_1,\delta_2)$$

and consider $2\sqrt{\frac{12}{5}}+\frac{12}{5}<\frac{11}{2}$, then the proof is beeing comlated.

Theorem 2. 3 : If $f\in D_B(\mathbb{R}^2_{0,+})$ with $D_Bf\in B(\mathbb{R}^2_{0,+})$, then for any $(x,y)\in\Delta$ and $n>30$,

$$|F_n(f(u,v);x,y)-f(x,y)| \leq \frac{36\|D_Bf\|_\infty}{5(n+3)}+\frac{17}{10}.\,\omega_{mixed}(D_Bf;\sqrt{\frac{1}{n+3}},\sqrt{\frac{1}{n+3}}).$$

Proof: Applying Theorem 1.4 and Lemma 1.1 (11^* and 12^*) ,we have

$$\begin{aligned} |F_n(f(u,v);x,y)-f(x,y)| &\leq 3\|D_Bf\|_\infty\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}. \\ &\quad \times\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}. \\ &\quad +\left[\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}\right. \\ &\quad +\frac{1}{\delta_1}\sqrt{\frac{12[a_nx^4-2b_nx^3+a_nx^2+6(n-2)x+2]}{(n+3)(n+4)(n+5)(n+6)}}\sqrt{\frac{2[1+(n-4)y-2(n-6)y^2]}{(n+3)(n+4)}} \\ &\quad +\frac{1}{\delta_2}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}.\sqrt{\frac{12[a_ny^4-2b_ny^3+a_ny^2+6(n-2)y+2]}{(n+3)(n+4)(n+5)(n+6)}} \\ &\quad \left.+\frac{1}{\delta_1\delta_2}.\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}\frac{2[1+(n-4)y-2(n-6)y^2]}{(n+3)(n+4)}\right]\omega_{mixed}(D_Bf,\,\delta_1,\delta_2). \end{aligned}$$

$$\begin{aligned} |F_n(f(u,v);x,y)-f(x,y)| &\leq \frac{36\|D_Bf\|_\infty}{5(n+3)}+\left[\frac{12}{5(n+3)}+\right. \\ &\quad +\frac{1}{\delta_1}\sqrt{\frac{12}{17(n+3)}\frac{12}{5(n+3)}}+\frac{1}{\delta_2}\sqrt{\frac{12}{17(n+3)}\frac{12}{5(n+3)}} \\ &\quad \left.+\frac{1}{\delta_1\delta_2}.\frac{12}{5(n+3)}\frac{12}{5(n+3)}\right]\omega_{mixed}(D_Bf,\,\delta_1,\delta_2) \end{aligned}$$

If we choose $\delta_1=\sqrt{\frac{1}{n+3}}$ and $\delta_2=\sqrt{\frac{1}{n+3}}$ and take into account that $\frac{1}{n+3}<\frac{1}{33}$ for $n>30$, then we get desired result.

3. Conclusion

As a result, it is seen that the operator defined in GBS format takes a better approximation. In order to more visibly show that this approximation is better, a numerical value table with the margin of error and a graph can be drawn at different points.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] J. L. Durrmeyer, ″*une formule d′ inversion de la transformee de Laplace:Applicationsa la theorie de moments*″ , these de 3e cycle, Faculte des sciences de 1′universite de Paris,1967.

[2] M. M. Derriennic,*Sur l′ approximation de fonctions integrables sur [0, 1] par de polynomes de Bernstein modifies*, J. Approx. Theory, **31**(1981),325-343.

[3] S.P. Singh,*On approximation Of Integrable Functions By Modified Bernstein Polynomials*, Publications De L'Institut Mathematicque, **41**(55), (1987), 91-95.

[4] K. Bögel,*Mehrdimensionale Differentitation von Funktionen mehrerer Veränderhicher*, J. Reine agnew. Math., **170**, (1934) 197-217.

[5] K. Bögel,*Über die mehrdimensionale Differentitation, Integration and beschränkte variation* , J. Reine agnew. Math., (1935) 173, 5-29.

[6] K. Bögel,*Über die mehrdimensionale Differentitation*, Jber. DMV, **65**, (1962) 45-71.

[7] E. Dobrescu, I. Matei,*Approximarea prin polinoame de tip Bernstein a functiilor bidimensional continue*, Anal. Univ. Timişoara , Şeria Ştiinte matematici-fizice, **IV** (1966), 85-90.

[8] C. Badea, I.Badea, H. H. Gonska,*A test function theorem and approximation by pseudopolynomials*, Bull. Austral. Math. Soc.,**34**, (1986) 53-64.

[9] I. Badea, *Modul de continuitate in sens Bögel şi unele applicatii in approximarea printr-un operator Bernstein*, Studia Univ. "Babeş-Bolyai" Ser. Math-Mech., **18**(2), (1973)69-78, (Romanian).

[10] H. H. Gonska,*Quantitative approximation in C(X), Habilitationsschrift*, Universitaät Disburg, (1985).

[11] C. Badea, C. Cottin, *Korovkin-type theorems for Generalized Boolean Sum operators*, Colloquia Mathematica Sociekatis Janos Bolyai, 58, Approximation Theory , Kecskemet(Hungary), (1990) 51-68.

[12] V. Volkov,*I.On the convergence of sequences of linear positive operators in the space of continuous functions of two variable*, Math. Sb. N. S. **43**(85) (1957) 504 (Russian).

[13] O. T. Pop,*Approximation of B-differentiable functions by GBS operators*, Analele Univ. Oradea. Fac. Mathematica, Tom. **XIV**, (2007) 15-31.