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Some Generalized Special Functions and their Properties

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Abstract

In the present paper, first, we investigate a generalized Pochhammer's symbol and its various properties in terms of a new symbol (s, k), where $s, k > 0$. Then, we define a generalization of gamma and beta functions and their various associated properties in the form of (s, k). Also, we define new generalization of hypergeometric functions and develop differential equations for generalized hypergeometric functions in the form of (s, k). We present that generalized hypergeometric functions are the solution of the said differential equation. Furthermore, some useful results, properties and integral representation related to these generalized Pochhammer's symbol, gamma function, beta function, and hypergeometric functions are presented.

Keywords: Pochhammer symbol beta function Gamma function generalized Pochhammer symbol generalized hypergeometric function.
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1. Introduction

The theory of special functions comprises a major part of mathematics. In last three centuries, the essential of solving the problems take place in the fields of classical mechanics, hydrodynamics and control

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theory, motivated the development of the theory of special functions. This field also has wide applications in both pure mathematics and applied mathematics. Numerous extensions of special functions have been introduced by many authors (see [1, 2, 3, 4]).

Agarwal *et al.* [5] proved new differential equations for the extended Mittag-Leffler function by using Saigo-Maeda fractional differential operators. Mdallal *et al.* [6] also worked on the differential equations of fractional order with variables coefficients. They found the eigenvalues by applying associated boundary conditions. Also, they present the eigenfunctions in the form of Mittag-Leffler functions. Babakhani *et al.* [7] used thye fixed point theorems to find the existence of positive solutions for a non-autonomous fractional differential equation with integral boundary conditions. They also presented some examples related to differential equations. Jarad *et al.* [8] investigated the modified Laplace transform and related properties. They used these results to find the solution of some ordinary differential equations of a certain type generalized fractional derivatives.

Diaz *et al.* [9, 10, 11] have introduced gamma and beta k-functions and proved a number of their properties. They have also studied zeta k-functions and hypergeometric k-functions based on Pochhammer k-symbols for factorial functions. In [12, 13, 14, 15], the researchers studied the generalized gamma k-function and proposed its various properties. Later on, Mubeen and Habibullah [32] proposed the so-called k-fractional integral based on gamma k-function and its applications. In [33], Mubeen and Habibullah defined the integral representation of generalized confluent hypergeometric k-functions and hypergeometric k-functions by utilizing the properties of Pochhammer k-symbols, k-gamma, and k-beta functions. In [34], Mubeen *et al.* proposed the following second order linear differential equation for hypergergeometric k-functions as

$$k\omega(1-kx)\omega'' + [\gamma - (\alpha+\beta+k)\,kx]\,\omega' - \alpha\beta\omega = 0.$$

The solution in the form of the so-called k-hypergeometric series of k-hypergeometric differential equation by utilizing the Frobenius method can be found in the work of Mubeen *et al.* [35, 36]. Recently, Li and Dong [37] investigated the hypergeometric series solutions for the second order non-homogeneous k-hypergeometric differential equation with the polynomial term. Rahman *et al.* [38, 39] proposed the generalization of Wright type hypergeometric k-functions and derived its various basic properties.

Qi and Wang [40] worked on Young's integral Inequalities and discussed its geometric interpretation. Adjimi and Benbachir [41] worked on Katugampola fractional differential equation with Erdelyi-Kober integral boundary conditions. Furthermore, Mubeen and Iqbal [16] investigated the generalized version of Grüss-type inequalities by considering k-fractional integrals. Agarwal *et al.* [18] established certain Hermite-Hadamard type inequalities involving k-fractional integrals. Set *et al.* [27] proposed generalized HermiteâĂŞHadamard type inequalities for RiemannâĂŞLiouville k-fractional integral. Östrowski type k-fractional integral inequalities can be found in the work of Mubeen *et al.* [17]. Many researchers have established further the generalized versions of Riemann-Liouville k-fractional integrals, and defined a large number of various inequalities by using different kind of generalized fractional integrals. The interesting readers may consult [19, 31, 25, 20]. The Hadamard k-fractional integrals can be found in the work of Farid *et al.* [21]. In [23] Farid proposed the idea of Hadamard-type inequalities for k-fractional Riemann-Liouville integrals. In [24, 22], the authors have introduced the inequalities by employing Hadamard-type inequalities for k-fractional integrals. Nisar *et al.* [28] investigated Gronwall type inequalities by utilizing Riemann-Liouville k-fractional derivatives and Hadamard k-fractional derivatives [28]. In [28], they presented dependence solutions of certain k-fractional differential equations of arbitrary real order with initial conditions. Samraiz *et al.* [26] proposed Hadamard k-fractional derivative and related properties. Recently, Rahman *et al.* [29] defined generalized k-fractional derivative operator. Jangid *et al.* [30] worked on the generalization of integral inequalities. By using the Saigo k-fractional integral operators, they found some new inequalities for the Chebyshev functional for two synchronous functions.

By getting motivation from Diaz and other researchers working on the generalization and extensions of the special functions, we get the idea to obtain the more generalized and extended form of special functions. The structure of the paper is organized as follows:
In Section 2, we study the generalized Pochhammer's symbol, gamma function and beta function in terms of (s,k). Also, some basic properties are presented. In Section 3, generalized hypergeometric functions, differential equation for hypergeometric function and their associated properties are discussed. In Section 4, concluding remarks are given.

2. Main Results

In this section, we introduce generalized Pochhammer's symbol and its various associated properties. Also, generalization of gamma and beta functions and their associated properties are presented.

Definition 2.1. *The generalized Pochhammer's symbol in term of (s,k) is defined as*

$$^{s}(\alpha)_n^k = \alpha(\alpha+(s/k))\cdots(\alpha+((n-1)s/k)), \quad \alpha\in\mathbb{C}, \qquad s,k\in\mathbb{R}^+,\ n\in\mathbb{N},\ s>k>0. \quad (1)$$

$^{s}(b)_{nm}^k = 2^{mn}\ {}^{s}(\frac{b}{m})_n^k\ {}^{s}(\frac{bk+s}{mk})_n^k \cdots\ {}^{s}(\frac{bk+(m-1)s}{mk})_n^k$

Just as for the usual Γ, the function $^{s}\Gamma^k$ admits an infinite product expression given by

$$\frac{1}{^{s}\Gamma^k(z)} = z(s/k)^{-(kz)/s}e^{(kz\gamma)/s}\prod_{n=1}^{\infty}\left[\left(1+\frac{kz}{ns}\right)\exp\left(-\frac{kz}{ns}\right)\right], \quad (2)$$

where γ is Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty}(H_n-\log(n)) \approx 0.57721566490,$$

where $H_n = \sum\limits_{m=1}^{n}\frac{1}{m}$.

Lemma 2.2. *For all z, the Euler product is given as*

$$^{s}\Gamma^k(z) = \frac{1}{z(s/k)^{-\frac{kz}{s}}}\prod_{n=1}^{\infty}\left[\left(\frac{n+1}{n}\right)^{\frac{kz}{s}}\left(1+\frac{kz}{sn}\right)^{-1}\right]. \quad (3)$$

Proof. Using equation (2), we have

$$\frac{1}{^{s}\Gamma^k(z)} = z(s/k)^{-\frac{-kz}{s}}e^{\frac{k\gamma z}{s}}\prod_{n=1}^{\infty}\left[\left(1+\frac{kz}{ns}\right)\exp\left(\frac{-kz}{ns}\right)\right].$$

$$z(s/k)^{-\frac{kz}{s}}\ {}^{s}\Gamma^k(z) = e^{-\frac{k\gamma z}{s}}\lim_{n\to\infty}\prod_{m=1}^{n}\left[\left(1+\frac{kz}{sm}\right)^{-1}\exp\left(\frac{kz}{sm}\right)\right]. \quad (4)$$

Since

$$\gamma = \lim_{n\to\infty}\big[H_n-\log(n+1)\big]. \quad (5)$$

Therefore equation (5) becomes

$$\gamma = \lim_{n\to\infty}\left[H_n-\sum_{m=1}^{n}\log\left(\frac{m+1}{m}\right)\right]. \quad (6)$$

Multiply the whole equation (6) by $-\frac{kz}{s}$, we obtain

$$-\frac{kz\gamma}{s} = \lim_{n\to\infty}\Big[-\frac{kz}{s}H_n + \sum_{m=1}^{n}\log\Big(\frac{m+1}{m}\Big)^{\frac{kz}{s}}\Big].$$

By taking exponential, we get

$$e^{-\frac{kz\gamma}{s}} = \lim_{n\to\infty}\Big[\exp\Big(-\frac{kz}{s}H_n\Big)\exp\sum_{m=1}^{n}\log\Big(\frac{m+1}{m}\Big)^{\frac{kz}{s}}\Big]. \tag{7}$$

Since $H_n = \sum\limits_{m=1}^{n}\frac{1}{m}$, put in equation (7), we have

$$\begin{aligned} e^{-\frac{kz\gamma}{s}} &= \lim_{n\to\infty}\Big[\exp\Big(-\frac{kz}{s}\sum_{m=1}^{n}\frac{1}{m}\Big)\exp\sum_{m=1}^{n}\log\Big(\frac{m+1}{m}\Big)^{\frac{kz}{s}}\Big] \\ &= \lim_{n\to\infty}\Big[\exp\Big(\sum_{m=1}^{n}\frac{-kz}{sm}\Big)\exp\sum_{m=1}^{n}\log\Big(\frac{m+1}{m}\Big)^{\frac{kz}{s}}\Big]. \end{aligned}$$

And so

$$e^{-\frac{kz\gamma}{s}} = \lim_{n\to\infty}\exp\Big(\sum_{m=1}^{n}\log\Big(\exp\big(\frac{-kz}{sm}\big)\Big)\Big)\Big(\exp\sum_{m=1}^{n}\log\big(\frac{m+1}{m}\big)^{\frac{kz}{s}}\Big)$$

$$e^{-\frac{kz\gamma}{s}} = \lim_{n\to\infty}\prod_{m=1}^{n}\exp\Big(\frac{-kz}{sm}\Big)\Big(\frac{m+1}{m}\Big)^{\frac{kz}{s}}. \tag{8}$$

This implies that

$$^{s}\Gamma^{k}(z) = \frac{1}{z(s/k)^{-\frac{kz}{s}}}\prod_{n=1}^{\infty}\Big[\big(\frac{n+1}{n}\big)^{\frac{kz}{s}}\big(1+\frac{kz}{sn}\big)^{-1}\Big].$$

□

Lemma 2.3. *For all finite z, the difference equation is given below*

$$^{s}\Gamma^{k}(z+(s/k)) = z\ {}^{s}\Gamma^{k}(z). \tag{9}$$

Proof. Using equation (3), we have

$$\begin{aligned} \frac{^{s}\Gamma^{k}(z+(s/k))}{^{s}\Gamma^{k}(z)} &= \frac{\frac{1}{(z+(s/k))(s/k)^{-\frac{k(z+(s/k))}{s}}}\prod\limits_{n=1}^{\infty}[(\frac{n+1}{n})^{\frac{k(z+(s/k))}{s}}(1+\frac{kz+k}{sn})^{-1}]}{\frac{1}{z(s/k)^{-\frac{kz}{s}}}\prod\limits_{n=1}^{\infty}[(\frac{n+1}{n})^{\frac{kz}{s}}(1+\frac{kz}{ns})^{-1}]} \\ &= \frac{z(s/k)}{z+(s/k)}\lim_{n\to\infty}\prod_{m=1}^{n}(\frac{m+1}{m})(\frac{sm+kz}{sm+kz+k}) \\ &= \frac{z(s/k)}{z+(s/k)}\lim_{n\to\infty}\Big[(\frac{n+1}{1})(\frac{s+zk}{sn+kz+k})\Big] = z. \end{aligned}$$

Therefore

$$^{s}\Gamma^{k}(z+(s/k)) = z\ {}^{s}\Gamma^{k}(z).$$

□

Definition 2.4. *The **Gamma** (s,k)-**function** is defined as*

$$^{s}\Gamma^{k}(z) = \lim_{n\to\infty} \frac{n! s^{n} (\frac{ns}{k})^{\frac{kz}{s}}}{k^{n}\ {}^{s}(z)_{n}^{k}}$$

$$, \quad z \in \mathbb{C} - k\mathbb{Z}, \tag{10}$$

$$s,k \in \mathbb{R}^{+},\ s > k > 0,\ \Re(z) > 0. \tag{11}$$

Lemma 2.5. *For $z \in \mathbb{C}$, $\Re(z) > 0$, we have*

$$^{s}\Gamma^{k}(z) = \int_{0}^{\infty} t^{z-1} e^{-\frac{kt^{s/k}}{s}} dt.$$

Proof. By equation (10), we have

$$\begin{aligned} ^{s}\Gamma^{k}(z) &= \int_{0}^{\infty} t^{z-1} e^{-\frac{kt^{s/k}}{s}} dt \\ &= \lim_{n\to\infty} \int_{0}^{(\frac{ns}{k})^{\frac{k}{s}}} t^{z-1} \bigg(1 - \frac{kt^{\frac{s}{k}}}{ns}\bigg) dt. \end{aligned}$$

Let $A_{n,i}(z)$, $i = 0, \cdots, n$, be given by

$$A_{n,i}(z) = \int_{0}^{(\frac{ns}{k})^{\frac{k}{s}}} t^{z-1} \bigg(1 - \frac{kt^{\frac{s}{k}}}{ns}\bigg) dt.$$

The following recursive formula is proven using integration by parts

$$A_{n,i}(z) = \frac{i}{nz} A_{n,i}(z + \frac{s}{k}).$$

Also,

$$A_{n,0}(z) = \int_{0}^{(\frac{ns}{k})^{\frac{k}{s}}} t^{z-1} dt = \frac{(\frac{ns}{k})^{\frac{kz}{s}}}{z}.$$

Therefore,

$$A_{n,n}(z) = \frac{n! s^{n} (\frac{ns}{k})^{\frac{kz}{s}}}{k^{n}\ {}^{s}(z)_{n}^{k} (1 + \frac{zk}{ns})}$$

and

$$^{s}\Gamma^{k}(z) = \lim_{n\to\infty} A_{n,n}(z) = \lim_{n\to\infty} \frac{n! s^{n} (\frac{ns}{k})^{\frac{kz}{s}}}{k^{n}\ {}^{s}(z)_{n}^{k}}.$$

$$^{s}\Gamma^{k}(z) = \int_{0}^{\infty} t^{z-1} e^{-\frac{kt^{s/k}}{s}} dt, \quad z \in \mathbb{C},\ s,k \in \mathbb{R}^{+},\ s > k > 0,\ \Re(z) > 0. \tag{12}$$

$\square$

Lemma 2.6. *If α is neither zero nor a negative integer, then*

$$ {}^{s}(\alpha)_n^k = \frac{{}^{s}\Gamma^k\left(\alpha + \frac{ns}{k}\right)}{{}^{s}\Gamma^k(\alpha)}. \tag{13}$$

Proof. Consider

$$ {}^{s}\Gamma^k\left(\alpha + \frac{ns}{k}\right) = {}^{s}\Gamma^k\left(\alpha + \frac{(n-1+1)s}{k}\right).$$

Using equation (9), for n a positive integer, we have

$$\begin{aligned} {}^{s}\Gamma^k\left(\alpha + \frac{ns}{k}\right) &= \left(\alpha + \frac{ns-s}{k}\right) {}^{s}\Gamma^k\left(\alpha + \frac{ns-s}{k}\right) \\ &= \left(\alpha + \frac{ns-s}{k}\right)\left(\alpha + \frac{ns-2s}{k}\right)\cdots\alpha\ {}^{s}\Gamma^k(\alpha) \end{aligned}$$

i.e.,

$$ {}^{s}\Gamma^k\left(\alpha + \frac{ns}{k}\right) = \alpha\left(\alpha + \frac{s}{k}\right)\cdots\left(\alpha + \frac{ns-2s}{k}\right)\left(\alpha + \frac{ns-s}{k}\right) {}^{s}\Gamma^k(\alpha).$$

Since

$$ {}^{s}(\alpha)_n^k = \alpha\left(\alpha + \frac{s}{k}\right)\cdots\left(\alpha + \frac{ns-2s}{k}\right)\left(\alpha + \frac{ns-s}{k}\right),$$

therefore we have

$$ {}^{s}\Gamma^k\left(\alpha + \frac{ns}{k}\right) = {}^{s}(\alpha)_n^k\ {}^{s}\Gamma^k(\alpha).$$

This implies that

$$ {}^{s}(\alpha)_n^k = \frac{{}^{s}\Gamma^k\left(\alpha + \frac{ns}{k}\right)}{{}^{s}\Gamma^k(\alpha)}.$$

□

Definition 2.7. *The* ***beta*** (s,k)***-function*** *is defined as*

$$ {}^{s}\beta^k(x,y) = \frac{k}{s}\int_0^1 t^{\frac{kx}{s}-1}(1-t)^{\frac{ky}{s}-1}dt, \quad s,k\in\mathbb{R}^+, \quad s>k>0, \quad \Re(x)>0,\ \Re(y)>0. \tag{14}$$

The beta (s,k)-function can be defined in terms of algebraic functions by putting $t=\sin^2\Phi$, given as

$$ {}^{s}\beta^k(x,y) = \frac{2k}{s}\int_0^{\frac{\pi}{2}} \sin^{\frac{2kx}{s}-1}\Phi \cos^{\frac{2ky}{s}-1}\Phi d\Phi, \quad \Re(x)>0, \quad \Re(y)>0. \tag{15}$$

Lemma 2.8. *If $Re(p)>0$ and $Re(q)>0$, then*

$$ {}^{s}\beta^k(p,q) = \frac{{}^{s}\Gamma^k(p)\ {}^{s}\Gamma^k(q)}{{}^{s}\Gamma^k(p+q)}. \tag{16}$$

Proof. Using equation (12), We have

$$ {}^{s}\Gamma^k(p)\ {}^{s}\Gamma^k(q) = \int_0^\infty e^{-\frac{ku^{\frac{s}{k}}}{s}} u^{\frac{kp}{s}-1}du \int_0^\infty e^{-\frac{kv^{\frac{s}{k}}}{s}} v^{\frac{kq}{s}-1}dv. \tag{17}$$

In equation (17), use $u = x^2$ and $v = y^2$. Thus $du = 2xdx$ and $dv = 2ydy$.
Now if $u \to 0$, then $x \to 0$. If $v \to 0$, then $y \to 0$. If $u \to \infty$, then $x \to \infty$. If $v \to \infty$, then $y \to \infty$. We have

$$\begin{aligned} {}^s\Gamma^k(p)\ {}^s\Gamma^k(q) &= \int_0^\infty e^{-\frac{kx^{\frac{2s}{k}}}{s}} x^{\frac{2kp}{s}-2} 2xdx \int_0^\infty e^{-\frac{ky^{\frac{2s}{k}}}{s}} y^{\frac{2kq}{s}-2} 2ydy \\ &= 4\int_0^\infty\int_0^\infty e^{-(\frac{kx^{\frac{2s}{k}}}{s}+\frac{ky^{\frac{2s}{k}}}{s})} x^{\frac{2kp}{s}-1} y^{\frac{2kq}{s}-1} dxdy. \end{aligned} \tag{18}$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$, where $0 < r < \infty$ and $0 < \theta < \frac{\pi}{2}$, and $dxdy = rdrd\theta$. Put in the equation (18), we have

$$\begin{aligned} {}^s\Gamma^k(p)\ {}^s\Gamma^k(q) &= 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-\frac{kr^{\frac{2s}{k}}}{s}} r^{2p-1} \cos^{2\frac{kp}{s}-1}\theta r^{2q-1} \sin^{\frac{2kq}{s}-1}\theta r dr d\theta \\ &= 2\int_0^\infty e^{-\frac{kr^{\frac{2s}{k}}}{s}} r^{2p+2q-2} 2rdr \int_0^{\frac{\pi}{2}} \cos^{\frac{2pk}{s}-1}\theta \sin^{\frac{2kq}{s}-1}\theta d\theta. \end{aligned} \tag{19}$$

Put the following in equation (19)

$$\begin{aligned} r^2 &= t & \theta &= \frac{\pi}{2} - \phi \\ 2rdr &= dt & d\theta &= -d\phi, \end{aligned}$$

as $r \to 0$, $t \to 0$ and as $r \to \infty$, $t \to \infty$. Also if $\theta = 0$, then $\phi = \frac{\pi}{2}$ and if $\theta = \frac{\pi}{2}$, then $\phi = 0$, so we have

$$\begin{aligned} {}^s\Gamma^k(p)\ {}^s\Gamma^k(q) &= \int_0^\infty e^{-\frac{kt^{\frac{s}{k}}}{s}} t^{p+q-1} dt \quad 2\int_0^{\frac{\pi}{2}} \sin^{\frac{2kp}{s}-1}\phi \cos^{\frac{2kq}{s}-1}\phi d\phi \\ &= {}^s\Gamma^k(p+q)\ {}^s\beta^k(p,q). \end{aligned}$$

By using equation (12) and equation (15) we have

$${}^s\beta^k(p,q) = \frac{{}^s\Gamma^k(p)\ {}^s\Gamma^k(q)}{{}^s\Gamma^k(p+q)}.$$

□

Lemma 2.9. *Prove that*

$$(1 - \frac{sy}{k})^{-\frac{ka}{s}} = \sum_{n=0}^\infty \frac{{}^s(a)_n^k\ y^n}{n!}. \tag{20}$$

Proof. The binomial theorem states that

$$(1 - \frac{sy}{k})^{-\frac{ka}{s}} = \sum_{n=0}^\infty \frac{(-\frac{ka}{s})(-\frac{ka}{s}-1)(-\frac{ka}{s}-2)...(-\frac{ka}{s}-n+1)(-1)^n(\frac{sy}{k})^n}{n!},$$

which may be written as

$$(1 - \frac{sy}{k})^{-\frac{ka}{s}} = \sum_{n=0}^\infty \frac{(a)(a+\frac{s}{k})(a+2\frac{s}{k})...(a+(n-1)\frac{s}{k})\ y^n}{n!}.$$

Therefore, in factorial function notation,

$$(1-\frac{sy}{k})^{-\frac{ka}{s}}=\sum_{n=0}^{\infty}\frac{{}^{s}(a)_{n}^{k}\ y^{n}}{n!}.$$

□

2.1. Some important results of Pochhammer (s,k)-symbol.

In this section, some important results of Pochhammer's (s,k)-symbol are presented.

$$\begin{aligned}&\frac{{}^{s}\Gamma^{k}(\frac{s-ns}{k}-\alpha)}{{}^{s}\Gamma^{k}(\frac{s}{k}-\alpha)}\\&=\frac{{}^{s}\Gamma^{k}(\frac{s-ns}{k}-\alpha)}{(\frac{s-s}{k}-\alpha)(\frac{s-2s}{k}-\alpha)\cdots(\frac{s-ns}{k}-\alpha)\ {}^{s}\Gamma^{k}(\frac{s-ns}{k}-\alpha)}\\&=\frac{(-1)^{n}}{\alpha(\alpha+\frac{s}{k})\cdots(\alpha+(n-1)\frac{s}{k})}=\frac{(-1)^{n}}{{}^{s}(\alpha)_{n}^{k}}.\end{aligned}$$

Thus,

$$\frac{{}^{s}\Gamma^{k}(\frac{s-ns}{k}-\alpha)}{{}^{s}\Gamma^{k}(\frac{s}{k}-\alpha)}=\frac{(-1)^{n}}{{}^{s}(\alpha)_{n}^{k}}.$$

$$\begin{aligned}&{}^{s}(\alpha)_{m}^{k}\ {}^{s}(\alpha+\frac{ms}{k})_{n}^{k}=\alpha(\alpha+\frac{s}{k})\cdots(\alpha+\frac{(m-1)s}{k})(\alpha+\frac{ms}{k})(\alpha+\frac{(m+1)s}{k})\cdots\\&(\alpha+\frac{ms+(n-1)s}{k})={}^{s}\ (\alpha)_{n+m}^{k}.\end{aligned}$$

Thus,

$${}^{s}(\alpha)_{m}^{k}\ {}^{s}(\alpha+\frac{ms}{k})_{n}^{k}={}^{s}\ (\alpha)_{n+m}^{k}.$$

$$\begin{aligned}&\frac{(-1)^{m}\ {}^{s}(\alpha)_{n}^{k}}{{}^{s}(\frac{s-ns}{k}-\alpha)_{m}^{k}}\\&=\frac{(-1)^{m}\alpha(\alpha+\frac{s}{k})\cdots(\alpha+\frac{(n-1)s}{k})}{(\frac{s-ns}{k}-\alpha)(\frac{2s-ns}{k}-\alpha)\cdots(\frac{s-nk+(m-1)s}{k}-\alpha)}\\&=\alpha(\alpha+\frac{s}{k})\cdots(\alpha+\frac{(n-m-1)s}{k})={}^{s}\ (\alpha)_{n-m}^{k}.\end{aligned}$$

Thus,

$$\frac{(-1)^{m}\ {}^{s}(\alpha)_{n}^{k}}{{}^{s}(\frac{s-ns}{k}-\alpha)_{m}^{k}}={}^{s}\ (\alpha)_{n-m}^{k}.$$

Example 2.10. *Prove that*

$$(n-m)!=\frac{(-s)^{m}n!}{k^{m}\ {}^{s}(-\frac{ns}{k})_{m}^{k}}.$$

Proof.

$$^{s}(\alpha)^{k}_{n-m} = \frac{(-1)^{m}\ {}^{s}(\alpha)^{k}_{n}}{^{s}(\frac{s-ns}{k}-\alpha)^{k}_{m}},$$

taking $\alpha = \frac{s}{k}$, we have

$$^{s}(\frac{s}{k})^{k}_{n-m} = \frac{(-1)^{m}\ {}^{s}(\frac{s}{k})^{k}_{n}}{^{s}(\frac{-ns}{k})^{k}_{m}},$$

$$(\frac{s}{k})^{n-m}(n-m)! = \frac{(-1)^{m}\ (\frac{s}{k})^{n}n!}{^{s}(\frac{-ns}{k})^{k}_{m}},$$

$$(n-m)! = \frac{(-s)^{m}n!}{k^{m}\ {}^{s}(-\frac{ns}{k})^{k}_{m}}.$$

□

2.1.1. Legender's (s,k)-Duplication formula.

Lemma 2.11. *Prove that*

$$\sqrt{\frac{k\pi}{s}}\ {}^{s}\Gamma^{k}(2z) = 2^{\frac{2kz}{s}-1}\ {}^{s}\Gamma^{k}(z)\ {}^{k}\Gamma^{k}(z+\frac{s}{2k}).$$

Proof. Since $^{s}(\alpha)^{k}_{2n} = 2^{2n}\ {}^{s}(\frac{\alpha}{2})^{k}_{n}\ {}^{s}(\frac{k\alpha+s}{2k})^{k}_{n}$ and $\alpha = 2z$, we have

$$^{s}(2z)^{k}_{2n} = 2^{2n}\ {}^{s}(\frac{2z}{2})^{k}_{n}\ {}^{s}(\frac{2kz+s}{2k})^{k}_{n}$$
$$\frac{^{s}\Gamma^{k}(2z+\frac{2ns}{k})}{^{s}\Gamma^{k}(2z)} = \frac{2^{2n}\ {}^{s}\Gamma^{k}(\frac{2z}{2}+\frac{ns}{k})\ {}^{s}\Gamma^{k}(\frac{2kz+s+2ns}{2k})}{^{s}\Gamma^{k}(\frac{2z}{2})\ {}^{s}\Gamma^{k}(\frac{2kz+s}{2k})}.$$

Using

$$^{s}\Gamma^{k}(z) = \lim_{n\to\infty}\frac{n!s^{n}(\frac{ns}{k})^{\frac{kz}{k}}}{k^{n}\ {}^{s}(z)^{k}_{n}}$$

and

$$1 = \lim_{n\to\infty}\frac{n!s^{n}(\frac{ns}{k})^{\frac{kz}{k}}}{k^{n}\ {}^{s}\Gamma^{k}(z+\frac{ns}{k})}.$$

$$\frac{^{s}\Gamma^{k}(2z)}{^{s}\Gamma^{k}(z)\ {}^{s}\Gamma^{k}(z+\frac{s}{2k})} = \lim_{n\to\infty}\frac{^{s}\Gamma^{k}(2z+\frac{2ns}{k})n!k^{2n}\ s^{n}(\frac{ns}{k})^{\frac{kz}{s}}}{(2n)!s^{2n}k^{n}(\frac{2ns}{k})^{\frac{2kz}{s}}\ {}^{s}\Gamma^{k}(z+\frac{ns}{k})}$$
$$\times\frac{n!s^{n}(\frac{ns}{k})^{\frac{kz}{s}}(2n)!s^{2n}(\frac{2ns}{k})^{\frac{2kz}{s}}k^{2n}}{(\frac{ns}{k})^{\frac{4kz-3s}{s}}k^{2n}k^{n}n!n!2^{2n}\ {}^{s}\Gamma^{k}(z+\frac{s+2ns}{2k})}$$
$$= \sqrt{\frac{s}{k}}\lim_{n\to\infty}\frac{(2n)!\sqrt{n}}{2^{2n}(n!)^{2}} = \sqrt{\frac{s}{k}}C.$$

Put $z = \frac{s}{2k}$ in above equation and use ${}^s\Gamma^k(\frac{s}{2k}) = \sqrt{\frac{k\pi}{s}}$. This implies that

$$C = \sqrt{\frac{1}{\pi}}.$$

Therefore

$$\sqrt{\frac{k\pi}{s}}\ {}^s\Gamma^k(2z) = 2^{\frac{2kz}{s}-1}\ {}^s\Gamma^k(z)\ {}^k\Gamma^k(z+\frac{s}{2k}).$$

□

Definition 2.12. *The* ***binomial*** (s,k)***-theorem*** *states that*

$$(1+sz/k)^n = \binom{n}{0}(sz/k)^0 + \binom{n}{1}(sz/k)^1 + \cdots \binom{n}{n}(sz/k)^n + \cdots, \tag{21}$$

i.e.

$$(1+sz/k)^n = \sum_{m=0}^{\infty}\binom{n}{m}(sz/k)^m \tag{22}$$

and

$$(1+sz/k)^n = 1 + (nsz/k) + \frac{n(n-1)(sz/k)^2}{2!} + \cdots . \tag{23}$$

Also

$$(1-sz/k)^{-\frac{k\alpha}{s}} = \sum_{n=0}^{\infty}\frac{{}^s(\alpha)_n^k z^n}{n!}, \quad \alpha\in\mathbb{R},\ s>k>0. \tag{24}$$

3. Generalized hypergeometric functions and their properties

In this section, a new generalization of hypergeometric functions are introduced with the help of new generalized Pochhammer's symbol (1). We also introduce the generalized hypergeometric differential equations and derive that the new generalized hypergeometric function is the solutions of the said differential equations. Certain basic properties are also presented.

Theorem 3.1. *Prove that* $sz(k-sz)\omega'' + [k^2c - (ak+bk+s)sz]\omega' - abk^2\omega = 0$ *if and only if* ${}^s_2F^k_1\left(\begin{matrix}(a;s,k),(b;s,k)\\ (c;s,k)\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty}\frac{{}^s(a)_n^k\ {}^s(b)_n^k z^n}{{}^s(c)_n^k n!}$ *is a solution.*

Proof. Let $\omega = {}^s_2F^k_1\left(\begin{matrix}(a;s,k),(b;s,k)\\ (c;s,k)\end{matrix}\middle| z\right)$ be a solution of the differential equation. Now Consider $\theta = z\frac{d}{dz}$

$$\omega = \sum_{n=0}^{\infty}\frac{{}^s(a)_n^k\ {}^s(b)_n^k z^n}{{}^s(c)_n^k n!}. \tag{25}$$

$$\begin{aligned}\frac{s}{k}\theta\left(\frac{s}{k}\theta + c - \frac{s}{k}\right)\omega &= \frac{s}{k}\theta\left(\frac{s}{k}\theta + c - \frac{s}{k}\right)\sum_{n=0}^{\infty}\frac{{}^s(a)_n^k\ {}^s(b)_n^k z^n}{{}^s(c)_n^k\ n!}\\ &= \frac{sz}{k}\theta\left(a+\frac{s\theta}{k}\right)\left(b+\frac{s\theta}{k}\right)\omega.\end{aligned}$$

Thus

$$\frac{s}{k}\theta\left(\frac{s}{k}\theta + c - \frac{s}{k}\right)\omega - \frac{sz}{k}\theta\left(a+\frac{s\theta}{k}\right)\left(b+\frac{s\theta}{k}\right)\omega = 0, \tag{26}$$

after a little simplification, we get

$$sz(k-sz)\omega'' + [k^2c - (ak+bk+s)sz]\omega' - abk^2\omega = 0.$$

Conversely:
Let $sz(k-sz)\omega'' + [k^2c - (ak+bk+s)sz]\omega' - abk^2\omega = 0$, suppose that

$$\begin{aligned}
\omega &= \sum_{n=0}^{\infty} d_n z^n \\
\omega' &= \sum_{n=1}^{\infty} n d_n z^{n-1} \\
\omega'' &= \sum_{n=2}^{\infty} n(n-1) d_n z^{n-2},
\end{aligned}$$

put these values in differential equation

$$sz(k-sz)\sum_{n=2}^{\infty} n(n-2)d_n z^{n-1} + [k^2c - (ak+bk+s)sz]\sum_{n=1}^{\infty} n d_n z^{n-1} - abk^2 \sum_{n=0}^{\infty} d_n z^n = 0,$$

therefore

$$\begin{aligned}
sk\sum_{n=2}^{\infty} n(n-1)d_n z^{n-1} - s\sum_{n=2}^{\infty} n(n-1)d_n z^n + k^2c\sum_{n=1}^{\infty} n d_n z^{n-1} \\
- [abk^2 + (ak+bk+s)s]\sum_{n=0}^{\infty} d_n z^n = 0
\end{aligned}$$

replace n by $(n+1)$

$$\begin{aligned}
sk\sum_{n=1}^{\infty} n(n+1)d_{n+1} z^{n} - s\sum_{n=2}^{\infty} n(n-1)d_n z^n + k^2c\sum_{n=0}^{\infty} (n+1) d_{n+1} z^{n} \\
- [abk^2 + (ak+bk+s)s]\sum_{n=0}^{\infty} d_n z^n = 0.
\end{aligned}$$

comparing the coefficient of z^n on both sides

$$skn(n+1)d_{n+1} - sn(n-1)d_n + k^2c(n+1)d_{n+1} - [abk^2 + (ak+bk+s)s]d_n = 0.$$

Therefore

$$\begin{aligned}
d_{n+1} &= \frac{sn(n-1) + [abk^2 + (ak+bk+s)s]}{skn(n+1) + k^2c(n+1)} d_n \\
&= \frac{(a+(ns/k))(b+(ns/k))}{(n+1)(c+(sn/k))} d_n.
\end{aligned}$$

This implies

$$d_n = \frac{{}_s(a)_n^k \; {}_s(b)_n^k}{{}_s(c)_n^k n!}$$

$$\begin{aligned}
\omega &= \sum_{n=0}^{\infty} \frac{{}_s(a)_n^k \; {}_s(b)_n^k z^n}{{}_s(c)_n^k n!} \\
\omega &= {}_2^sF_1^k\left(\left.\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\right| z\right)
\end{aligned}$$

$$
{}_{2}^{s}F_{1}^{k}\left(\left.\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\right|z\right)=\sum_{n=0}^{\infty}\frac{{}^{s}(a)_{n}^{k}\ {}^{s}(b)_{n}^{k}z^{n}}{{}^{s}(c)_{n}^{k}n!}.
$$

□

Definition 3.2. *The* ***hypergeometric*** (s,k)***-function*** *is defined as*

$$
{}_{2}^{s}F_{1}^{k}\left(\left.\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\right|z\right)=\sum_{n=0}^{\infty}\frac{{}^{s}(a)_{n}^{k}\ {}^{s}(b)_{n}^{k}z^{n}}{{}^{s}(c)_{n}^{k}n!}, \tag{27}
$$

provided $a,\ b,\ c\in\mathbb{C},\ s>k>0,\ |z|<\frac{k}{s}$ *and* $\Re(c-b-a)>0$.

Theorem 3.3. *If* $\Re(c)>\Re(b)>0,\ s>k>0$, *then for all finite* $z<\frac{k}{s}$

$$
\begin{aligned}
&{}_{2}^{s}F_{1}^{k}\left[\left.\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\right|z\right]\\
&=\frac{k\ {}^{s}\Gamma^{k}(c)}{s\ {}^{s}\Gamma^{k}(b)\ {}^{s}\Gamma^{k}(c-b)}\int_{0}^{1}t^{\frac{kb}{s}-1}(1-t)^{\frac{kc-kb}{s}-1}\left(1-\frac{sxt}{k}\right)^{-\frac{ka}{s}}dt.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
{}_{2}^{s}F_{1}^{k}\left[\left.\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\right|z\right] &= \sum_{n=0}^{\infty}\frac{{}^{s}(a)_{n}^{k}\ {}^{s}(b)_{n}^{k}\ \ z^{n}}{{}^{s}(c)_{n}^{k}\ n!}\\
&= \sum_{n=0}^{\infty}\frac{{}^{s}(a)_{n}^{k}\ {}^{s}(b)_{n}^{k}\ z^{n}}{{}^{s}(c)_{n}^{k}n!},
\end{aligned}
$$

by using ${}^{s}(a)_{n}^{k}=\frac{{}^{s}\Gamma^{k}(a+(ns)/k)}{{}^{s}\Gamma^{k}(a)}$, we have

$$
\begin{aligned}
&{}_{2}^{s}F_{1}^{k}\left[\left.\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\right|z\right]\\
&=\sum_{n=0}^{\infty}\frac{{}^{s}(a)_{n}^{k}\ {}^{s}\Gamma^{k}(b+(ns)/k)\ {}^{s}\Gamma^{k}(c)\ z^{n}}{{}^{s}\Gamma^{k}(c+(ns)/k)\ {}^{s}\Gamma^{k}(b)\ n!}\\
&=\frac{{}^{s}\Gamma^{k}(c)}{{}^{s}\Gamma^{k}(b)\ {}^{s}\Gamma^{k}(c-b)}\sum_{n=0}^{\infty}\frac{{}^{s}(a)_{n}^{k}\ {}^{s}\Gamma^{k}(b+(ns)/k)\ {}^{s}\Gamma^{k}(c-b)\ z^{n}}{{}^{s}\Gamma^{k}(c+(ns)/k)\ n!}\\
&=\frac{k\ {}^{s}\Gamma^{k}(c)}{s\ {}^{s}\Gamma^{k}(b)\ {}^{s}\Gamma^{k}(c-b)}\int_{0}^{1}t^{\frac{kb}{s}-1}(1-t)^{\frac{kc-kb}{s}-1}\left(1-\frac{szt}{k}\right)^{-\frac{ka}{s}}dt.
\end{aligned}
$$

□

Theorem 3.4. *If* $\Re(c-b-a)>0,\ s>k>0$, *then*

$$
{}_{2}^{s}F_{1}^{k}\left[\left.\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\right|\frac{k}{s}\right]=\frac{{}^{s}\Gamma^{k}(c)\ {}^{s}\Gamma^{k}(c-b-a)}{{}^{s}\Gamma^{k}(c-b)\ {}^{s}\Gamma^{k}(c-a)}.
$$

Proof. Let $|z| < \frac{k}{s}$ and consider

$$
\begin{aligned}
&{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\middle|\frac{k}{s}\right]\\
&= \frac{k\ {}^s\Gamma^k(c)}{s\ {}^s\Gamma^k(b)\ {}^s\Gamma^k(c-b)}\int_0^1 t^{\frac{kb}{s}-1}(1-t)^{\frac{kc-kb}{s}-1}\big(1-t\big)^{-\frac{ka}{s}}dt\\
&= \frac{{}^s\Gamma^k(c)\ {}^s\Gamma^k(c-b-a)}{{}^s\Gamma^k(c-b)\ {}^s\Gamma^k(c-a)}.
\end{aligned}
$$

□

Theorem 3.5. *If* $|z| < \frac{k}{s}$ *and* $\frac{|z|}{|(1-(s/k)z)|} < 1$, $s > k > 0$, *then*

$$
{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\ z\right] = (1-(s/k)z)^{-\frac{ka}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(c-b;s,k)\\(c;s,k)\end{matrix}\ \frac{-kz}{k-sz}\right].
$$

Proof. Let $|z| < \frac{k}{s}$ and consider

$$
\begin{aligned}
&(1-(s/k)z)^{-\frac{ka}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(c-b;s,k)\\(c;s,k)\end{matrix}\ \frac{-kz}{k-sz}\right]\\
&= \sum_{m=0}^{\infty}\frac{{}^s(a)_m^k\ {}^s(c-b)_m^k\ z^m(1-(s/k)z)^{-\frac{ka}{s}-m}}{{}^s(c)_m^k\ m!}\\
&= \sum_{n=0}^{\infty}\frac{{}^s(a)_n^k\ {}^s(b)_n^k\ z^n}{{}^s(c)_n^k\ n!}\\
&= {}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\ z\right].
\end{aligned}
$$

□

Theorem 3.6. *If* $|z| < \frac{k}{s}$, $s > k > 0$, *then*

$$
{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\middle|z\right] = (1-(s/k)z)^{-\frac{kc-ka-kb}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(c-a;s,k),(c-b;s,k)\\(c;s,k)\end{matrix}\middle|z\right].
$$

Proof. Let $|z| < \frac{k}{s}$ and consider

$$
{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\middle|z\right] = (1-(s/k)z)^{-\frac{ka}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(c-b;s,k)\\(c;s,k)\end{matrix}\middle|\frac{-kz}{k-sz}\right].
$$

Suppose that $y = \frac{-kz}{k-sz}$, we have

$$
\begin{aligned}
&(1-(s/k)z)^{\frac{ka}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\ z\right]\\
&= {}_2^sF_1^k\left[\begin{matrix}(a;s,k),(c-b;s,k)\\(c;s,k)\end{matrix}\middle|y\right]\\
&= (1-(s/k)y)^{\frac{kb-kc}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(c-b;s,k),(c-a;s,k)\\(c;s,k)\end{matrix}\middle|\frac{-ky}{k-sy}\right].
\end{aligned}
$$

This implies that

$$
{}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\ z\right] = (1-(s/k)z)^{-\frac{kc-ka-kb}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(c-a;s,k),(c-b;s,k)\\(c;s,k)\end{matrix}\middle|z\right].
$$

□

Lemma 3.7. *Show that*

$$ {}_2^sF_1^k\left[\begin{matrix}(\frac{-ns}{k};s,k),(\frac{s}{k}-b-\frac{ns}{k};s,k)\\ (a;s,k)\end{matrix}\middle|\frac{k}{s}\right]=\frac{{}^s(a+b-\frac{s}{k})_n^k}{{}^s(a)_n^k\ {}^s(a+b-\frac{s}{k})_n^k}. $$

Proof. Since Gauss summation (s,k)-theorem

$$ \begin{aligned} &{}_2^sF_1^k\left[\begin{matrix}(\frac{-ns}{k};s,k),(\frac{s}{k}-b-\frac{ns}{k};s,k)\\ (a;s,k)\end{matrix}\middle|\frac{k}{s}\right]\\ &=\frac{{}^s\Gamma^k(a)\ {}^s\Gamma^k(a+b+\frac{2ns-s}{k})}{{}^s\Gamma^k(a+\frac{sn}{k})\ {}^s\Gamma^k(a+b+\frac{ns-s}{k})}\\ &=\frac{{}^s(a+b-\frac{s}{k})^k}{{}^s(a)^k\ {}^s(a+b-\frac{s}{k})^k}. \end{aligned} $$

□

Theorem 3.8. *If $2b$ is neither zero nor a negative integer and if $|sy|<\frac{k}{2}$ and $\left|\frac{sy}{k-sy}\right|<1$, $s>k>0$, then*

$$ k^{\frac{ka}{s}}(k-sy)^{-\frac{ka}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(\frac{a}{2};s,k),(\frac{ka+s}{2k};s,k)\\ (\frac{s}{2k}+b;s,k)\end{matrix}\middle|\frac{sky^2}{(k-sy)^2}\right]={}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\ (2b;s,k)\end{matrix}\middle|2y\right]. $$

Proof. Let $|sy|<\frac{k}{2}$ and $\left|\frac{sy}{k-sy}\right|<1$, we have

$$ \begin{aligned} &k^{\frac{ka}{s}}(k-sy)^{-\frac{ka}{s}}\,{}_2^sF_1^k\left[\begin{matrix}(\frac{a}{2};s,k),(\frac{ka+s}{2k};s,k)\\ (\frac{s}{2k}+b;s,k)\end{matrix}\middle|\frac{sky^2}{(k-sy)^2}\right]\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{{}^s(\frac{a}{2})_m^k\ {}^s(\frac{ka+s}{2k})_m^k\ {}^s(a+\frac{2ms}{k})_n^k\ y^{n+2m}s^m}{k^m\ {}^s(b+\frac{s}{2k})_m^k\ n!m!}\\ &={}_2^sF_1^k\left[\begin{matrix}(a;s,k),(b;s,k)\\ (2b;s,k)\end{matrix}\middle|2y\right]. \end{aligned} $$

□

Definition 3.9. ***Confluent hypergeometric*** (s,k)***-function*** *is a solution of confluent hypergeometric differential (s,k)-equation which is a degenerate form of a hypergeometric differential (s,k)-equation where two regular singularities out of three merge into an irregular singularity.*
We define the ***confluent hypergeometric*** (s,k)***-function*** *as*

$$ {}_1^sF_1^k\left(\begin{matrix}(a;s,k)\\ (c;s,k)\end{matrix}\middle|z\right)=\sum_{n=0}^{\infty}\frac{{}^s(a)_n^k\ z^n}{{}^s(c)_n^k\ n!}, \tag{28} $$

for $|z|<\infty$, $c\neq 0,-s/k,-2s/k,\cdots$.

Theorem 3.10. *If $\Re(b)>\Re(a)>0$, $s>k>0$, then for all finite z*

$$ \begin{aligned} &{}_1^sF_1^k\left[\begin{matrix}(a;s,k)\\ (b;s,k)\end{matrix}\middle|z\right]\\ &=\frac{k\ {}^s\Gamma^k(b)}{s\ {}^s\Gamma^k(a)\ {}^s\Gamma^k(b-a)}\int_0^1 t^{\frac{ka}{s}-1}(1-t)^{\frac{kb-ka}{s}-1}e^{xt}dt. \end{aligned} $$

Proof.

$$ {}_{1}^{s}F_{1}^{k}\left[\begin{matrix}(a;s,k)\\(b;s,k)\end{matrix}\middle|z\right]=\sum_{n=0}^{\infty}\frac{{}^{s}(a)_{n}^{k}\ z^{n}}{{}^{s}(b)_{n}^{k}\ n!}. $$

By using ${}^{s}(a)_{n}^{k}=\frac{{}^{s}\Gamma^{k}(a+(ns)/k)}{{}^{s}\Gamma^{k}(a)}$, we have

$$ \begin{aligned} &{}_{1}^{s}F_{1}^{k}\left[\begin{matrix}(a;s,k)\\(b;s,k)\end{matrix}\middle|z\right]\\ &=\frac{k\ {}^{s}\Gamma^{k}(b)}{s\ {}^{s}\Gamma^{k}(a)\ {}^{s}\Gamma^{k}(b-a)}\int_{0}^{1}t^{\frac{ka}{s}-1}(1-t)^{\frac{kb-ka}{s}-1}e^{zt}dt. \end{aligned} $$

□

Definition 3.11. *The **Generalized hypergeometric** (s,k)**-functions** is defined as*

$$ {}_{r}^{s}F_{q}^{k}\left[\begin{matrix}(\alpha_1;s,k),(\alpha_2;s,k),\cdots,(\alpha_r;s,k)\\(\beta_1;s,k),(\beta_2;s,k),\cdots,(\beta_q;s,k)\end{matrix}\middle|z\right]=\sum_{n=0}^{\infty}\frac{{}^{s}(\alpha_1)_{n}^{k}\ {}^{s}(\alpha_2)_{i}^{k}\ \cdots\ {}^{s}(\alpha_r)_{n}^{k}\ z^{n}}{{}^{s}(\beta_1)_{n}^{k}\ {}^{s}(\beta_2)_{n}^{k}\ \cdots\ {}^{s}(\beta_q)_{n}^{k}n!}, \tag{29} $$

for all $\alpha_i,\ \beta_j\in\mathbb{C},\ \beta_j\neq 0,-s/k,-2s/k,\cdots,\quad |z|<\frac{k}{s}$ *where* $i=1,2,\cdots,r$ *and* $j=1,2,\cdots,q$.
It is known that
(i) if $r\leq q$*, the series converges for all finite* z*,*
(ii) if $r=q+1$*, the series converges for* $|z|<\frac{k}{s}$ *and diverges for* $|z|>\frac{k}{s}$*,*
(iii) if $r>q+1$*, the series diverges for* $z\neq 0$ *unless the series terminates.*

Theorem 3.12. *If* $r\leq s+1$*, if* $\Re(b_1)>\Re(a_1)>0$*, if no one of* $b_1,\ b_2,\ \cdots\ b_s$ *is zero or a negative integer,* $s>k>0$ *and if* $|z|\frac{k}{s}$*, then*

$$ \begin{aligned} {}_{r}^{s}F_{q}^{k}\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle|z\right]&=\frac{k\ {}^{s}\Gamma^{k}(b_1)}{s\ {}^{s}\Gamma^{k}(a_1){}^{s}\Gamma^{k}(b_1-a_1)}\\ \times\int_{0}^{1}t^{\frac{ka_1}{s}-1}(1-t)^{\frac{kb_1-ka_1}{s}-1}\ {}_{r-1}^{s}F_{q-1}^{k}&\left[\begin{matrix}(a_2;s,k),\cdots,(a_r;s,k)\\(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle|zt\right]dt. \end{aligned} $$

Proof.

$$ \begin{aligned} &{}_{r}^{s}F_{q}^{k}\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle|z\right]\\ &=\sum_{n=0}^{\infty}\frac{{}^{s}(a_1)_{n}^{k}\ {}^{s}(a_2)_{n}^{k}\cdots{}^{s}(a_r)_{n}^{k}\ z^{n}}{{}^{s}(b_1)_{n}^{k}\ {}^{s}(b_2)_{n}^{k}\cdots{}^{s}(b_q)_{n}^{k}\ n!}\\ &=\frac{k\ {}^{s}\Gamma^{k}(b_1)}{s\ {}^{s}\Gamma^{k}(a_1){}^{s}\Gamma^{k}(b_1-a_1)}\\ &\quad\times\int_{0}^{1}t^{\frac{ka_1}{s}-1}(1-t)^{\frac{kb_1-ka_1}{s}-1}\ {}_{r-1}^{s}F_{q-1}^{k}\left[\begin{matrix}(a_2;s,k),\cdots,(a_r;s,k)\\(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle|zt\right]dt. \end{aligned} $$

□

Theorem 3.13. *If $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and if m and p are positive integers, $s > k > 0$, then inside the region of convergence of the resultant series*

$$\int_0^t x^{\frac{k\alpha}{s}-1}(t-x)^{\frac{k\beta}{s}-1} {}_rF_q^{s\,k}\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| cx^m(t-x)^p\right]dx$$
$$= \frac{s}{k}\, {}^sB^k(\alpha,\beta)t^{\frac{k\alpha+k\beta}{s}-1}$$
$$\times {}_{r+m+p}^{\ \ \ s}F_{q+m+p}^k\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k),(\frac{\alpha}{m};s,k),(\frac{k\alpha+s}{mk};s,k),\\ \cdots,(\frac{k\alpha+(m-1)s}{mk};s,k),(\frac{k\beta+s}{pk};s,k),\cdots,(\frac{k\beta+(p-1)s}{pk};s,k)\\ (b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k),\\ (\frac{\alpha+\beta}{(m+p)};s,k),(\frac{k\alpha+k\beta+s}{(m+p)k};s,k),\cdots,(\frac{k\alpha+k\beta+(m+p-1)s}{(m+p)k};s,k)\end{matrix}\middle| \frac{m^mp^pct^{m+p}}{m+p^{m+p}}\right].$$

Proof. Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and if m and p are positive integers and consider

$$\int_0^t x^{\frac{k\alpha}{s}-1}(t-x)^{\frac{k\beta}{s}-1} {}_rF_q^{s\,k}\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| cx^m(t-x)^p\right]dx.$$

Suppose that $x = tv \Rightarrow \frac{dx}{dv} = t$ when $x = 0 \Rightarrow v = 0$ when $x = t \Rightarrow v = 1$, we have

$$\int_0^t x^{\frac{k\alpha}{s}-1}(t-x)^{\frac{k\beta}{s}-1} {}_rF_q^{s\,k}\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| cx^m(t-x)^p\right]dx$$
$$= \int_0^1 (tv)^{\frac{k\alpha}{s}-1}(t-tv)^{\frac{k\beta}{s}-1} {}_rF_q^{s\,k}\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| cv^mt^{m+p}(1-v)^p\right]tdv$$
$$= \frac{s}{k}\, {}^sB^k(\alpha,\beta)t^{\frac{k\alpha+k\beta}{s}-1}$$
$$\times {}_{r+m+p}^{\ \ \ s}F_{q+m+p}^k\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k),(\frac{\alpha}{m};s,k),(\frac{k\alpha+s}{mk};s,k),\\ \cdots,(\frac{k\alpha+(m-1)s}{mk};s,k),(\frac{k\beta+s}{pk};s,k),\cdots,(\frac{k\beta+(p-1)s}{pk};s,k)\\ (b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k),\\ (\frac{\alpha+\beta}{(m+p)};s,k),(\frac{k\alpha+k\beta+s}{(m+p)k};s,k),\cdots,(\frac{k\alpha+k\beta+(m+p-1)s}{(m+p)k};s,k)\end{matrix}\middle| \frac{m^mp^pct^{m+p}}{m+p^{m+p}}\right].$$

□

Theorem 3.14. *If $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and if m is a positive integers, $s > k > 0$, then inside the region of convergence of the resultant series*

$$\int_0^t x^{\frac{k\alpha}{s}-1}(t-x)^{\frac{k\beta}{s}-1} {}_rF_q^{s\,k}\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| cx^m\right]dx$$
$$= \frac{s}{k}\, {}^sB^k(\alpha,\beta)t^{\frac{k\alpha+k\beta}{s}-1}$$
$$\times {}_{r+m}^{\ \ s}F_{q+m}^k\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k),\\ (\frac{\alpha}{m};s,k),(\frac{k\alpha+s}{mk};s,k),\cdots,(\frac{k\alpha+(m-1)s}{mk};s,k)\\ (b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k),\\ (\frac{\alpha+\beta}{m};s,k),(\frac{k\alpha+k\beta+s}{mk};s,k),\cdots,(\frac{k\alpha+k\beta+(m-1)s}{mk};s,k)\end{matrix}\middle| ct^m\right].$$

Proof. Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and if m is a positive integers and consider

$$\int_0^t x^{\frac{k\alpha}{s}-1}(t-x)^{\frac{k\beta}{s}-1} {}_rF_q^{s\,k}\left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| cx^m\right]dx.$$

Suppose that $x = tv \Rightarrow \frac{dx}{dv} = t$ when $x = 0 \Rightarrow v = 0$ when $x = t \Rightarrow v = 1$, we have

$$\begin{aligned}
&\int_0^t x^{\frac{k\alpha}{s}-1}(t-x)^{\frac{k\beta}{s}-1} {}_r^sF_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| cx^m\right] dx\\
&= \int_0^1 (tv)^{\frac{k\alpha}{s}-1}(t-tv)^{\frac{k\beta}{s}-1} {}_r^sF_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| cv^mt^m\right] tdv\\
&= \frac{s}{k}\ {}^sB^k(\alpha,\beta)t^{\frac{k\alpha+k\beta}{s}-1}\\
&\quad\times {}_{r+m}^{\ \ s}F_{q+m}^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k),\\ (\frac{\alpha}{m};s,k),(\frac{k\alpha+s}{mk};s,k),\cdots,(\frac{k\alpha+(m-1)s}{mk};s,k)\\ (b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k),\\ (\frac{\alpha+\beta}{m};s,k),(\frac{k\alpha+k\beta+s}{mk};s,k),\cdots,(\frac{k\alpha+k\beta+(m-1)s}{mk};s,k)\end{matrix}\middle| ct^m\right].
\end{aligned}$$

□

Lemma 3.15. *Show that*

$$\begin{aligned}
\frac{d}{dz} {}_r^sF_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| z\right] &= \frac{\prod_{m=1}^{r} a_m}{\prod_{j=1}^{q} b_j}\\
&\times {}_r^sF_s^k \left[\begin{matrix}(a_1+\frac{s}{k};s,k),(a_2+\frac{s}{k};s,k),\cdots,(a_r+\frac{s}{k};s,k)\\(b_1+\frac{s}{k};s,k),(b_2+\frac{s}{k};s,k),\cdots,(b_q+\frac{s}{k};s,k)\end{matrix}\middle| z\right].
\end{aligned}$$

Proof. Consider

$$\begin{aligned}
&\frac{d}{dz} {}_r^sF_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| z\right]\\
&= \frac{\prod_{m=1}^{r} a_m}{\prod_{j=1}^{s} b_j} {}_r^sF_s^k \left[\begin{matrix}(a_1+\frac{s}{k};s,k),(a_2+\frac{s}{k};s,k),\cdots,(a_r+\frac{s}{k};s,k)\\(b_1+\frac{s}{k};s,k),(b_2+\frac{s}{k};s,k),\cdots,(b_q+\frac{s}{k};s,k)\end{matrix}\middle| z\right].
\end{aligned}$$

□

Lemma 3.16. *Show that*

$$\begin{aligned}
\frac{d^m}{dz^m} {}_r^sF_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| z\right] &= \frac{\prod_{i=1}^{r} {}^s(a_i)_n^k}{\prod_{j=1}^{q} {}^s(b_j)_n^k}\\
&\times {}_r^sF_q^k \left[\begin{matrix}(a_1+\frac{ms}{k};s,k),(a_2+\frac{ms}{k};s,k),\cdots,(a_r+\frac{ms}{k};s,k)\\(b_1+\frac{ms}{k};s,k),(b_2+\frac{ms}{k};s,k),\cdots,(b_q+\frac{ms}{k};s,k)\end{matrix}\middle| z\right].
\end{aligned}$$

Proof. Consider

$$\begin{aligned}
&\frac{d^m}{dz^m} {}_r^sF_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_s;s,k)\end{matrix}\middle| z\right]\\
&= \frac{\prod_{i=1}^{r} {}^s(a_i)_m^k}{\prod_{j=1}^{q} {}^s(b_j)_m^k} {}_r^sF_q^k \left[\begin{matrix}(a_1+\frac{ms}{k};s,k),(a_2+\frac{ms}{k};s,k),\cdots,(a_r+\frac{ms}{k};s,k)\\(b_1+\frac{ms}{k};s,k),(b_2+\frac{ms}{k};s,k),\cdots,(b_q+\frac{ms}{k};s,k)\end{matrix}\middle| z\right].
\end{aligned}$$

□

Theorem 3.17. *Show that*

$$\int_0^\infty e^{-pt} t^{\frac{k\alpha}{s}-1} {}_r^sF_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| xt\right] dt$$
$$= \frac{k^{\frac{k\alpha}{s}-1}\ {}^s\Gamma^k(\alpha)}{s^{\frac{k\alpha}{s}-1} p^{\frac{k\alpha}{s}}} {}_{r+1}^{\ \ s}F_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k),(\alpha;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| \frac{kx}{sp}\right],$$

where $r \le q$, $\Re(p) > 0$, $\Re(\alpha) > 0$, $s > k > 0$.

Proof. Consider

$$\int_0^\infty e^{-pt} t^{\frac{k\alpha}{s}-1} {}_r^sF_q^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| xt\right] dt$$
$$= {}_{r+1}^{\ \ s}F_s^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k),(\alpha;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| \frac{kx}{sp}\right] \int_0^\infty e^{-pt} t^{\frac{k\alpha}{s}-1} dt.$$

Since $L\{t^n\} = \frac{n!}{p^{n+1}}$

$$F(s) = \int_0^\infty e^{-pt} f(t) dt.$$

So,

$$\int_0^\infty e^{-pt} t^{\frac{k\alpha}{s}-1} {}_r^sF_s^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| xt\right] dt$$
$$= \frac{k^{\frac{k\alpha}{s}-1}\ {}^s\Gamma^k(\alpha)}{s^{\frac{k\alpha}{s}-1} p^{\frac{k\alpha}{s}}} {}_{r+1}^{\ \ s}F_s^k \left[\begin{matrix}(a_1;s,k),(a_2;s,k),\cdots,(a_r;s,k),(\alpha;s,k)\\(b_1;s,k),(b_2;s,k),\cdots,(b_q;s,k)\end{matrix}\middle| \frac{kx}{sp}\right].$$

□

3.0.1. Saalschütz's (s,k)-theorem.

Theorem 3.18. *If n is a non-negative integer and if a, b, c are independent of n, $s > k > 0$, then*

$${}_3^sF_2^k \left[\begin{matrix}(\frac{-ns}{k};s,k),(a;s,k),(b;s,k)\\(c;s,k),(\frac{s}{k}-c+a+b-\frac{ns}{k})\end{matrix}\middle| \frac{k}{s}\right] = \frac{{}^s(c-b)_n^k\ {}^s(c-a)_n^k}{{}^s(c)_n^k\ {}^s(c-a-b)_n^k}.$$

Proof. Since theorem (3.6), we have

$${}_2^sF_1^k \left[\begin{matrix}(c-a;s,k),(c-b;s,k)\\(c;s,k)\end{matrix}\middle| z\right] = (1-(s/k)z)^{\frac{kc-ka-kb}{s}} {}_2^sF_1^k \left[\begin{matrix}(a;s,k),(b;s,k)\\(c;s,k)\end{matrix}\middle| z\right].$$

$$\sum_{n=0}^\infty \frac{{}^s(c-a)_n^k\ {}^s(c-b)_n^k\ z^n}{{}^s(c)_n^k\ n!}$$
$$= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{{}^s(a)_m^k\ {}^s(b)_m^k\ {}^s(c-a-b)_n^k\ z^{m+n}}{{}^s(c)_m^k\ n!m!}$$
$$= \sum_{n=0}^\infty \frac{{}^s(c-a-b)_n^k\ z^n}{n!} {}_3^sF_2^k \left[\begin{matrix}(\frac{-ns}{k};s,k),(a;s,k),(b;s,k)\\(c;s,k),(\frac{s}{k}-c+a+b-\frac{ns}{k})\end{matrix}\middle| \frac{k}{s}\right].$$

Comparing coefficient of $\frac{z^n}{n!}$, we get

$$ {}_3^sF_2^k\left[\begin{matrix}(\frac{-ns}{k};s,k),(a;s,k),(b;s,k)\\(c;s,k),(\frac{s}{k}-c+a+b-\frac{ns}{k})\end{matrix}\middle|\frac{k}{s}\right]=\frac{{}^s(c-b)_n^k\ {}^s(c-a)_n^k}{{}^s(c)_n^k\ {}^s(c-a-b)_n^k}. $$

□

Theorem 3.19. *If $\Re(b)>\Re(a)>0$, $s>k>0$, $m\geq 1$, $m\in\mathbb{Z}^+$, then for all finite z*

$$ {}_m^sF_m^k\left[\begin{matrix}(a/m;s,k),((a+(s/k))/m;s,k),\cdots,((a+((m-1)s/k))/m;s,k)\\(b/m;s,k),((b+(s/k))/m;s,k),\cdots,((b+((m-1)s/k))/m;s,k)\end{matrix}\middle|z\right] $$
$$ =\frac{k\ {}^s\Gamma^k(b)}{s\ {}^s\Gamma^k(a)\ {}^s\Gamma^k(b-a)}\int_0^1 t^{\frac{ka}{s}-1}(1-t)^{\frac{kb-ka}{s}-1}e^{xt^m}dt. $$

Proof.

$$ {}_m^sF_m^k\left[\begin{matrix}(a/m;s,k),((a+(s/k))/m;s,k),\cdots,((a+((m-1)s/k))/m;s,k)\\(b/m;s,k),((b+(s/k))/m;s,k),\cdots,((b+((m-1)s/k))/m;s,k)\end{matrix}\middle|z\right] $$
$$ =\sum_{n=0}^{\infty}\frac{{}^s(\frac{a}{m})_n^k\ {}^s(\frac{ak+s}{mk})_n^k\ \cdots\ {}^s(\frac{ak+(m-1)s}{mk})_n^k\ z^n}{{}^s(\frac{b}{m})_n^k\ {}^s(\frac{bk+s}{mk})_n^k\cdots\ {}^s(\frac{bk+(m-1)s}{mk})_n^k n!}. $$

Since ${}^s(b)_{nm}^k=2^{mn}\ {}^s(\frac{b}{m})_n^k\ {}^s(\frac{bk+s}{mk})_n^k\cdots\ {}^s(\frac{bk+(m-1)s}{mk})_n^k$. Therefore

$$ {}_m^sF_m^k\left[\begin{matrix}(a/m;s,k),((a+(s/k))/m;s,k),\cdots,((a+((m-1)s/k))/m;s,k)\\(b/m;s,k),((b+(s/k))/m;s,k),\cdots,((b+((m-1)s/k))/m;s,k)\end{matrix}\middle|z\right] $$
$$ =\sum_{n=0}^{\infty}\frac{{}^s(a)_{nm}^k\ z^n}{{}^s(b)_{nm}^k n!}. $$

By using ${}^s(a)_{nm}^k=\frac{{}^s\Gamma^k(a+(nms)/k)}{{}^s\Gamma^k(a)}$, we have

$$ {}_m^sF_m^k\left[\begin{matrix}(a/m;s,k),((a+(s/k))/m;s,k),\cdots,((a+((m-1)s/k))/m;s,k)\\(b/m;s,k),((b+(s/k))/m;s,k),\cdots,((b+((m-1)s/k))/m;s,k)\end{matrix}\middle|z\right] $$
$$ =\sum_{n=0}^{\infty}\frac{{}^s\Gamma^k(a+(nms)/k)\ {}^s\Gamma^k(b)\ z^n}{{}^s\Gamma^k(b+(nms)/k)\ {}^s\Gamma^k(a)\ n!} $$
$$ =\frac{k\ {}^s\Gamma^k(b)}{s\ {}^s\Gamma^k(a)\ {}^s\Gamma^k(b-a)}\int_0^1 t^{\frac{ka}{s}-1}(1-t)^{\frac{kb-ka}{s}-1}e^{zt^m}dt. $$

□

Theorem 3.20. *If $\Re(b)>\Re(a)>0$, $s>k>0$, $m\geq 1$, $m\in\mathbb{Z}^+$, then for all finite $z<\frac{k}{s}$*

$$ {}_{m+1}^{\ \ s}F_m^k\left[\begin{matrix}(c;s,k),(a/m;s,k),((a+(s/k))/m;s,k),\cdots,((a+((m-1)s/k))/m;s,k)\\(b/m;s,k),((b+(s/k))/m;s,k),\cdots,((b+((m-1)s/k))/m;s,k)\end{matrix}\middle|z\right] $$
$$ =\frac{k\ {}^s\Gamma^k(b)}{s\ {}^s\Gamma^k(a)\ {}^s\Gamma^k(b-a)}\int_0^1 t^{\frac{ka}{s}-1}(1-t)^{\frac{kb-ka}{s}-1}\left(1-\frac{sxt^m}{k}\right)^{-\frac{kc}{s}}dt. $$

Proof.

$$
\begin{aligned}
&{}_{m+1}^{\ \ s}F_m^k\left[\begin{matrix}(c;s,k),(a/m;s,k),((a+(s/k))/m;s,k),\cdots,((a+((m-1)s/k))/m;s,k)\\ (b/m;s,k),((b+(s/k))/m;s,k),\cdots,((b+((m-1)s/k))/m;s,k)\end{matrix}\middle| z\right]\\
&=\sum_{n=0}^{\infty}\frac{{}^s(c)_n^k\ {}^s(\frac{a}{m})_n^k\ {}^s(\frac{ak+s}{mk})_n^k\ \cdots\ {}^s(\frac{ak+(m-1)s}{mk})_n^k\ z^n}{{}^s(\frac{b}{m})_n^k\ {}^s(\frac{bk+s}{mk})_n^k\cdots\ {}^s(\frac{bk+(m-1)s}{mk})_n^k n!}.
\end{aligned}
$$

Since ${}^s(b)_{nm}^k = 2^{mn}\ {}^s(\frac{b}{m})_n^k\ {}^s(\frac{bk+s}{mk})_n^k\cdots\ {}^s(\frac{bk+(m-1)s}{mk})_n^k$

$$
\begin{aligned}
&{}_{m+1}^{\ \ s}F_m^k\left[\begin{matrix}(c;s,k),(a/m;s,k),((a+(s/k))/m;s,k),\cdots,((a+((m-1)s/k))/m;s,k)\\ (b/m;s,k),((b+(s/k))/m;s,k),\cdots,((b+((m-1)s/k))/m;s,k)\end{matrix}\middle| z\right]\\
&=\sum_{n=0}^{\infty}\frac{{}^s(c)_n^k\ {}^s(a)_{nm}^k\ z^n}{{}^s(b)_{nm}^k n!}.
\end{aligned}
$$

By using ${}^s(a)_{nm}^k = \frac{{}^s\Gamma^k(a+(nms)/k)}{{}^s\Gamma^k(a)}$, we have

$$
\begin{aligned}
&{}_{m+1}^{\ \ s}F_m^k\left[\begin{matrix}(c;s,k),(a/m;s,k),((a+(s/k))/m;s,k),\cdots,((a+((m-1)s/k))/m;s,k)\\ (b/m;s,k),((b+(s/k))/m;s,k),\cdots,((b+((m-1)s/k))/m;s,k)\end{matrix}\middle| z\right]\\
&=\sum_{n=0}^{\infty}\frac{{}^s(c)_n^k\ {}^s\Gamma^k(a+(nms)/k)\ {}^s\Gamma^k(b)\ z^n}{{}^s\Gamma^k(b+(nms)/k)\ {}^s\Gamma^k(a)\ n!}\\
&=\frac{k\ {}^s\Gamma^k(b)}{s\ {}^s\Gamma^k(a)\ {}^s\Gamma^k(b-a)}\int_0^1 t^{\frac{ka}{s}-1}(1-t)^{\frac{kb-ka}{s}-1}\left(1-\frac{sxt^m}{k}\right)^{-\frac{kc}{s}}dt.
\end{aligned}
$$

□

4. Conclusions

In this paper, we have defined the certain generalized special functions in terms of a new symbol (s,k), where $s,k>0$ and called them special (s,k)-functions. We developed differential equation for hypergeometric (s,k)-functions in the form of (s,k) and their integral representations. Furthermore, we have obtained some useful results and properties related to these special (s,k)-functions. If we take $s=1$, then the obtained results are reduced to the special k-functions cited in literature. Similarly if we letting $k=1$ in all definitions and results, then certain new results in term of s can be obtained. If we letting both $s=k=1$, then all the results will reduce to the classical results.

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