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Fixed points of Suzuki $\mathcal{Z}$-contraction type maps in b-metric spaces

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Abstract

In this paper, we introduce Suzuki $\mathcal{Z}$-contraction type (I) maps, Suzuki $\mathcal{Z}$-contraction type (II) maps, for a single selfmap and prove the existence and uniqueness of fixed points. Our results extend / generalize the results of Kumam, Gopal and Budhia [22] and Padcharoen, Kumam, Saipara and Chaipunya [25] from the metric space setting to b-metric spaces. We provide examples in support of our results.

Keywords: Fixed points; b-metric space; b-continuous; Suzuki $\mathcal{Z}$-contraction type maps.
2010 MSC: 47H10, 54H25.

1. Introduction

In 1975, in the direction of generalization of contraction condition, Dass and Gupta [18] initiated a contraction condition involving rational expression and established the existence of fixed points in complete metric spaces. In 2008, Suzuki [28] proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness.

On the other hand, in the direction of generalization of metric spaces, Bourbaki [15] and Bakhtin [9] initiated the idea of b-metric spaces. The concept of b-metric space or metric type space was introduced by Czerwik [16] as a generalization of metric space. Afterwards, many authors studied the existence of fixed points for a single-valued and multi-valued mappings in b-metric spaces under certain contraction conditions. For more details, we refer [1, 3, 4, 5, 6, 10, 11, 12, 13, 14, 17, 20, 23, 27].

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Definition 1.1. [16] Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$

(i) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 1.2. [11] Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in $X$is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) A b-metric space (X, d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent in X.

(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.
In this paper, we denote $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N}$ is the set of all natural numbers.

Example 1.3. [19] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(m, n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.4. [11] Let (X, d_X) and (Y, d_Y) be two b-metric spaces. A function $f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x we have fx_n is b-convergent to fx.

The following lemmas are useful in proving our main results.

Lemma 1.5. [8] *Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_k-1}) < \epsilon$. In this case,*

(i) $\lim\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$,

(ii) $\lim\limits_{k\to\infty} d(x_{n_k-1}, x_{m_k}) = \epsilon$,

(iii) $\lim\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k}) = \epsilon$, *(iv)* $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k-1}) = \epsilon$.

Lemma 1.6. *[26] Suppose (X,d) is a b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is a not Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_k-1}) < \epsilon$ and*

(i) $\epsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\epsilon$,

(ii) $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq s^2\epsilon$,

(iii) $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq s^2\epsilon$,

(iv) $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\epsilon$.

Lemma 1.7. *[2] Let (X,d) be a b-metric space with coefficient $s \geq 1$.*
Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively. Then we have
$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$
In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have
$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

In 2015, Khojasteh, Shukla and Radenović [21] introduced simulation function and defined $\mathcal{Z}$-contraction with respect to a simulation function.

Definition 1.8. [21] A simulation function is a mapping
$\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ satisfying the following conditions:

(i) $\zeta(0,0) = 0$;

(ii) $\zeta(t,s) < s - t$ for all $s, t > 0$;

(iii) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then
$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Remark 1.9. [7] Let ζ be a simulation function. If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(kt_n, s_n) < 0$ for any $k > 1$.

The following are examples of simulation functions.

Example 1.10. [7] Let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ be defined by

(i) $\zeta(t,s) = \lambda s - t$ for all $t, s \in \mathbb{R}^+$, where $\lambda \in [0,1)$;

(ii) $\zeta(t,s) = \frac{s}{1+s} - t$ for all $s, t \in \mathbb{R}^+$;

(iii) $\zeta(t,s) = s - kt$ for all $t, s \in \mathbb{R}^+$, where $k > 1$;

(iv) $\zeta(t,s) = \frac{1}{1+s} - (1+t)$ for all $s, t \in \mathbb{R}^+$;

(v) $\zeta(t,s) = \frac{1}{k+s} - t$ for all $s, t \in \mathbb{R}^+$ where $k > 1$.

Definition 1.11. [21] Let (X, d) be a metric space and $f : X \to X$ be a selfmap of X. We say that f is a $\mathcal{Z}$-contraction with respect to ζ if there exists a simulation function ζ such that

$$\zeta(d(fx, fy), d(x, y)) \geq 0$$

for all $x, y \in X$.

Theorem 1.12. [21] *Let (X, d) be a complete metric space and $f : X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function ζ. Then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in X and $\lim\limits_{n\to\infty} f^n x_0 = u$(say) in X and u is the unique fixed point of f in X.*

Recently, Olgun, Bicer and Alyildiz [24] proved the following result in complete metric spaces.

Theorem 1.13. *[24] Let (X, d) be a complete metric space and $f : X \to X$ be a selfmap on X. If there exists a simulation function ζ such that*

$$\zeta(d(fx, fy), M(x, y)) \geq 0$$

for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}\}$, then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in X and $\lim\limits_{n\to\infty} f^n x_0 = u$(say) in X and u is the unique fixed point of f in X.

The following theorem is due to Kumam, Gopal and Budhia [22].

Theorem 1.14. *[22] Let (X, d) be a complete metric space and $f : X \to X$ be a selfmap on X. If there exists a simulation function ζ such that*

$$\frac{1}{2}d(x, fx) < d(x, y) \implies \zeta(d(fx, fy), d(x, y)) \geq 0$$

for all $x, y \in X$, then for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ converges to the unique fixed point of f.

In 2018, Padcharoen, Kumam, Saipara and Chaipunya [25] proved the following theorem in complete metric spaces.

Theorem 1.15. [25] *Let (X, d) be a complete metric space and $f : X \to X$ be a selfmap on X. If there exists a simulation function ζ such that*

$$\frac{1}{2}d(x, fx) < d(x, y) \implies \zeta(d(fx, fy), M(x, y)) \geq 0$$

for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}\}$, then for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ converges to the unique fixed point of f.

Motivated by the works of Kumam, Gopal and Budhia [23] and Padcharoen, Kumam, Saipara and Chaipunya [25], we extend Theorem 1.14 and Theorem 1.15 to b-metric spaces for the maps satisfying Suzuki $\mathcal{Z}$-contraction type maps.

In Section 2, we introduce Suzuki $\mathcal{Z}$-contraction type (I) maps, Suzuki $\mathcal{Z}$-contraction type (II) maps, for a single selfmap and provide examples of these maps. In Section 3, we prove the existence and uniqueness of fixed points of Suzuki $\mathcal{Z}$-contraction type maps. Examples are provided in support of our results in Section 4.

2. Suzuki $\mathcal{Z}$-contraction type maps

The following we introduce Suzuki $\mathcal{Z}$-contraction type (I) and Suzuki $\mathcal{Z}$-contraction type (II) maps for a single selfmap in b-metric spaces as follows:

Definition 2.1. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a selfmap. We say that f is a Suzuki $\mathcal{Z}$-contraction type (I) map, if there exists a simulation function ζ such that

$$\frac{1}{2s}d(x,fx) < d(x,y) \text{ implies that } \zeta(s^4 d(fx,fy), M_1(x,y)) \geq 0 \tag{2.1.1}$$

for all distinct $x, y \in X$, where

$$M_1(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\}.$$

Remark 2.2. It is clear that from definition of simulation function that $\zeta(u,v) < 0$, for all $u \geq v > 0$. Therefore if f satisfies (2.1.1), then

$$\frac{1}{2s}d(x,fx) < d(x,y) \text{ implies that } s^4 d(fx,fy) < M_1(x,y),$$

for all distinct $x, y \in X$.

Example 2.3. Let $X = (0,1)$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a b-metric space with coefficient $s = 2$.
We define $f : X \to X$ by $f(x) = \frac{x}{16(1+x)}$ for all $x \in (0,1)$ and $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ by $\zeta(t,s) = \frac{1}{4}s - t, t, s \geq 0$. Without loss of generality, we assume that $y \leq x$. We have

$$\frac{1}{2s}d(x,fx) = \frac{1}{4}(x + \frac{x}{16(1+x)})^2 \leq \frac{1}{4}(x + \frac{x}{(1+x)})^2 \leq (x+y)^2 = d(x,y).$$

Here

$$\begin{aligned} M_1(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy), \tfrac{d(x,fy)+d(y,fx)}{2s}\} \\ &= \max\{(x+y)^2, (x+\tfrac{x}{16(1+x)})^2, (y+\tfrac{y}{16(1+y)})^2, \\ &\quad \tfrac{(x+\frac{y}{16(1+y)})^2+(y+\frac{x}{16(1+x)})^2}{4}\}. \end{aligned}$$

Now we consider

$$\begin{aligned} s^4 d(fx,fy) &= 16(\tfrac{x}{16(1+x)} + \tfrac{y}{16(1+y)})^2 = \tfrac{1}{16}(\tfrac{x}{(1+x)} + \tfrac{y}{(1+y)})^2 \\ &\leq \tfrac{1}{16}(\tfrac{x}{(1+x)} + x)^2 \leq \tfrac{1}{4}(x+y)^2 \\ &\leq \tfrac{1}{4}d(x,y) \leq \tfrac{1}{4}M_1(x,y). \end{aligned}$$

Therefore f is a Suzuki $\mathcal{Z}$-contraction type (I) map.

Definition 2.4. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a selfmap. We say that f is a Suzuki $\mathcal{Z}$-contraction type (II) map, if there exists a simulation function ζ such that

$$\frac{1}{2s}d(x,fx) < d(x,y) \text{ implies that } \zeta(s^4 d(fx,fy), M_2(x,y)) \geq 0 \tag{2.4.1}$$

for all distinct $x, y \in X$, where

$$M_2(x,y) = \max\{d(x,y), \frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}.$$

Remark 2.5. It is clear that from definition of simulation function that $\zeta(u,v) < 0$, for all $u \geq v > 0$. Therefore if f satisfies (2.4.1), then

$$\frac{1}{2s}d(x,fx) < d(x,y) \text{ implies that } s^4 d(fx,fy) < M_2(x,y),$$

for all distinct $x, y \in X$.

Example 2.6. *Let $X = (0,1)$ and let $d : X \times X \to \mathbb{R}^+$ defined by*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

It is clear that (X,d) is a b-metric space with coefficient $s = 2$.

Let $f : X \to X$ by $f(x) = \frac{x(10+x)}{256}$ for all $x \in (0,1)$ and $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{1}{4}s - t, t \geq 0, s \geq 0$. Without loss of generality, we assume that $y \leq x$.

We have

$$\frac{1}{2s}d(x,fx) = \frac{1}{4}(x + \frac{x(10+x)}{256})^2 \leq \frac{1}{4}(x + \frac{x(10+x)}{16})^2 \leq (x+y)^2 = d(x,y).$$

Here

$$\begin{aligned} M_2(x,y) &= \max\{d(x,y), \frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\} \\ &= \max\{(x+y)^2, \frac{(y+\frac{y(10+y)}{256})^2[1+(x+\frac{x(10+x)}{256})^2]}{1+(x+y)^2}, \frac{(y+\frac{x(10+x)}{256})^2[1+(x+\frac{x(10+x)}{256})^2]}{4(1+(x+y)^2)}\}. \end{aligned}$$

Now we consider

$$\begin{aligned} s^4 d(fx,fy) &= 16(\frac{x(10+x)}{256} + \frac{y(10+y)}{256})^2 = \frac{1}{16}(\frac{x(10+x)}{16} + \frac{y(10+y)}{16})^2 \\ &\leq \tfrac{1}{16}(\tfrac{x(10+x)}{16} + y)^2 \leq \tfrac{1}{4}(x+y)^2 \leq \tfrac{1}{4}d(x,y) \leq \tfrac{1}{4}M_2(x,y). \end{aligned}$$

Therefore f is a Suzuki $\mathcal{Z}$-contraction type (II) map.

3. Main results

Theorem 3.1. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a Suzuki $\mathcal{Z}$-contraction type (I) map. Then f has a unique fixed point in X.*

Proof. We take $x_0 \in X$ and let $\{x_n\}$ be the Picard sequence, that is, $x_n = fx_{n-1} = f^n x_0$ for $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $d(x_n, fx_n) = 0$ then $x = x_n$ becomes a fixed point of f, which completes the proof. So, without loss of generality, we suppose that $d(x_n, fx_n) > 0$
for all $n = 0, 1, 2, \ldots$.
Since

$$\frac{1}{2s}d(x_n, fx_n) \leq d(x_n, x_{n+1}),$$

from (2.1.1), we have

$$\zeta(s^4 d(x_{n+1}, x_{n+2}), M_1(x_n, x_{n+1})) = \zeta(s^4 d(fx_n, fx_{n+1}), M_1(x_n, x_{n+1})) \geq 0, \tag{3.1.1}$$

where

$$\begin{aligned} M_1(x_n, x_{n+1}) &= \max\{d(x_n,x_{n+1}), d(x_n,fx_n), d(x_{n+1},fx_{n+1}), \tfrac{1}{2s}[d(x_n,fx_{n+1}) + d(x_{n+1},fx_n)]\} \\ &= \max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \tfrac{d(x_n,x_{n+2})}{2s}\} \\ &= \max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}. \end{aligned}$$

If $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$ then $M_1(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$. Therefore from (3.1.1), we have

$$\begin{aligned} 0 &\leq \zeta(s^4 d(x_{n+1}, x_{n+2}), M_1(x_n, x_{n+1})) = \zeta(s^4 d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})) \\ &< d(x_{n+1}, x_{n+2}) - s^4 d(x_{n+1}, x_{n+2}), \end{aligned}$$

which is a contradiction. Therefore $d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$ for all $n = 0, 1, 2, \ldots$.
Hence $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real sequence. Thus there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$.
Suppose that $r > 0$. By using the condition (ζ_3) with $t_n = d(x_{n+1}, x_{n+2})$ and $s_n = d(x_n, x_{n+1})$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(s^4 d(x_{n+1}, x_{n+2}), M_1(x_n, x_{n+1})) \\ &= \limsup_{n\to\infty} \zeta(s^4 d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) < 0, \end{aligned}$$

a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.1.2}$$

Now we prove that $\{x_n\}$ is a b-Cauchy sequence.
On the contrary, suppose that $\{x_n\}$ is not b-Cauchy.
Case (i). $s = 1$.
In this case, by Lemma 1.5 there exist an $\epsilon > 0$ and sequence of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{n_k-1}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.5.
Suppose that there exists a $k \geq k_1$ such that

$$\frac{1}{2} d(x_{m_k}, x_{m_k+1}) > d(x_{m_k}, x_{n_k}). \tag{3.1.3}$$

On letting as $k \to \infty$ in (3.1.3), we get that $\epsilon \leq 0$,
which is a contradiction.
Therefore $\frac{1}{2} d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{n_k})$ and from (2.1.1), we have

$$\zeta(d(fx_{m_k}, fx_{n_k}), M_1(x_{m_k}, x_{n_k})) \geq 0,$$

where

$$M_1(x_{m_k}, x_{n_k}) = \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, fx_{m_k}), d(x_{n_k}, fx_{n_k}), \frac{1}{2}[d(x_{n_k}, fx_{m_k}) + d(x_{m_k}, fx_{n_k})]\}.$$

On taking limits as $k \to \infty$ and using (3.1.2), we get

$$\lim_{n\to\infty} M_1(x_{m_k}, x_{n_k}) = \max\{\epsilon, 0, 0, \epsilon\} = \epsilon.$$

By using (ζ_3) with $t_n = d(x_{m_k+1}, x_{n_k+1})$ and $s_n = M_1(x_{m_k}, x_{n_k})$, we have

$$0 \leq \limsup_{k\to\infty} \zeta(d(x_{m_k+1}, x_{n_k+1}), M_1(x_{m_k}, x_{n_k})) < 0,$$

a contradiction.
Case (ii). $s > 1$.
In this case, by Lemma 1.6 there exist an $\epsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{n_k-1}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.6. Suppose that there exists a $k \geq k_1$ such that

$$\frac{1}{2s} d(x_{m_k}, x_{m_k+1}) > d(x_{m_k}, x_{n_k}). \tag{3.1.4}$$

On letting limit superior as $k \to \infty$ in (3.1.4), we get that $\epsilon \leq 0$, which is a contradiction. Therefore $\frac{1}{2s}d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{n_k})$ and from (2.1.1), we have

$$\zeta(s^4 d(fx_{m_k}, fx_{n_k}), M_1(x_{m_k}, x_{n_k})) \geq 0,$$

where

$$M_1(x_{m_k}, x_{n_k}) = \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, fx_{m_k}), d(x_{n_k}, fx_{n_k}), \frac{1}{2s}[d(x_{n_k}, fx_{m_k}) + d(x_{m_k}, fx_{n_k})]\}.$$

On taking limit superior as $k \to \infty$ and using (3.1.2), we get

$$\lim_{n\to\infty} M_1(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0, 0, s\epsilon\} = s\epsilon.$$

Now we have

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(s^4 d(fx_{m_k}, fx_{n_k}), M_1(x_{m_k}, x_{n_k})) \\ &\leq \limsup_{k\to\infty}[M_1(x_{m_k}, x_{n_k}) - s^4 d(x_{m_k+1}, x_{n_k+1})] \\ &= \limsup_{k\to\infty} M_1(x_{m_k}, x_{n_k}) - s^4 \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \\ &\leq s\epsilon - s^4 \tfrac{\epsilon}{s^2}, \end{aligned}$$

which is a contradiction. Therefore by Case (i) and Case (ii), we have $\{x_n\}$ is a b-Cauchy sequence in X. Since X is b-complete, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$.
Now we prove that x is a fixed point of f. Suppose that $x \neq fx$. We now show that

$$\text{either } (a) : \frac{1}{2s}d(x_n, x_{n+1}) \leq d(x_n, x) \ \text{ (or) } (b) : \frac{1}{2s}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x) \tag{3.1.5}$$

hold.
On the contrary, suppose that

$$\frac{1}{2s}d(x_n, x_{n+1}) > d(x_n, x) \text{ and } \frac{1}{2s}d(x_{n+1}, x_{n+2}) > d(x_{n+1}, x) \text{ hold for some } n = \{0, 1, 2, \ldots\}.$$

By b-triangular property, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq s[d(x_n, x) + d(x, x_{n+1})] \\ &< s\tfrac{1}{2s}[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &= \tfrac{1}{2}[d(x_n, x_{n+1}) + d(x_n, x_{n+1})] \\ &= d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Therefore the inequality (3.1.5) holds.
Subcase (a). Suppose $\frac{1}{2s}d(x_n, x_{n+1}) \leq d(x_n, x)$.
Since $\frac{1}{2s}d(x_n, fx_n) \leq d(x_n, x)$, from the inequality (2.1.1), we have

$$\zeta(s^4 d(fx_n, fx), M_1(x_n, x)) \geq 0,$$

where

$$M_1(x_n, x) = \max\{d(x_n, x), d(x_n, fx_n), d(x, fx), \frac{1}{2s}[d(x_n, fx) + d(x, fx_n)]\}.$$

On taking limit superior as $n \to \infty$, we get

$$\limsup_{n\to\infty} M_1(x_n, x) \leq \max\{0, 0, d(x, fx), \frac{1}{2s}sd(x, fx)\} = d(x, fx).$$

Therefore

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(s^4 d(fx_n, fx), M_1(x_n, x)) \\ &= \limsup_{n\to\infty} M_1(x_n, x) - \liminf_{n\to\infty} s^4 d(x_{n+1}, fx) \\ &\leq d(x, fx) - s^4 \tfrac{d(x,fx)}{s}, \end{aligned}$$

a contradiction. Therefore $x = fx$.

Subcase (b). Suppose $\frac{1}{2s} d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x)$.

Since $\frac{1}{2s} d(x_{n+1}, fx_{n+1}) \leq d(x_{n+1}, x)$, from the inequality (2.1.1), we have

$$\zeta(s^4 d(fx_{n+1}, fx), M_1(x_{n+1}, x)) \geq 0.$$

Following on the similar lines as in Subcase (a), we have x is a fixed point of f.

We now show that f has unique fixed point in X. Let x and y be two fixed points of f with $x \neq y$. Since $\frac{1}{2s} d(x, fx) < d(x, y)$, from the inequality (2.1.1), we have

$$\zeta(s^4 d(fx, fy), M_1(x, y)) \geq 0,$$

where

$$M_1(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2s}[d(x, fy) + d(y, fx)]\} = d(x, y).$$

Therefore

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(s^4 d(fx, fy), M_1(x, y)) \\ &= \limsup_{n\to\infty} M(x, y) - \liminf_{n\to\infty} s^4 d(x, y) \\ &\leq d(x, y) - s^4 d(x, y), \end{aligned}$$

a contradiction.

Therefore x is the unique fixed point of f in X. □

Even though, the proof of the following theorem is as that of Theorem 3.1, the importance of the rational term $\frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}$ in the inequality (2.4.1) is established in Example 4.3.

Theorem 3.2. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a Suzuki $\mathcal{Z}$-contraction type (II) map. Then f has a unique fixed point in X.*

Proof. Take $x_0 = x \in X$ and let $\{x_n\}$ be the Picard sequence, that is, $x_n = fx_{n-1} = f^n x_0$ for all $n \in \mathbb{N}$. Without loss of generality, we suppose that $d(x_n, fx_n) > 0$ for $n = 0, 1, 2, \ldots$.

We have $\frac{1}{2s} d(x_n, fx_n) \leq d(x_n, x_{n+1})$. From (2.4.1), we have

$$\zeta(s^4 d(x_{n+1}, x_{n+2}), M_2(x_n, x_{n+1})) = \zeta(s^4 d(fx_n, fx_{n+1}), M_2(x_n, x_{n+1})) \geq 0 \tag{3.2.1}$$

where

$$\begin{aligned} M_2(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), \tfrac{d(x_{n+1},fx_{n+1})[1+d(x_n,fx_n)]}{1+d(x_n,x_{n+1})}, \tfrac{d(x_{n+1},fx_n)[1+d(x_n,fx_n)]}{s^2(1+d(x_n,x_{n+1}))}\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

If $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$ then $M_2(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$.

Therefore from (3.2.1), we have

$$\begin{aligned} 0 &\leq \zeta(s^4 d(x_{n+1}, x_{n+2}), M_2(x_n, x_{n+1})) = \zeta(s^4 d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})) \\ &< d(x_{n+1}, x_{n+2}) - s^4 d(x_{n+1}, x_{n+2}), \end{aligned}$$

a contradiction. Therefore $d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$ for all $n = 0, 1, 2, \ldots$. Hence $\{d(x_n, x_{n+1})\}$ is a decreasing nonnegative sequence of reals.

Thus there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$.
Suppose that $r > 0$. By using the condition (ζ_3) with $t_n = d(x_{n+1}, x_{n+2})$ and $s_n = d(x_n, x_{n+1})$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(s^4 d(x_{n+1}, x_{n+2}), M_2(x_n, x_{n+1})) = \limsup_{n\to\infty} \zeta(s^4 d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) < 0,$$

a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.2.2}$$

We now prove that $\{x_n\}$ is a b-Cauchy sequence. On the contrary suppose that $\{x_n\}$ is not b-Cauchy.
Case (i). $s = 1$.
In this case, by Lemma 1.5 there exist an $\epsilon > 0$ and sequence of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{n_k-1}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.5.
Suppose that there exists a $k \geq k_1$ such that

$$\frac{1}{2} d(x_{m_k}, x_{m_k+1}) > d(x_{m_k}, x_{n_k}). \tag{3.2.3}$$

On letting as $k \to \infty$ in (3.2.3), we get that $\epsilon \leq 0$,
which is a contradiction.
Therefore $\frac{1}{2} d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{n_k})$ and from (2.4.1), we have

$$\zeta(d(fx_{m_k}, fx_{n_k}), M_2(x_{m_k}, x_{n_k})) \geq 0,$$

where

$$\begin{aligned} M_2(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}), \tfrac{d(x_{n_k}, fx_{n_k})[1+d(x_{m_k}, fx_{m_k})]}{1+d(x_{m_k}, x_{n_k})}, \\ &\quad \tfrac{d(x_{n_k}, fx_{m_k})[1+d(x_{m_k}, fx_{m_k})]}{1+d(x_{m_k}, x_{n_k})}\}. \end{aligned}$$

On taking limits as $k \to \infty$ and using (3.2.2), we get

$$\lim_{n\to\infty} M(x_{m_k}, x_{n_k}) = \max\{\epsilon, 0, \frac{\epsilon}{1+\epsilon}\} = \epsilon.$$

By using (ζ_3) with $t_n = d(x_{m_k+1}, x_{n_k+1})$ and $s_n = M_2(x_{m_k}, x_{n_k})$, we have

$$0 \leq \limsup_{k\to\infty} \zeta(d(x_{m_k+1}, x_{n_k+1}), M_2(x_{m_k}, x_{n_k})) < 0,$$

which is a contradiction.
Case (ii). $s > 1$.
In this case, by Lemma 1.6 there exist an $\epsilon > 0$ and and sequence of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{n_k-1}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.6.
Suppose that there exists a $k \geq k_1$ such that

$$\frac{1}{2s} d(x_{m_k}, x_{m_k+1}) > d(x_{m_k}, x_{n_k}). \tag{3.2.4}$$

On taking limit superior as $k \to \infty$ in (3.2.4), we get that $\epsilon \leq 0$,
which is a contradiction.
Therefore $\frac{1}{2s} d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{n_k})$ and from (2.4.1), we have

$$\zeta(s^4 d(fx_{m_k}, fx_{n_k}), M_2(x_{m_k}, x_{n_k})) \geq 0,$$

where

$$\begin{aligned} M_2(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}), \tfrac{d(x_{n_k}, fx_{n_k})[1+d(x_{m_k}, fx_{m_k})]}{1+d(x_{m_k}, x_{n_k})}, \\ &\quad \tfrac{d(x_{n_k}, fx_{m_k})[1+d(x_{m_k}, fx_{m_k})]}{s^2(1+d(x_{m_k}, x_{n_k}))}\}. \end{aligned}$$

On taking limit superior as $k \to \infty$ and using (3.2.2), we get

$$\lim_{k\to\infty} M_2(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0, \frac{s^2\epsilon}{s^2(1+\epsilon)}\} = s\epsilon.$$

Now we have

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(s^4 d(fx_{m_k}, fx_{n_k}), M_2(x_{m_k}, x_{n_k})) \\ &\leq \limsup_{k\to\infty} [M_2(x_{m_k}, x_{n_k}) - s^4 d(x_{m_k+1}, x_{n_k+1})] \\ &= \limsup_{k\to\infty} M_2(x_{m_k}, x_{n_k}) - s^4 \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \\ &\leq s\epsilon - s^4 \tfrac{\epsilon}{s^2}, \end{aligned}$$

which is a contradiction. Therefore by Case (i) and Case (ii), we have $\{x_n\}$ is a b-Cauchy sequence in X. Since X is b-complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.
Now we prove that x is a fixed point of f. Suppose that $x \neq fx$. We now show that either

$$\text{(a)} : \frac{1}{2s} d(x_n, x_{n+1}) \leq d(x_n, x) \text{ or (b)} : \frac{1}{2s} d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x) \tag{3.2.5}$$

hold.
On the contrary suppose that

$$\frac{1}{2s} d(x_n, x_{n+1}) > d(x_n, x) \text{ and } \frac{1}{2s} d(x_{n+1}, x_{n+2}) > d(x_{n+1}, x) \text{ for some } n = \{0, 1, 2, \ldots\}.$$

By b-triangular property, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq s[d(x_n, x) + d(x, x_{n+1})] < s\tfrac{1}{2s}[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &= \tfrac{1}{2}[d(x_n, x_{n+1}) + d(x_n, x_{n+1})] = d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Therefore the inquality (3.2.5) holds.
Subcase (a). Suppose $\frac{1}{2s} d(x_n, x_{n+1}) \leq d(x_n, x)$.
Since $\frac{1}{2s} d(x_n, fx_n) \leq d(x_n, x)$, from the inequality (2.4.1), we have

$$\zeta(s^4 d(fx_n, fx), M_2(x_n, x)) \geq 0,$$

where

$$M_2(x_n, x) = \max\{d(x_n, x), \frac{d(x, fx)[1 + d(x_n, fx_n)]}{1 + d(x_n, x)}, \frac{d(x, fx_n)[1 + d(x_n, fx_n)]}{s^2(1 + d(x_n, x))}\}.$$

On taking limit superior as $n \to \infty$, we get

$$\limsup_{n\to\infty} M_2(x_n, x) \leq \max\{0, d(x, fx), \frac{d(x, fx)}{s}\} = d(x, fx).$$

Therefore

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(s^4 d(fx_n, fx), M_2(x_n, x)) \\ &= \limsup_{n\to\infty} M_2(x_n, x) - \liminf_{n\to\infty} s^4 d(x_{n+1}, fx) \\ &\leq d(x, fx) - s^4 \tfrac{d(x, fx)}{s}, \end{aligned}$$

a contradiction. Therefore $x = fx$.
Subcase (b). Suppose $\frac{1}{2s} d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x)$.
Since $\frac{1}{2s} d(x_{n+1}, fx_{n+1}) \leq d(x_{n+1}, x)$, from the inequality (2.4.1), we have

$$\zeta(s^4 d(fx_{n+1}, fx), M_2(x_{n+1}, x)) \geq 0.$$

On the similar lines as in Subcase (a), here also it follows that x is a fixed point of f.

Uniqueness of fixed point of f follows as in the proof of Theorem 3.1. □

4. Examples

The following is an example in support of Theorem 3.1.

Example 4.1. *Let $X = \mathbb{R}^+$ and let $d : X \times X \to \mathbb{R}^+$ defined by*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in [0,1], \\ 5 + \frac{1}{x+y} & \text{if } x, y \in (1,\infty), \\ \frac{66}{25} & \text{otherwise.} \end{cases}$$

Then clearly (X,d) is a complete b-metric space with coefficient $s = \frac{25}{24}$. Here we observe that when $x = \frac{10}{9}, z = 1 \in [1,\infty)$ and $y \in (0,1)$, we have $d(x,z) = 5 + \frac{1}{x+z} = \frac{104}{19}$ and $d(x,y) + d(y,z) = \frac{66}{25} + \frac{66}{25} = \frac{132}{25}$ so that $d(x,z) \neq d(x,y) + d(y,z)$. Hence d is a b-metric with $s = \frac{25}{24}$ but not a metric.
We define $f : X \to X$ by $f(x) = \begin{cases} 2 & \text{if } x \in [0,1) \\ \frac{1}{x} & \text{if } x \in [1,\infty). \end{cases}$ and $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{99}{100}s - t$, *$t, s \in \mathbb{R}^+$.*
Then ζ is a simulation function. Without loss of generality, we assume that $y \leq x$.
Case (i). $x, y \in [0,1)$.
Since $\frac{1}{2s}d(x,fx) = \frac{12}{25}(\frac{66}{25}) \leq 4 = d(x,y)$, we have $d(fx,fy) = 0$ and clearly the inequality (2.1.1) holds in this case.
Case (ii). $x, y \in (1,\infty)$.
Since $\frac{1}{2s}d(x,fx) = \frac{12}{25}(\frac{66}{25}) \leq 5 + \frac{1}{(x+y)} = d(x,y)$, we have $d(fx,fy) = 4, d(x,y) = 5 + \frac{1}{(x+y)}, d(x,fx) = \frac{66}{25}, d(y,fy) = \frac{66}{25}, d(x,fy) = \frac{66}{25}, d(y,fx) = \frac{66}{25}$ and

$$\begin{aligned} M_1(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy, \tfrac{1}{2s}[d(x,fy) + d(y,fx)])\} \\ &= \max\{5 + \tfrac{1}{(x+y)}, \tfrac{66}{25}, \tfrac{66}{25}, \tfrac{12}{25}[\tfrac{66}{25} + \tfrac{66}{25}]\} = 5 + \tfrac{1}{(x+y)}. \end{aligned}$$

We consider

$$\zeta(s^4 d(fx,fy), M_1(x,y)) = \frac{99}{100}M_1(x,y) - s^4 d(fx,fy) = \frac{99}{100}(5 + \frac{1}{(x+y)}) - (\frac{25}{24})^4(4) \geq 0.$$

Case (iii). $x \in (1,\infty), y \in [0,1)$.
Since $\frac{1}{2s}d(x,fx) = \frac{12}{25}(\frac{66}{25}) \leq \frac{66}{25} = d(x,y)$.
$d(fx,fy) = \frac{66}{25}, d(x,y) = \frac{66}{25}, d(x,fx) = \frac{66}{25}, d(y,fy) = \frac{66}{25}, d(x,fy) = 5 + \frac{1}{(x+y)}, d(y,fx) = 4$ *and*

$$\begin{aligned} M_1(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy), \tfrac{1}{2s}[d(x,fy) + d(y,fx)]\} \\ &= \max\{\tfrac{66}{25}, \tfrac{66}{25}, \tfrac{66}{25}, \tfrac{12}{25}[5 + \tfrac{1}{(x+y)} + 4]\} = \tfrac{12}{25}[9 + \tfrac{1}{(x+y)}]. \end{aligned}$$

We consider

$$\begin{aligned} \zeta(s^4 d(fx,fy), M_1(x,y)) &= \tfrac{99}{100}M_1(x,y) - s^4 d(fx,fy) \\ &= \tfrac{99}{100}(\tfrac{12}{25}[9 + \tfrac{1}{(x+y)}]) - (\tfrac{25}{24})^4(\tfrac{66}{25}) \geq 0. \end{aligned}$$

Case (iv). $x = 1, y \in [0,1)$.
Since $\frac{1}{2s}d(x,fx) = 0 < 4 = d(x,y)$.
$d(fx,fy) = \frac{66}{25}, d(x,y) = 4, d(x,fx) = 0, d(y,fy) = \frac{66}{25}, d(x,fy) = \frac{66}{25}, d(y,fx) = 4$ *and*

$$\begin{aligned} M_1(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy), \tfrac{1}{2s}[d(x,fy) + d(y,fx)]\} \\ &= \max\{4, 0, \tfrac{66}{25}, \tfrac{12}{25}[\tfrac{66}{25} + 4]\} = 4. \end{aligned}$$

We consider

$$\begin{aligned} \zeta(s^4 d(fx,fy), M_1(x,y)) &= \tfrac{99}{100}M_1(x,y) - s^4 d(fx,fy) \\ &= \tfrac{99}{100}(4) - (\tfrac{25}{24})^4(\tfrac{66}{25}) \geq 0. \end{aligned}$$

From all the above cases, f is a Suzuki $\mathcal{Z}$-contraction type (I) map. Therefore f satisfies all the hypotheses of Theorem 3.1 and 1 is the unique fixed point of f.

Remark 4.2. Theorem 3.1 and Example 4.1 extend and generalize Theorem 1.14 to b-metric spaces. Also Theorem 3.1 extends Theorem 1.15 to b-metric spaces.

The following is an example in support of Theorem 3.2.

Example 4.3. *Let $X = [0, \infty)$ and let $d : X \times X \to \mathbb{R}^+$ defined by*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in [0, 1], \\ 5 + \frac{1}{x+y} & \text{if } x, y \in (1, \infty), \\ \frac{27}{10} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{489}{480}$. Here we observe that when $x = \frac{11}{10}, z = \frac{12}{10} \in (1, \infty)$ and $y \in (0, 1]$, we have

$$d(x, z) = 5 + \frac{1}{x + z} = \frac{125}{23} \text{ and } d(x, y) + d(y, z) = \frac{27}{10} + \frac{27}{10} = \frac{54}{10}$$

so that $d(x, z) \neq d(x, y) + d(y, z)$. Hence d is a b-metric with $s = \frac{489}{480}$ but not a metric.
We define $f : X \to X$ by $f(x) = \begin{cases} 2 & \text{if } x \in [0, 1) \\ \frac{2}{x^2+1} & \text{if } x \in [1, \infty). \end{cases}$
We define $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ by $\zeta(s, t) = \frac{99}{100}t - s, t \geq 0, s \geq 0$. Then ζ is a simulation function.
Without loss of generality, we assume that $x \geq y$.
Case (i). *$x, y \in [0, 1)$.*
$\frac{1}{2s}d(x, fx) = (\frac{480}{978})(\frac{27}{10}) \leq 4 = d(x, y)$. Since $d(fx, fy) = 0$ the inequality (2.4.1) holds in this case.
Case (ii). *$x, y \in (1, \infty)$.*
We have $\frac{1}{2s}d(x, fx) = (\frac{480}{978})(\frac{27}{10}) \leq 5 + \frac{1}{x+y} = d(x, y)$,

$$d(fx, fy) = 4, d(x, y) = 5 + \frac{1}{x + y}, d(x, fx) = \frac{27}{10}, d(y, fy) = \frac{27}{10}, d(y, fx) = \frac{27}{10}$$

and

$$\begin{aligned} M_2(x, y) &= \max\{d(x, y), \tfrac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}, \tfrac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\} \\ &= \max\{5 + \tfrac{1}{x+y}, \tfrac{\frac{27}{10}[1+\frac{27}{10}]}{6+\frac{1}{x+y}}, \tfrac{\frac{27}{10}[1+\frac{27}{10}]}{(\frac{489}{480})^2(6+\frac{1}{x+y})}\} \\ &= 5 + \tfrac{1}{x+y}. \end{aligned}$$

We consider

$$\begin{aligned} \zeta(s^4 d(fx, fy), M_2(x, y)) &= \tfrac{99}{100}M_2(x, y) - s^4 d(fx, fy) \\ &= \tfrac{99}{100}(5 + \tfrac{1}{x+y}) - (\tfrac{489}{480})^4(4) \geq 0. \end{aligned}$$

Case (iii). *$x \in (1, \infty), y \in [0, 1)$.*
We have $\frac{1}{2s}d(x, fx) = (\frac{480}{978})(\frac{27}{10}) \leq \frac{27}{10} = d(x, y)$,

$$d(fx, fy) = \frac{27}{10}, d(x, y) = \frac{27}{10}, d(x, fx) = \frac{27}{10}, d(y, fy) = \frac{27}{10}, d(y, fx) = 4$$

and

$$\begin{aligned} M_2(x, y) &= \max\{d(x, y), \tfrac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}, \tfrac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\} \\ &= \max\{\tfrac{27}{10}, \tfrac{\frac{27}{10}[1+\frac{27}{10}]}{1+\frac{27}{10}}, \tfrac{4[1+\frac{27}{10}]}{(\frac{489}{480})^2(1+\frac{27}{10})}\} \\ &= \tfrac{4}{(\frac{489}{480})^2}. \end{aligned}$$

We consider

$$\zeta(s^4 d(fx,fy), M_2(x,y)) = \frac{99}{100}M_2(x,y) - s^4 d(fx,fy) = \frac{99}{100}(\frac{4}{\frac{489}{480}^2}) - (\frac{489}{480})^4(\frac{27}{10}) \geq 0.$$

Case (iv). $x = 1, y \in [0,1)$.
We have $\frac{1}{2s}d(x,fx) = 0 \leq 4 = d(x,y)$,

$$d(fx,fy) = \frac{27}{10}, d(x,y) = 4, d(x,fx) = 0, d(y,fy) = \frac{27}{10}, d(y,fx) = 4,$$

and

$$\begin{aligned} M_2(x,y) &= \max\{d(x,y), \tfrac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}, \tfrac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\} \\ &= \max\{4, \tfrac{27}{50}, \tfrac{4}{(\frac{489}{480})^2(5)}\} = 4. \end{aligned}$$

We consider

$$\zeta(s^4 d(fx,fy), M_2(x,y)) = \frac{99}{100}M_2(x,y) - s^4 d(fx,fy) = \frac{99}{100}(4) - (\frac{489}{480})^4(\frac{27}{10}) \geq 0.$$

From all the above cases, f is a Suzuki $\mathcal{Z}$-contraction type (II) map. Therefore f satisfies all the hypotheses of Theorem 3.2 and 1 *is the unique fixed point of f.*

Here we observe from Case (iii) that, if we omit the term $\frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}$ from the inequality (2.4.1), then the inequality (2.4.1) fails to hold so that Theorem 3.2 is not possible to apply.

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