Universal Journal of Mathematics and Applications, 7(2) (2024) 68-75
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1393569

Some Identities and Generating Functions for Bidimensional Balancing and Cobalancing Sequences

José Chimpanzo[1*], Paula Catarino[2] and M. Victoria Otero-Espinar[3]

[1]*Department of Exact Sciences, Higher Polytechnic Institute of Soyo, Zaire, Angola*
[2]*Department of Mathematics, University of Trás-os-Montes e Alto Douro, Vila Real, Portugal*
[3]*Department of Statistics, Mathematical Analysis and Optimization, University of Santiago de Compostela and Galician Center for Mathematical Research and Technology (CITMAga), 15782 Santiago de Compostela, Spain*
[*]*Corresponding author*

Article Info

Keywords: *Bidimensional balancing, Bidimensional cobalancing, Bidimensional Lucas-balancing, Bidimensional Lucas-cobalancing, Generating functions, Some identities*
2010 AMS: *11B39, 11B83*
Received: *20 November 2023*
Accepted: *24 March 2024*
Available online: *30 March 2024*

Abstract

This article studies the bidimensional versions of four number sequences: balancing, Lucas-balancing, Lucas-cobalancing and cobalancing. Some of the identities and generating functions were presented.

1. Introduction

The study of numerical sequences has been the subject of a lot of research in recent years. Much emphasis has been placed on the Fibonacci, balancing, Lucas-balancing, cobalancing and Lucas-cobalancing sequences, among others. In [1], some identities were deduced for the Pell, Pell-Lucas and balancing numbers and some relationships between them. Some formulas for sums, divisibility properties, perfect squares and Pythagorean triples involving these numbers were also studied. In [2], some combinatorial expressions of balancing and Lucas-balancing numbers were established and some of their properties were investigated. Finally, in [3], a brief study was made of the limits for reciprocal sums involving terms from balancing and Lucas-balancing sequences. Many properties and identities of sequences are established using the so-called Binet formula for these sequences. This formula is known to be an explicit formula used to determine any term of a specific numerical sequence without having to resort to its previous terms. In 1985, the mathematician Levesque, in [4], deduced this formula for a linear recurrence of m-th order ($m \in \mathbb{N}$). Binet's formula for the sequence of balancing numbers B_n, Lucas-balancing numbers C_n, Lucas-cobalancing numbers c_n and cobalancing numbers b_n, is given, respectively, by:

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

$$C_n = \frac{r_1^n + r_2^n}{2},$$

$$c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2},$$

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2},$$

***Email addresses and ORCID numbers*: jalchimpanzo@gmail.com, 0009-0002-8658-5929 (J. Chimpanzo), pcatarin@utad.pt, 0000-0001-6917-5093 (P. Catarino), mvictoria.otero@usc.es, 0000-0002-0201-0523 (M. V. Otero-Espinar)**
Cite as: J. Chimpanzo, P. Catarino, M. V. Otero-Espinar, *Some identities and generating functions for bidimensional balancing and cobalancing sequences*, Univers. J. Math. Appl., 7(2) (2024), 68-75.

with $r_1=\alpha_1^2=3+2\sqrt{2}$ and $r_2=\alpha_2^2=3-2\sqrt{2}$, where $\alpha_1=\sqrt{3+2\sqrt{2}}$ and $\alpha_2=\sqrt{3-2\sqrt{2}}$.
In this article, our objective is to study the bidimensional versions of these four number sequences, presenting some identities and generating functions.
There are already many works on the bidimensional, tridimensional and n-dimensional versions of some number sequences. For example, in [5], we can find a brief approach to the Leonardo sequence, as well as a discussion related to the bidimensional recurrence relations of this type of number from its unidimensional model. In [6], we have a study of bidimensional and tridimensional recursive relations defined from the unidimensional recursive model of the Narayana sequence. In [7], we can find a study of bidimensional and tridimensional identities for the Fibonacci numbers in complex form. In [8], the authors introduced the gaussian Fibonacci numbers and the bidimensional recurrence relations of this sequence. In [9], we have a brief study on the bidimensional extensions of the balancing and Lucas-balancing numerical sequences and, in particular, in this study, the authors define, respectively, the bidimensional recurrence relations of these two numerical sequences, as follows:
• The bidimensional numerical sequence balancing $B_{(n,m)}$ satisfies the following recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} B_{(n+1,m)}=6B_{(n,m)}-B_{(n-1,m)}, \\ B_{(n,m+1)}=6B_{(n,m)}-B_{(n,m-1)}, \end{cases} \tag{1.1}$$

with the initial conditions $B_{(0,\ 0)}=0$, $B_{(1,\ 0)}=1$, $B_{(0,\ 1)}=i$, $B_{(1,\ 1)}=1+i$, where $i^2=-1$.
• The bidimensional numerical sequence Lucas-balancing $C_{(n,m)}$ satisfies the following recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} C_{(n+1,m)}=6C_{(n,m)}-C_{(n-1,m)}, \\ C_{(n,m+1)}=6C_{(n,m)}-C_{(n,m-1)}, \end{cases}$$

with the initial conditions $C_{(0,\ 0)}=1$, $C_{(1,\ 0)}=3$, $C_{(0,\ 1)}=1+i$, $C_{(1,1)}=3+i$, where $i^2=-1$.
Finally, in [10], we have the introduction of a new bidimensional version of the cobalancing and Lucas-cobalancing numbers, where we can find the study of some properties and identities satisfied by these new bidimensional sequences. In this study, the authors define, respectively, the bidimensional recurrence relations of these sequences, as follows:
• The bidimensional numerical sequence Lucas-cobalancing $c_{(m,n)}$ satisfies the following recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} c_{(n+1,m)}=6c_{(n,m)}-c_{(n-1,m)}, \\ c_{(n,m+1)}=6c_{(n,m)}-c_{(n,m-1)}, \end{cases} \tag{1.2}$$

with the initial conditions $c_{(0,0)}=1$, $c_{(1,0)}=7$, $c_{(0,1)}=1+i$, $c_{(1,1)}=7+i$, where $i^2=-1$.
• The bidimensional sequence of the cobalancing numbers $b_{(m,n)}$, satisfies the recurrence relation

$$b_{(n,m)}=\frac{1}{8}c_{(n+1,m)}-\frac{3}{8}c_{(n,m)}-\frac{1}{2},\ \forall n,m\in\mathbb{N}_0, \tag{1.3}$$

with the initial conditions $b_{(0,0)}=0$, $b_{(1,0)}=2$, $b_{(0,1)}=-\frac{1}{4}i$, $b_{(1,1)}=2+\frac{1}{4}i$, where $i^2=-1$.

It was, therefore, these and other works that served as motivation for our study, which consists of continuing to approach the bidimensional version of balancing and cobalancing numbers, investigating topics related to these sequences.
The following results are included in [9] and they are one of the main tools of this work, being used in the various proofs of this paper. To clarify this article, we have decided to include these results in what follows. Thus:

Lemma 1.1 (Lemma 3.2 in [9]). *The following properties are true for every non-negative integers n and m:*

1. $B_{(n,0)}=B_n$;
2. $B_{(0,m)}=B_m i$;
3. $B_{(n,1)}=B_n+(B_n-B_{n-1})i$;
4. $B_{(1,m)}=(B_m-B_{m-1})+B_m i$.

Lemma 1.2 (Theorem 3.3 in [9]). *For non-negative integers n, m, the bidimensional balancing numbers are described as follows:*

$$B_{(n,m)}=B_n(B_m-B_{m-1})+(B_n-B_{n-1})B_m i. \tag{1.4}$$

Lemma 1.3 (Theorem 5.3 in [9]). *For non-negative integers n, m, the bidimensional Lucas-balancing numbers are described as follows:*

$$C_{(n,m)}=C_n(B_m-B_{m-1})+(B_n-B_{n-1})B_m i. \tag{1.5}$$

This article is structured as follows: in the next section, the Catalan and Cassini identities of the bidimensional versions of the balancing, Lucas-balancing, Lucas-cobalancing and cobalancing numerical sequences will be presented. Section 3 is dedicated to the study of the generating functions of these four sequences. Finally, Section 4 presents a brief conclusion on this subject.

2. Some Identities

In this section we study some identities involving the bidimensional sequences already mentioned in the previous section. These identities are the Catalan identity (sometimes called the Simson identity) and the Cassini identity. It should be noted that Cassini's identity is a particular case of Catalan's identity. Each subsection is dedicated to one of the bidimensional versions of the sequences mentioned above.

2.1. For the sequence $\{B_{(n,m)}\}_{n,m\geq 0}$

In this subsection we present the results for the bidimensional balancing sequence relating to Catalan's identity and Cassini's identity.

Proposition 2.1 (Catalan's identity). *For any nonzero positive integer r and for all positive integers n,m, the following identity concerning the numerical sequence balancing in the bidimensional version is valid:*

$$B_{(n-r,m)}B_{(n+r,m)} - \left(B_{(n,m)}\right)^2 = -B_r^2X_m^2 - d_rB_m^2 + e_rX_mB_mi, \tag{2.1}$$

where $d_r = X_{n+r}X_{n-r} - X_n^2$ e $e_r = X_{n+r}B_{n-r} + X_{n-r}B_{n+r} - 2B_nX_n$.

Proof. Applying Lemma 1.2 (Theorem 3.3 in [9]) to the expressions $B_{(n-r,m)}$ and $B_{(n+r,m)}$, calling $X_i = B_i - B_{i-1}$, $i \geq 0$, multiplying both terms and using expression (2.1) from Proposition 2.1 in [**?**], we get

$$B_{(n-r,m)}B_{(n+r,m)} = \left(B_n^2 - B_r^2\right)X_m^2 - X_{n+r}X_{n-r}B_m^2 + (X_{n+r}B_{n-r} + X_{n-r}B_{n+r})X_mB_mi. \tag{2.2}$$

Once again, taking into account Lemma 1.2 (Theorem 3.3 in [9] and using the perfect square trinomial, we obtain

$$\left(B_{(n,m)}\right)^2 = B_n^2X_m^2 - X_n^2B_m^2 + 2B_nX_nX_mB_mi. \tag{2.3}$$

Subtracting the identities (2.2) and (2.3), the result follows. □

As we have already mentioned, Cassini's identity is a particular case of Catalan's identity, making $r = 1$. So we get:

Corollary 2.2 (Cassini's identity). *When $r = 1$, for the bidimensional balancing sequence, we have Cassini's identity which consists of the following identity:*

$$B_{(n-1,m)}B_{(n+1,m)} - \left(B_{(n,m)}\right)^2 = -X_m^2 - d_1B_m^2 + e_1X_mB_mi, \tag{2.4}$$

where $d_1 = X_{n+1}X_{n-1} - X_n^2$ and $e_1 = X_{n+1}B_{n-1} + X_{n-1}B_{n+1} - 2B_nX_n$.

2.2. For the sequence $\{C_{(n,m)}\}_{n,m\geq 0}$

In this subsection, we present results similar to those in the previous subsection for the bidimensional Lucas-balancing sequence related to the Catalan's identity and Cassini's identity.

Proposition 2.3 (Catalan's identity). *For any nonzero positive integer r and for all positive integers n,m, the following identity related to the bidimensional Lucas-balancing number is true:*

$$C_{(n-r,m)}C_{(n+r,m)} - \left(C_{(n,m)}\right)^2 = \left(C_r^2 - 1\right)X_m^2 - d_rB_m^2 + \tilde{e}_rX_mB_mi,$$

where d_r has already been previously defined and $\tilde{e}_r = C_{n-r}X_{n+r} + C_{n+r}X_{n-r} - 2C_nX_n$.

Proof. Using Lemma 1.2 (Theorem 3.3 in [9]) in the expressions $B_{(n-r,m)}$ and $B_{(n+r,m)}$, considering $X_i = B_i - B_{i-1}$, $i \geq 0$, multiplying both terms and taking into account expression (2.2) of Proposition 2.1 in [**?**], we obtain

$$C_{(n-r,m)}C_{(n+r,m)} = \left(C_n^2 + C_r^2 - 1\right)X_m^2 - X_{n-r}X_{n+r}B_m^2 + X_mB_mi. \tag{2.5}$$

Considering Lemma 1.3 (Theorem 5.3 in [9]) and using the perfect square trinomial, we get

$$\left(C_{(n,m)}\right)^2 = C_n^2X_m^2 - X_n^2B_m^2 + 2C_nX_nX_mB_mi. \tag{2.6}$$

The result follows, subtracting identities (2.5) and (2.6). □

The Cassini's identity for this sequence of numbers is defined by:

Corollary 2.4 (Cassini's identity). *When $r = 1$ we have the Cassini's identity for the bidimensional Lucas-balancing sequence:*

$$C_{(n-1,m)}C_{(n+1,m)} - \left(C_{(n,m)}\right)^2 = 8X_m^2 - d_1B_m^2 + \tilde{e}_1X_mB_mi,$$

where d_1 has already been defined and $\tilde{e}_1 = C_{n-1}X_{n+1} + C_{n+1}X_{n-1} - 2C_nX_n$.

2.3. For the sequence $\{c_{(n,m)}\}_{n,m\geq 0}$

In this subsection we present results similar to those in the previous subsections on the bidimensional Lucas-cobalancing sequence involving Catalan's and Cassini's identities. We will omit the respective proofs, since they are similar to the previous ones.

Proposition 2.5 (Catalan's identity). *For any nonzero positive integer r and for all positive integers n,m, the following identity related to bidimensional Lucas-cobalancing numbers is valid:*

$$c_{(n-r,m)}c_{(n+r,m)}-\left(c_{(n,m)}\right)^2=f_rX_m^2-d_rB_m^2+g_rX_mB_mi,$$

where $f_r=c_{n-r+1}c_{n+r+1}-c_{n+1}^2$, d_r already known and $g_r=c_{n-r+1}X_{n+r}+c_{n+r+1}X_{n-r}-2c_{n+1}X_n$.

Cassini's identity for this numerical sequence is given by:

Corollary 2.6 (Cassini's identity). *When $r=1$ we have the identity for the bidimensional Lucas-cobalancing sequence:*

$$c_{(n-1,m)}c_{(n+1,m)}-\left(c_{(n,m)}\right)^2=f_1X_m^2-d_1B_m^2+g_1X_mB_mi,$$

where $f_1=c_nc_{n+2}-c_{n+1}^2$, d_1 already presented and $g_1=c_nX_{n+1}-2c_{n+1}X_n+c_{n+2}X_{n-1}$.

2.4. For the sequence $\{b_{(n,m)}\}_{n,m\geq 0}$

In this subsection we present the results for the bidimensional cobalancing numerical sequence involving Catalan's and Cassini's identities.

Proposition 2.7 (Catalan's identity). *For any nonzero positive integer r and for all positive integers n,m, the following identity for bidimensional cobalancing numbers is true:*

$$b_{(n-r,m)}b_{(n+r,m)}-\left(b_{(n,m)}\right)^2=\frac{1}{64}s_rX_m^2-\frac{1}{8}t_rX_m-\frac{1}{64}u_rB_m^2+\frac{1}{32}v_rX_mB_mi-\frac{1}{8}z_rB_mi,$$

where

$$s_r=(c_{n-r+2}-3c_{n-r+1})(c_{n+r+2}-3c_{n+r+1})-c_{n+2}+3c_{n+1};$$

$$t_r=\frac{c_{n-r+2}c_{n+r+2}-3(c_{n-r+1}+c_{n+r+1})}{2}-c_{n+2}+3c_{n+1};$$

$$u_r=(X_{n-r+2}-3X_{n-r})+(X_{n+r+2}-3X_{n+r})-(X_{n+2}-3X_n)^2;$$

$$v_r=\frac{(c_{n-r+2}-3c_{n-r+1})(X_{n+r+2}-3X_{n+r})+(c_{n+r+2}-3c_{n+r+1})(X_{n-r+2}-3X_{n-r})}{2}-(c_{n+2}-3c_{n+1})(X_{n+2}-3X_n)$$

and

$$z_r=\frac{X_{n-r+2}+X_{n+r+2}-3(X_{n-r}+X_{n+r})}{2}-X_{n+2}+3X_n.$$

Proof. Using the identity (1.3) for the expressions $b_{(n-r,m)}$ and $b_{(n+r,m)}$, making $X_i=B_i-B_{i-1}$, $i\geq 0$ and multiplying both terms, we get

$$b_{(n-r,m)}b_{(n+r,m)}=\frac{1}{64}h_rX_m^2-\frac{1}{16}j_rX_m-\frac{1}{64}l_rB_m^2+\frac{1}{64}p_rX_mB_mi-\frac{1}{16}q_rB_mi+\frac{1}{4}, \tag{2.7}$$

where

$$h_r=(c_{n-r+2}-3c_{n-r+1})(c_{n+r+2}-3c_{n+r+1});$$

$$j_r=c_{n-r+2}c_{n+r+2}-3(c_{n-r+1}+c_{n+r+1});$$

$$l_r=(X_{n-r+2}-3X_{n-r})+(X_{n+r+2}-3X_{n+r});$$

$$p_r=(c_{n-r+2}-3c_{n-r+1})(X_{n+r+2}-3X_{n+r})+(c_{n+r+2}-3c_{n+r+1})(X_{n-r+2}-3X_{n-r})$$

and

$$q_r=X_{n-r+2}+X_{n+r+2}-3(X_{n-r}+X_{n+r}).$$

Once again, taking into account expression (1.3) and applying the perfect square trinomial, we obtain

$$\left(b_{(n,m)}\right)^2=\frac{1}{64}\tilde{h}_rX_m^2-\frac{1}{8}\tilde{j}_rX_m-\frac{1}{64}\tilde{l}_rB_m^2+\frac{1}{32}\tilde{p}_rX_mB_mi-\frac{1}{8}\tilde{q}_rB_mi+\frac{1}{4}, \tag{2.8}$$

where $\tilde{h}_r=(c_{n+2}-3c_{n+1})^2$, $\tilde{j}_r=c_{n+2}-3c_{n+1}$, $\tilde{l}_r=(X_{n+2}-3X_n)^2$, $\tilde{p}_r=(c_{n+2}-3c_{n+1})(X_{n+2}-3X_n)$ and $\tilde{q}_r=X_{n+2}-3X_n$.
Subtracting (2.7) from (2.8), the result follows. □

The Cassini's identity for this number sequence is given by:

Corollary 2.8 (Cassini's identity)**.** *When $r=1$ we have the identity for the bidimensional cobalancing sequence:*

$$b_{(n-1,m)}b_{(n+1,m)}-\left(b_{(n,m)}\right)^2=\frac{1}{64}s_1X_m^2-\frac{1}{8}t_1X_m-\frac{1}{64}u_1B_m^2+\frac{1}{32}v_1X_mB_mi-\frac{1}{8}z_1B_mi,$$

where

$$s_1=(c_{n+1}-3c_n)c_{n+3}-(c_{n+1}-3c_n+1)c_{n+2}+3c_{n+1};$$

$$t_1=\frac{c_{n+2}+(c_{n+3}+6)c_{n+1}-3c_n}{2};$$

$$u_1=X_{n+3}-2X_{n+1}-3X_{n-1}-(X_{n+2}-3X_n)^2;$$

$$v_1=\frac{(c_{n+1}-3c_n)X_{n+3}-2(c_{n+2}-3c_{n+1})(X_{n+2}-3X_n)-(c_{n+3}-3(c_{n+2}-c_{n+1}-3))X_{n+1}}{2}$$

and

$$z_1=\frac{X_{n+3}-2(X_{n+2}+X_{n+1})+3(2X_n-X_{n-1})}{2}.$$

3. Generating Functions

In this section we study the generating functions of the four sequences mentioned above, in their bidimensional versions. We use the definition of the ordinary generating function of any sequence $\{a_n\}_n$ which is given by $G_{(a_n;x)}=\sum_{n=0}^{\infty}a_nx^n$, where x is any positive integer and n is any natural number, except for the case of bidimensional cobalancing numbers. So let's start with the case of balancing numbers in their bidimensional version.

Proposition 3.1. *For any positive integer x and for all natural numbers m,n, the following generating functions for the balancing numbers in the bidimensional version are valid:*

1. $G_{(B_{(n,\mathbf{m})};x)}=\frac{B_mi+((B_m-B_{m-1})-5B_mi)x}{1-6x+x^2}$, *where m is fixed;*
2. $G_{(B_{(\mathbf{n},m)};x)}=\frac{B_n-(5B_n-(B_n-B_{n-1})i)x}{1-6x+x^2}$, *where n is fixed.*

Proof. 1 The proof is done by fixing m. By the definition of the ordinary generating function, we obtain

$$G_{(B_{(n,\mathbf{m})};x)}=\sum_{n=0}^{\infty}B_{(n,m)}x^n=B_{(0,m)}x^0+B_{(1,m)}x+B_{(2,m)}x^2+\sum_{n=3}^{\infty}B_{(n,m)}x^n.$$

Thus, by Lemma 1.1, items 2 and 4 (Lemma 3.2, items 2 and 4 in [9]) and by (1.1), we get

$$G_{(B_{(n,\mathbf{m})};x)}=B_mi+((B_m-B_{m-1})+B_mi)x+\left(6B_{(1,m)}-B_{(0,m)}\right)x^2+x^3\sum_{n=3}^{\infty}B_{(n,m)}x^{n-3}.$$

Once again, by Lemma 1.1, items 4 and 2 (Lemma 3.2, items 4 and 2 in [9] and the fact that $\sum_{i=s}^{t}x_i=\sum_{i=s+p}^{t+p}x_{i-p}$, we obtain

$$G_{(B_{(n,\mathbf{m})};x)}=B_mi+(B_m-B_{m-1})x+B_mix+6((B_m-B_{m-1})+B_mi)x^2-B_mix^2+x^3\sum_{n=2}^{\infty}B_{(n+1,m)}x^{n-2}.$$

Again, by (1.1) and using one of the properties of summation, we get

$$\begin{aligned}G_{(B_{(n,\mathbf{m})};x)}&=B_mi+(B_m-B_{m-1})x+B_mix+6(B_m-B_{m-1})x^2+5B_mix^2+x^3\sum_{n=2}^{\infty}\left(6B_{(n,m)}-B_{(n-1,m)}\right)x^{n-2}\\&=B_mi+(B_m-B_{m-1})x+B_mix+6(B_m-B_{m-1})x^2+5B_mix^2+6x^3\sum_{n=2}^{\infty}B_{(n,m)}x^{n-2}-x^3\sum_{n=2}^{\infty}B_{(n-1,m)}x^{n-2}\\&=B_mi+(B_m-B_{m-1})x+B_mix+6(B_m-B_{m-1})x^2+5B_mix^2+6x^3x^{-2}\sum_{n=2}^{\infty}B_{(n,m)}x^n-x^3x^{-1}\sum_{n=2}^{\infty}B_{(n-1,m)}x^{n-1}\\&=B_mi+(B_m-B_{m-1})x+B_mix+6(B_m-B_{m-1})x^2+5B_mix^2+6x\left(\sum_{n=0}^{\infty}B_{(n,m)}x^n-B_{(0,m)}-B_{(1,m)}x\right)-x^2\sum_{n=1}^{\infty}B_{(n,m)}x^n.\end{aligned}$$

And once again, by items 2 and 4 of Lemma 1.1 (items 2 and 4 of Lemma 3.2 in [9]), we obtain

$$\begin{aligned}G_{(B_{(n,\mathbf{m})};x)}&=B_mi+(B_m-B_{m-1})x+B_mix+6(B_m-B_{m-1})x^2+5B_mix^2+6xG_{(B_{(n,\mathbf{m})};x)}-6xB_mi-6((B_m-B_{m-1})+B_mi)x^2\\&\quad-x^2\left(\sum_{n=0}^{\infty}B_{(n,m)}x^n-B_{(0,m)}\right)\\&=B_mi+(B_m-B_{m-1})x+B_mix+6(B_m-B_{m-1})x^2+5B_mix^2+6xG_{(B_{(n,\mathbf{m})};x)}-6xB_mi-6(B_m-B_{m-1})x^2-6B_mix^2\\&\quad-x^2G_{(B_{(n,\mathbf{m})};x)}+B_mix^2.\end{aligned}$$

Then come:

$$\left(1-6x+x^2\right)G_{(B_{(n,\mathbf{m})};x)}=B_m i+(B_m-B_{m-1})x-5B_m ix.$$

Therefore,

$$G_{(B_{(n,\mathbf{m})};x)}=\frac{B_m i+((B_m-B_{m-1})-5B_m i)x}{1-6x+x^2}.$$

2 The proof is done by fixing n. Using the definition of the generating function, we obtain

$$G_{(B_{(\mathbf{n},m)};x)}=\sum_{m=0}^{\infty}B_{(n,m)}x^m=B_{(n,0)}x^0+B_{(n,1)}x+B_{(n,2)}x^2+\sum_{m=3}^{\infty}B_{(n,m)}x^m.$$

Thus, using items 1 and 3 of Lemma 1.1 (items 1 and 3 of Lemma 3.2 in [9]) and by (1.1), we get

$$G_{(B_{(\mathbf{n},m)};x)}=B_n+(B_n+(B_n-B_{n-1})i)x+\left(6B_{(n,1)}-B_{(n,0)}\right)x^2+\sum_{m=3}^{\infty}B_{(n,m)}x^m.$$

Once again, by items 3 and 1 of Lemma 1 (items 3 and 1 of Lemma 3.2 in [9]), by one of the properties of summations and also by (1.1), we get

$$\begin{aligned}G_{(B_{(\mathbf{n},m)};x)}&=B_n+B_nx+(B_n-B_{n-1})ix+(6(B_n+(B_n-B_{n-1})i)-B_n)x^2+x^3\sum_{m=3}^{\infty}B_{(n,m)}x^{m-3}\\&=B_n+B_nx+(B_n-B_{n-1})ix+6B_nx^2+6(B_n-B_{n-1})ix^2-B_nx^2+x^3\sum_{m=2}^{\infty}B_{(n,m+1)}x^{m-2}\\&=B_n+B_nx+(B_n-B_{n-1})ix+5B_nx^2+6(B_n-B_{n-1})ix^2+x^3\sum_{m=2}^{\infty}\left(6B_{(n,m)}-B_{(n,m-1)}\right)x^{m-2}.\end{aligned}$$

Hence,

$$\begin{aligned}G_{(B_{(\mathbf{n},m)};x)}&=B_n+B_nx+(B_n-B_{n-1})ix+5B_nx^2+6(B_n-B_{n-1})ix^2+x^3\left(6\sum_{m=2}^{\infty}B_{(n,m)}x^{m-2}-\sum_{m=2}^{\infty}B_{(n,m-1)}x^{m-2}\right)\\&=B_n+B_nx+(B_n-B_{n-1})ix+5B_nx^2+6(B_n-B_{n-1})ix^2+6x^3x^{-2}\sum_{m=2}^{\infty}B_{(n,m)}x^m-x^3\sum_{m=2}^{\infty}B_{(n,m-1)}x^{m-2}\\&=B_n+B_nx+(B_n-B_{n-1})ix+5B_nx^2+6(B_n-B_{n-1})ix^2+6x\sum_{m=2}^{\infty}B_{(n,m)}x^m-x^3x^{-1}\sum_{m=2}^{\infty}B_{(n,m-1)}x^{m-2}\\&=B_n+B_nx+(B_n-B_{n-1})ix+5B_nx^2+6(B_n-B_{n-1})ix^2+6x\left(\sum_{m=0}^{\infty}B_{(n,m)}x^m-B_{(n,0)}-B_{(n,1)}x\right)-x^2\sum_{m=2}^{\infty}B_{(n,m-1)}x^{m-1}.\end{aligned}$$

Once again, using items 1 and 3 of Lemma 1.1 (items 1 and 3 of Lemma 3.2 in [9]) and by one of the properties of summations, we get

$$\begin{aligned}G_{(B_{(\mathbf{n},m)};x)}&=B_n+B_nx+(B_n-B_{n-1})ix+5B_nx^2+6(B_n-B_{n-1})ix^2+6x\left(\sum_{m=0}^{\infty}B_{(n,m)}x^m-B_n-(B_n+(B_n-B_{n-1})i)x\right)\\&\quad-x^2\sum_{m=1}^{\infty}B_{(n,m)}x^m.\end{aligned}$$

So,

$$\begin{aligned}G_{(B_{(\mathbf{n},m)};x)}&=B_n+B_nx+(B_n-B_{n-1})ix+5B_nx^2+6(B_n-B_{n-1})ix^2+6xG_{(B_{(\mathbf{n},m)};x)}-6xB_n-6(B_n+(B_n-B_{n-1})i)x^2\\&\quad-x^2\left(\sum_{m=0}^{\infty}B_{(n,m)}x^m-B_{(n,0)}\right).\end{aligned}$$

And once again, by items Lemma 1.1, item 1 (Lemma 3.2, item 1 in [9]), we get

$$G_{(B_{(\mathbf{n},m)};x)}=B_n-5B_nx+(B_n-B_{n-1})ix+5B_nx^2+6(B_n-B_{n-1})ix^2+6xG_{(B_{(\mathbf{n},m)};x)}-6B_nx^2-6(B_n-B_{n-1})ix^2-x^2G_{(B_{(\mathbf{n},m)};x)}+B_nx^2.$$

Thus,

$$G_{(B_{(\mathbf{n},m)};x)}=\frac{B_n-(5B_n-(B_n-B_{n-1})i)x}{1-6x+x^2}.$$

□

The following results concern the generating functions for the cases of the bidimensional versions of the Lucas-balancing and Lucas-cobalancing numbers.

Since the demonstrations are similar to those in the previous case, we omit their proofs here.

Proposition 3.2. *For any positive integer x and for all natural numbers m,n, the following generating functions related to bidimensional Lucas-balancing numbers are true:*

1. $G_{(C_{(n,\boldsymbol{m})};x)} = \frac{((B_m-B_{m-1})+B_m i)-(3(B_m-B_{m-1})+5B_m i)x}{1-6x+x^2}$, *where m is fixed;*
2. $G_{(C_{(\boldsymbol{n},m)};x)} = \frac{C_n+((B_n-B_{n-1})-5C_n)x}{1-6x+x^2}$, *where n is fixed.*

Proposition 3.3. *For any positive integer x and for all natural numbers m,n, the following generating functions for bidimensional Lucas-cobalancing numbers are valid:*

1. $G_{(c_{(n,\boldsymbol{m})};x)} = \frac{B_{(1,m)}-\left(5B_{(1,m)}-6(B_m-B_{m-1})\right)x}{1-6x+x^2}$, *where m is fixed;*
2. $G_{(c_{(\boldsymbol{n},m)};x)} = \frac{C_n+((B_n-B_{n-1})-5C_n)x}{1-6x+x^2}$, *where n is fixed.*

The next result we present concerns the generating functions of cobalancing numbers in the bidimensional version. In this case, let's consider that x is a positive integer greater than 1.

Proposition 3.4. *For any positive integer x and for all natural numbers m,n, the following generating functions related to cobalancing numbers in their bidimensional version are valid:*

1. $G_{(b_{(n,\boldsymbol{m})};x)} = \frac{1}{8x}\left((1-3x)\,G_{(c_{(n,\boldsymbol{m})};x)} - B_{(1,\boldsymbol{m})}\right) - \frac{1}{2(1-x)}$, *where m is fixed;*
2. $G_{(b_{(\boldsymbol{n},m)};x)} = \frac{1}{8}\left(G_{(c_{(\boldsymbol{n}+1,m)};x)} - 3G_{(c_{(\boldsymbol{n},m)};x)} + c_{\boldsymbol{n}+2}\right) - \frac{1}{2(1-x)}$, *where n is fixed.*

Proof. 1 The proof is done by fixing m. By (1.2),

$$G_{(b_{(n,\mathbf{m})};x)} = \sum_{n=0}^{\infty}\left(\frac{1}{8}c_{(n+1,\mathbf{m})} - \frac{3}{8}c_{(n,\mathbf{m})} - \frac{1}{2}\right)x^n.$$

So,

$$G_{(b_{(n,\mathbf{m})};x)} = \frac{1}{8}\sum_{n=0}^{\infty}c_{(n+1,m)}x^n - \frac{3}{8}\sum_{n=0}^{\infty}c_{(n,m)}x^n - \frac{1}{2}\sum_{n=0}^{\infty}x^n.$$

By one of the properties of summations and the fact that $\sum_{k=0}^{\infty}ar^k = \frac{a}{1-r}$ for $|r|<1$, we get

$$G_{(b_{(n,\mathbf{m})};x)} = \frac{1}{8}\sum_{n=1}^{\infty}c_{(n,\mathbf{m})}x^{n-1} - \frac{3}{8}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{2}\left(\frac{1}{1-x}\right).$$

So come: 1 (Lemma 3.2, item 1 in [9]), we get

$$\begin{aligned}G_{(b_{(n,\mathbf{m})};x)} &= \frac{1}{8}x^{-1}\sum_{n=1}^{\infty}c_{(n,\mathbf{m})}x^n - \frac{3}{8}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{2(1-x)}\\ &= \frac{1}{8}x^{-1}\left(\sum_{n=0}^{\infty}c_{(n,\mathbf{m})}x^n - c_{(0,\mathbf{m})}\right) - \frac{3}{8}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{2(1-x)}.\end{aligned}$$

By Lemma 2.2, item (b) in [10], we get

$$G_{(b_{(n,\mathbf{m})};x)} = \frac{1}{8}x^{-1}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{8}x^{-1}B_{(1,\mathbf{m})} - \frac{3}{8}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{2(1-x)}.$$

Therefore,

$$G_{(b_{(n,\mathbf{m})};x)} = \frac{1}{8x}\left((1-3x)\,G_{(c_{(n,\mathbf{m})};x)} - B_{(1,\mathbf{m})}\right) - \frac{1}{2(1-x)}.$$

2 The proof is done by fixing n. By (1.2),

$$G_{(b_{(\mathbf{n},m)};x)} = \sum_{m=0}^{\infty}\left(\frac{1}{8}c_{(\mathbf{n}+1,m)} - \frac{3}{8}c_{(\mathbf{n},m)} - \frac{1}{2}\right)x^m.$$

Hence,

$$G_{(b_{(\mathbf{n},m)};x)} = \frac{1}{8}\sum_{m=0}^{\infty}c_{(\mathbf{n}+1,m)}x^m - \frac{3}{8}\sum_{m=0}^{\infty}c_{(\mathbf{n},m)}x^m - \frac{1}{2}\sum_{m=0}^{\infty}x^m.$$

Applying one of the properties of summations and the fact that $\sum_{k=0}^{\infty}ar^k = \frac{a}{1-r}$ for $|r|<1$, we obtain

$$G_{(b_{(\mathbf{n},m)};x)} = \frac{1}{8}\sum_{m=0}^{\infty}c_{(n+1,m+1)}x^{m+1} + \frac{1}{8}c_{(\mathbf{n}+1,0)} - \frac{3}{8}G_{(c_{(\mathbf{n},m)};x)} - \frac{1}{2}\left(\frac{1}{1-x}\right).$$

By item (a) of Lemma 2.2 in [10], we get

$$G_{(b_{(\mathbf{n},m)};x)} = \frac{1}{8}G_{(c_{(\mathbf{n+1},m+1)};x)} - \frac{3}{8}G_{(c_{(\mathbf{n},m)};x)} - \frac{1}{2(1-x)} + \frac{1}{8}c_{\mathbf{n}+1}.$$

Thus,

$$G_{(b_{(\mathbf{n},m)};x)} = \frac{1}{8}\left(G_{(c_{(\mathbf{n}+1,m)};x)} - 3G_{(c_{(\mathbf{n},m)};x)} + c_{\mathbf{n}+2}\right) - \frac{1}{2(1-x)}.$$

□

4. Conclusion

This article continues work related to the bidimensional version of some numerical sequences. The results presented in this paper are considered a contribution to the field of mathematics and offer an opportunity for researchers interested in this topic of number sequences to spend some time studying them. As future work, we plan to study the respective Binet's formulas.

Article Information

Acknowledgements: The first and seconde authors are members of the CMAT Research Center - UTAD. The last author is part of them Mathematics Research Group of the University of Santiago de Compostela - USC GI 2136. This research was partially funded by Portuguese Funds through FCT - Fundação para a Ciência e a Tecnologia, under Projects UIDB/00013/2020, UIDP/00013/2020 and UIDB/00194/2020. the Ministerio de Ciencia e Innovación, Agencia Estatal de Investigación (Spain), grant PID2020-115155GB-I00 and the Consellería de Educación, Universidade e Formación Profesional (Xunta de Galicia), grant ED431C 2023/31. The authors would like to thank to the editor and the anonymous reviewers for their helpful comments and suggestions in order to improve this research work.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism was detected.

References

[1] G. K. Gözeri, *On Pell, Pell-Lucas, and balancing numbers*, J. Inequal. Appl., **3** (2018). https://doi.org/10.1186/s13660-017-1599-1
[2] B. K. Patel, N. Irmak, P. K. Ray, A. Zaharescu, *Incomplete balancing and Lucas-balancing numbers*, Math. Rep. (Bucur.), **20**(1) (2018), 59-72.
[3] G. K. Panda, T. Komatsu, R. K. Davala, *Reciprocal sums of sequences involving balancing and Lucas-balancing numbers*, Math. Rep. (Bucur.), **20**(2) (2018), 201-214.
[4] C. Levesque, *On m-th order linear recurrences*, Fibonacci Quart., **23**(4) (1985), 290–293.
[5] M. Vieira, V. Alves, P. Catarino, *Relações bidimensionais e identidades da sequência de Leonardo*, ReviSeM, **2** (2019), 156–173. https://doi.org/10.34179/revisem.v4i2.11863
[6] C. Bezerra, V. Alves, M. Vieira, *Relações recorrentes bidimensionais e tridimensionais de Narayana*, C.Q.D-Revista Eletrônica Paulista de Matemática, **18** (2020), 12–28. https://doi.org/10.21167/cqdvol18ic202023169664fhcbfrva1228
[7] R. R. de Oliveira, V. Alves, B. Paiva, *Identidades bi e tridimensionais para os números de Fibonacci na forma complexa*, C.Q.D-Revista Eletrônica Paulista de Matemática, **11** (2017), 91–106. https://doi.org/10.21167/cqdvol11ic201723169664rrofrvarebp91106
[8] R. R. de Oliveira, V. Alves, *Os números gaussianos de Fibonacci e relações recorrentes bidimensionais*, Revista Thema, **16**(4) (2019), 745–754. http://dx.doi.org/10.15536/thema.V16.2019.745-754.1333
[9] J. Chimpanzo, P. Catarino, P. Vasco, A. Borges, *Bidimensional extensions of balancing and Lucas-balancing numbers*, J. Discrete Math. Sci. Cryptogr., **27**(1) (2024), 95–115.
[10] J. Chimpanzo, M. V. Otero-Espinar, A. Borges, P. Vasco, P. Catarino, *Bidimensional extensions of cobalancing and Lucas-cobalancing numbers*, Ann. Math. Sil., (2023), https://doi.org/10.2478/amsil-2023-0022
[11] P. Catarino, H. Campos, P. Vasco, *On some identities for balancing and cobalancing numbers*, Ann. Math. Inform., **45** (2015), 11–24.