Universal Journal of Mathematics and Applications, 7 (1) (2024) 1-11
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1381621

Miao-Tam Equation and Ricci Solitons on Three-Dimensional Trans-Sasakian Generalized Sasakian Space-Forms

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Article Info

Keywords: *Generalized Sasakian space-forms, Gradient Ricci solitons, Miao-Tam equation, Sasakian space-forms, Trans-Sasakian manifolds*
2010 AMS: *53C15, 53D25*
Received: *26 October 2023*
Accepted: *24 December 2023*
Available online: *9 January 2024*

Abstract

The aim of the present article is to characterize some properties of the Miao-Tam equation on three-dimensional generalized Sasakian space-forms with trans-Sasakian structures. It has been proved that in such space-forms if the Miao-Tam equation admits non-trivial solution, then the metric of the space form must be a gradient Ricci soliton. We have derived that there does not exist a non-trivial solution of the Fischer-Marsden equation on the said space-forms. We have also investigated certain features of Ricci solitons and gradient Ricci solitons. At the end of the article, we construct an example to verify the obtained results.

1. Introduction

Miao-Tam equation on f-cosymplectic manifolds was investigated by X. Chen [1]. He proved that under certain restrictions such a manifold is either locally the product of a Kähler manifold and an interval or a unit circle, or, the manifold is of constant scalar curvature. He also established that if the manifold is connected and satisfies the Miao-Tam equation, then the manifold is Einstein under certain conditions. Since an Einstein manifold or a manifold of constant curvature is model of some interesting physical systems, geometers are naturally motivated to find the conditions under which a manifold will be Einstein or, a manifold of constant scalar curvature. To this end we study Miao-Tam equation on generalized Sasakian space-forms with trans-Sasakian structure and established that if a generalized Sasakian space-form with trans-Sasakian structure admits a non-trivial solution of the Miao-Tam equation, then the scalar curvature is constant and the manifold is Einstein or the structure is β-Kenmotsu. Several researchers [2–10] have investigated the Miao-Tam equation for some classes of contact manifolds.
Let $(M^n, g), n>2$ be a compact orientable Riemannian manifold with a smooth boundary ∂M and $\lambda : M^n \to \mathbb{R}$ be a smooth function on the manifold. Then the Miao-Tam equation on M^n is given by

$$Hess\lambda = (\Delta\lambda)g + \lambda S + g, \tag{1.1}$$

on M and $\lambda = 0$ on ∂M, $Hess$, Δ being respectively the Hessian operator and Laplacian with respect to the metric g. S indicates the Ricci curvature and λ indicates the potential function. The metrics satisfying the equation (1.1) are known as Miao-Tam critical metrics [11].
A sub-class of the Miao-Tam equation is the Fischer-Marsden equation which is given by

$$Hess\lambda = (\Delta\lambda)g + \lambda S.$$

The Fischer-Marsden equation (FME, in short) was constructed by A.E. Fischer and J. Marsden in [12]. The authors [12] in their paper conjectured that a compact Riemannian manifold that admits a non-trivial solution of the FME is necessarily Einstein. This statement is known as Fischer-Marsden conjecture. Later Kobayashi [13] pointed out that the said conjecture is not true in general. They are valid only in some special cases. After that a huge number of works has been done to analyze Fischer-Marsden conjecture on Riemannian manifolds admitting several structures.
R. S. Hamilton [14] introduced the notion of the Ricci flow in 1988. On a Riemannian or semi-Riemannian manifold,

$$\frac{\partial g}{\partial t} + 2S = 0$$

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Cite as: U. C. De, T. Mandal, A. Sarkar, *Miao-Tam Equation and Ricci Solitons on Three-Dimensional Trans-Sasakian Generalized Sasakian Space-Forms*, Univers. J. Math. Appl., 7(1) (2024), 1-11.

denotes the Ricci flow equation. A self-similar solution of the above equation is called the Ricci soliton and the soliton equation is given by

$$\pounds_V g + 2S + 2\psi g = 0, \tag{1.2}$$

$\pounds$ denotes the Lie-derivative operator. Here V is called the potential vector field and ψ is the soliton constant. If the sign of ψ is positive then the soliton is known as expanding and for the cases where ψ is zero or negative, the soliton is steady or shrinking, respectively. For details about Ricci solitons see the articles [15–18]. If the potential vector field V is the gradient of a smooth function ζ, then it is called the gradient Ricci soliton. Thus the gradient Ricci soliton is given by

$$Hess(\zeta) + S + \psi g = 0, \tag{1.3}$$

here $Hess$ is the Hessian operator.
The theory of generalized Sasakian space-forms came into existence after the work of Alegre et al. [19]. A generalized Sasakian spce-form (GSSF, in short) is such a manifold whose Riemann curvature R is given by

$$R(V_1,V_2)V_3 = f_1R_1(V_1,V_2)V_3 + f_2R_2(V_1,V_2)V_3 + f_3R_3(V_1,V_2)V_3, \tag{1.4}$$

f_1, f_2 and f_3 are smooth functions on M and

$$R_1(V_1,V_2)V_3 = g(V_2,V_3)V_1 - g(V_1,V_3)V_2,$$
$$R_2(V_1,V_2)V_3 = g(V_1,\phi V_3)\phi V_2 - g(V_2,\phi V_3)\phi V_1 + 2g(V_1,\phi V_2)\phi V_3,$$
$$R_3(V_1,V_2)V_3 = \eta(V_1)\eta(V_3)V_2 - \eta(V_2)\eta(V_3)V_1 + g(V_1,V_3)\eta(V_2)\xi - g(V_2,V_3)\eta(V_1)\xi.$$

Such a manifold admitting different almost contact structures like Sasakian, K-contact, trans-Sasakian, etc. was analyzed by Alegre and Carriazo. GSSF is now drawing attention of several geometers. In [20], it is proved that any GSSF with dimension greater than or equal to five must be Sasakian-space-form. It is also proved in the same article that a K-contact GSSF is a Sasakian manifold. For more details we cite the papers [21–25].
The present paper is organized as follows: After the introduction, we give some preliminaries in the Section 2. In Section 3, we have studied Miao-Tam equation on three dimensional GSSFs with trans-Sasakian structure. In the same section we have proved that if a non-trivial solution of the Miao-Tam equation exists then the metric must be a gradient Ricci soliton and non-existences of the non-trivial solution of the Fischer-Marsden equation is also deduced. In the next section, we have derived some new results of Ricci solitons and gradient Ricci solitons on the same space-forms. In the last section, we give an example to verify the deduced results.

2. Preliminaries

A smooth manifold M^{2n+1} is known as an almost contact manifold (ACM) if there exists a structure (ϕ,θ,η), where ϕ, θ and η are, respectively, a $(1,1)$-tensor field, a $(1,0)$ type vector field and a 1-form, such that

$$\phi^2V_1 = -V_1 + \eta(V_1)\theta, \quad \eta(\theta) = 1, \quad \phi\theta = 0, \quad \eta.\phi = 0 \quad \text{rank}(\phi) = 2n,$$

for every vector field V_1 on M^{2n+1} [26,27].
An ACM M^{2n+1} is called an almost contact metric manifold (ACMM) if it admits a Riemannian metric g such that

$$g(\phi V_1,\phi V_2) = g(V_1,V_2) - \eta(V_1)\eta(V_2), \tag{2.1}$$

for every vector fields V_1, V_2 on M^{2n+1}. Equation (2.1) gives

$$g(\phi V_1,V_2) = -g(V_1,\phi V_2).$$

An ACMM is called a contact metric manifold if there exists a 2-form Φ such that $d\eta = \Phi$, where $\Phi(V_1,V_2) = g(V_1,\phi V_2)$. An ACMM is called normal if Nijenhuis torsion tensor $[\phi,\phi](V_1,V_2) + 2d\eta(V_1,V_2)\theta$ vanishes, where $[\phi,\phi](V_1,V_2) = \phi^2[V_1,V_2] + [\phi V_1,\phi V_2] - \phi[\phi V_1,V_2] - \phi[V_1,\phi V_2]$. A normal contact metric manifold is called a Sasakian manifold. An ACMM is called a trans-Sasakian manifold [28] if there exist two smooth functions α and β such that

$$(\nabla_{V_1}\phi)V_2 = \alpha(g(V_1,V_2)\theta - \eta(V_2)V_1) + \beta(g(\phi V_1,V_2)\theta - \eta(V_2)\phi V_1), \tag{2.2}$$

for every vector fields V_1, V_2 on M^{2n+1}. Actually, trans-Sasakian manifolds are the generalizations of Sasakian manifolds and Kenmotsu manifolds, that means, if $\beta = 0$ (res. $\alpha = 0$) then the manifold reduces to α-Sasakian (res. β-Kenmotsu) manifold. For more details please follow the articles [29–33]. From equation (2.2), one can obtain

$$\nabla_{V_1}\theta = -\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta). \tag{2.3}$$

In view of (1.4), we have

$$S(V_2,V_3) = (2f_1 + 3f_2 - f_3)g(V_2,V_3) - (3f_2 + f_3)\eta(V_2)\eta(V_3), \tag{2.4}$$

which gives

$$QV_2 = (2f_1 + 3f_2 - f_3)V_2 - (3f_2 + f_3)\eta(V_2)\theta, \tag{2.5}$$

Q is the Ricci operator. Again, contracting V_2 in the foregoing equation, we get the scalar curvature as

$$r = 2(3f_1 + 3f_2 - 2f_3). \tag{2.6}$$

Lemma 2.1. *For a trans-Sasakian GSSF M, the following relation holds:*

$$f_1 - f_3 + \theta(\alpha) + \theta(\beta) - \alpha^2 + \beta^2 = 0. \tag{2.7}$$

Proof. According to the equations (2.2) and (2.3), we obtain

$$R(V_1,\theta)\theta = (\theta(\alpha) + \alpha\beta)\phi V_1 + (-\theta(\beta) - \beta^2 + \alpha^2 + \alpha\beta)(V_1 - \eta(V_1)\theta). \tag{2.8}$$

On the other hand, from equation (1.4), it can be easily seen that

$$R(V_1,\theta)\theta = (f_1 - f_3)(V_1 - \eta(V_1)\theta). \tag{2.9}$$

Comparing (2.8) and (2.9), we have

$$\theta(\alpha) + \alpha\beta = 0$$

and

$$-\theta(\beta) - \beta^2 + \alpha^2 + \alpha\beta = f_1 - f_3.$$

Combining the last two equations, we obtain the equation (2.7). □

Definition 2.2 ([34, 35]). *A vector field V on a Riemannian manifold is called an infinitesimal contact transformation if*

$$\pounds_V \eta = \kappa\eta, \tag{2.10}$$

for some smooth function κ on the manifold. If κ = 0, then the vector field is called a strict infinitesimal contact transformation.

3. Miao-Tam Equation (MTE) on Trans-Sasakian Generalized Sasakian Space-forms

The prime aim of the present section is to study the Miao-Tam equation (MTE, in short) on three-dimensional trans-Sasakian GSSFs and make a bridge between MTE and Ricci solitons. Before going to main topic, we proof the following lemma.

Lemma 3.1. *Let M^3 be a trans-Sasakian GSSF of dimension three, then*

$$\begin{aligned}(\nabla_{V_1}Q)V_2 =& V_1(2f_1+3f_2-f_3)V_2 - V_1(3f_2+f_3)\eta(V_2)\theta - (3f_2+f_3)(-\alpha g(\phi V_1,V_2)\theta + \beta(g(V_1,V_2)\theta - \eta(V_1)\eta(V_2)\theta)) \\ &- (3f_2+f_3)(-\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta))\eta(V_2),\end{aligned} \tag{3.1}$$

$$\frac{1}{2}V_2(r) = V_2(2f_1+3f_2-f_3) - \theta(3f_2+f_3)\eta(V_2) - 2\beta(3f_2+f_3)\eta(V_2), \tag{3.2}$$

and

$$\theta(r) = 4(\theta(f_1-f_3) - \beta(3f_2+f_3)), \tag{3.3}$$

for every vector fields V_1, V_2 on M^3.

Proof. Differentiating the equation (2.5) covariantly and using (2.3), one can obtain the equation (3.1). Contracting the equation (3.1) with respect to V_1, we obtain (3.2). Putting $V_2 = \xi$ in (3.2), we get the equation (3.3). □

Theorem 3.2. *If a three-dimensional trans-Sasakian GSSF admits non-trivial solution of the Miao-Tam equation then the scalar curvature is a constant.*

Proof. Let us suppose that the said space form admits non-trivial solution of the Miao-Tam equation. Then, from (1.1), we obtain

$$(\Delta\lambda)g(V_1,V_2) = (Hess\lambda)(V_1,V_2) - \lambda S(V_1,V_2) - g(V_1,V_2). \tag{3.4}$$

Let $\{u_1,u_2,\xi\}$ be an orthonormal set of tangent vector fields on M^3. Substituting $V_1 = V_2 = u_i$ in the previous equation and summing over i, we have

$$(\Delta\lambda) = -(3f_1+3f_2-2f_3)\lambda - \frac{3}{2}. \tag{3.5}$$

Using (3.5) in (3.4), we obtain

$$\nabla_{V_1}D\lambda = \lambda QV_1 - (3f_1+3f_2-2f_3)\lambda V_1 - \frac{1}{2}V_1. \tag{3.6}$$

The covariant derivative of the equation (3.6) in the direction of V_2 gives

$$\nabla_{V_2}\nabla_{V_1}D\lambda = V_2(\lambda)QV_1 + \lambda\nabla_{V_2}QV_1 - V_2(3f_1+3f_2-2f_3)\lambda V_1 - (3f_1+3f_2-2f_3)(V_2(\lambda)V_1 + \lambda\nabla_{V_2}V_1) - \frac{1}{2}\nabla_{V_2}V_1. \tag{3.7}$$

Interchanging V_1 and V_2 in (3.7), one can obtain

$$\nabla_{V_1}\nabla_{V_2}D\lambda = V_1(\lambda)QV_2 + \lambda\nabla_{V_1}QV_2 - V_1(3f_1+3f_2-2f_3)\lambda V_2 - (3f_1+3f_2-2f_3)(V_1(\lambda)V_2 + \lambda\nabla_{V_1}V_2) - \frac{1}{2}\nabla_{V_1}V_2. \tag{3.8}$$

Again, equation (3.6) gives

$$\nabla_{[V_1,V_2]}D\lambda = \lambda Q[V_1,V_2]-(3f_1+3f_2-2f_3)\lambda[V_1,V_2]-\frac{1}{2}[V_1,V_2]. \tag{3.9}$$

Using (3.7)-(3.9), we get the curvature tensor as

$$\begin{aligned} R(V_1,V_2)D\lambda =& V_1(\lambda)QV_2-V_2(\lambda)QV_1+\lambda((\nabla_{V_1}Q)V_2-(\nabla_{V_2}Q)V_1)-V_1(3f_1+3f_2-2f_3)\lambda V_2+V_2(3f_1+3f_2-2f_3)\lambda V_1 \\ &-(3f_1+3f_2-2f_3)(V_1(\lambda)V_2-V_2(\lambda)V_1). \end{aligned} \tag{3.10}$$

Contracting (3.10) along the vector field V_1, we obtain

$$\begin{aligned} S(V_2,D\lambda) =& (2f_1+3f_2-f_3)V_2(\lambda)-(3f_2+f_3)\theta(\lambda)\eta(V_2)+\lambda\{V_2(2f_1+3f_2-f_3)-\theta(3f_2+f_3)\eta(V_2) \\ &-2\beta(3f_2+f_3)\eta(V_2)\}. \end{aligned} \tag{3.11}$$

According to (2.4), we find

$$S(V_2,D\lambda)=(2f_1+3f_2-f_3)V_2(\lambda)-(3f_2+f_3)\theta(\lambda)\eta(V_2). \tag{3.12}$$

Comparing (3.11) and (3.12), we get

$$V_2(2f_1+3f_2-f_3)-\theta(3f_2+f_3)\eta(V_2)-2\beta(3f_2+f_3)\eta(V_2)=0, \tag{3.13}$$

where we have used $\lambda \neq 0$. Substituting (3.13) in (3.2), we see that $V_2(r)=0$, that is, r is a constant.
This completes the proof. □

Theorem 3.3. *If a three-dimensional trans-Sasakian GSSF admits non-trivial solution of the Miao-Tam equation then either the structure is β-Kenmotsu or, the manifold is Einstein.*

Proof. Replacing V_1 by ξ and taking inner product with V_1 of the equation (3.10), we have

$$\begin{aligned} g(R(\theta,V_2)D\lambda,V_1) =& \theta(\lambda)\{-(f_1-f_3)g(V_1,V_2)-(3f_2+f_3)\eta(V_1)\eta(V_2)\}+(f_1+3f_3)V_2(\lambda)\eta(V_1) \\ &+\lambda\{-\theta(f_1-f_3)g(V_1,V_2)-\theta(3f_2+f_3)\eta(V_1)\eta(V_2)+V_2(f_1+3f_2)\eta(V_1) \\ &+(3f_2+f_3)(-\alpha g(V_1,\phi V_2)+\beta(g(V_1,V_2)-\eta(V_1)\eta(V_2)))\}. \end{aligned} \tag{3.14}$$

Putting $V_1=\xi$ in (1.4) and then taking inner product with $D\lambda$, one can obtain

$$g(R(\theta,V_2)V_1,D\lambda)=(f_1-f_3)(\theta(\lambda)g(V_1,V_2)-V_2(\lambda)\eta(V_1)). \tag{3.15}$$

Comparing (3.14) and (3.15), we find

$$\begin{aligned} &\theta(\lambda)\{-(f_1-f_3)g(V_1,V_2)-(3f_2+f_3)\eta(V_1)\eta(V_2)\}+(f_1+3f_2)V_2(\lambda)\eta(V_1) \\ &+\lambda\left\{\begin{array}{l} -\theta(f_1-f_3)g(V_1,V_2)-\theta(3f_2+f_3)\eta(V_1)\eta(V_2)+V_2(f_1+3f_2)\eta(V_1) \\ +(3f_2+f_3)(-\alpha g(V_1,\phi V_2)+\beta(g(V_1,V_2)-\eta(V_1)\eta(V_2))) \end{array}\right\}=(f_3-f_1)(\theta(\lambda)g(V_1,V_2)-V_2(\lambda)\eta(V_1)). \end{aligned} \tag{3.16}$$

Interchanging V_1 and V_2 in the foregoing equation, we find

$$\begin{aligned} &\theta(\lambda)\{-(f_1-f_3)g(V_1,V_2)-(3f_2+f_3)\eta(V_1)\eta(V_2)\}+(f_1+3f_3)V_1(\lambda)\eta(V_2) \\ &+\lambda\left\{\begin{array}{l} -\theta(f_1-f_3)g(V_1,V_2)-\theta(3f_2+f_3)\eta(V_1)\eta(V_2)+V_1(f_1+3f_2)\eta(V_2) \\ +(3f_2+f_3)(\alpha g(V_1,\phi V_2)+\beta(g(V_1,V_2)-\eta(V_1)\eta(V_2))) \end{array}\right\}=(f_3-f_1)(\theta(\lambda)g(V_1,V_2)-V_1(\lambda)\eta(V_2)). \end{aligned} \tag{3.17}$$

Subtracting (3.17) from (3.16), one can obtain

$$(3f_2+f_3)(V_2(\lambda)\eta(V_1)-V_1(\lambda)\eta(V_2))+\lambda\{V_2(f_1+3f_2)\eta(V_1)-V_1(f_1+3f_2)\eta(V_2)-2(3f_2+f_3)\alpha g(V_1,\phi V_2)\}=0.$$

Replacing V_1 and V_2 by ϕV_1 and ϕV_2, respectively, in the last equation, we obtain

$$(3f_2+f_3)\alpha g(V_1,\phi V_2)=0,$$

which implies that either $3f_2+f_3=0$ or, $\alpha=0$, i.e., the structure is β-Kenmotsu.
Let us now discuss the case when $3f_2+f_3=0$. Then from(2.6), we get $r=6(f_1-f_3)$. With the help of (2.4), (3.1), equation (3.16) can be written as

$$\theta(\lambda)(S(V_1,V_2)-(f_3-f_1)g(V_1,V_2))-3(f_1-f_3)V_2(\lambda)\eta(V_1)+\theta(f)g(V_1,V_2)-V_2(f)\eta(V_1)=0,$$

where $f=-\frac{r\lambda+1}{2}$ and

$$\nabla_{V_1}D\lambda=\lambda QV_1+fV_1. \tag{3.18}$$

As r is a constant, $2V_2(f)=-rV_2(\lambda)$ and so, $2\theta(f)=-r\theta(\lambda)$. Applying these relations in the above equation, we obtain

$$\theta(\lambda)\{S(V_1,V_2)-2(f_1-f_3)g(V_1,V_2)\}=0,$$

where we have used $r=6(f_1-f_3)$. From the foregoing equation we obtain either $\theta(\lambda)=0$ or, $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$. If we consider $\theta(\lambda)=0$, i.e., $g(\theta,D\lambda)=0$, then by covariant derivative

$$g(\nabla_{V_1}\theta,D\lambda)+g(\theta,\nabla_{V_1}D\lambda)=0.$$

Using (2.3) and (3.18) in the foregoing equation, we have

$$-\alpha\phi V_1(\lambda)+\beta V_1(\lambda)+\lambda S(V_1,\theta)+f\eta(V_1)=0, \tag{3.19}$$

where we have used $\theta(\lambda)=0$. Applying (1.5), $r=6(f_1-f_3)$ and $f=-\frac{r\lambda+1}{2}$ in (3.19), we obtain

$$-\alpha\phi V_1(\lambda)+\beta V_1(\lambda)-\{\lambda(f_1-f_3)+\frac{1}{2}\}\eta(V_1)=0. \tag{3.20}$$

Replacing V_1 by θ, equation (3.20) gives $\lambda(f_1-f_3)+\frac{1}{2}=0$, as $\theta(\lambda)=0$. Thus we find that $f=1$, a constant and hence λ is also a non-zero constant. Applying these data in (3.4), we see that $S(V_1,V_2)=-\frac{1}{\lambda}g(V_1,V_2)$, i.e, $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$, as $\lambda(f_1-f_3)+\frac{1}{2}=0$. Thus for every cases, the space-form obeys $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$. Hence the manifold is Einstein.
Thus the proof is completed. □

A consequence of the above theorem is

Corollary 3.4. *There does not exist a non-cosymplectic three-dimensional GSSF with β-Kenmotsu structure obeying non-trivial solution of the MTE, where β is a constant.*

Proof. Putting $V_1=V_2=u_i$ in (3.16), where $\{u_i\}$, $(i=1,2,3)$ being an orthonormal frame of the tangent space, and summing over i, we find

$$\theta(f_1-f_3)-\beta(3f_2+f_3)=0. \tag{3.21}$$

Comparing (3.3) and (3.21), we obtain $\theta(r)=0$. Using (2.7) in (3.21) and considering β as a constant, we find

$$\beta(3f_2+f_3)=0,$$

which gives $\beta=0$, as $3f_2+f_3\neq 0$. Hence the structure is cosymplectic. □

Corollary 3.5. *Let a trans-Sasakian GSSF be an Einstein manifold and the space form admit non-trivial solution of MTE. Then the metric is a gradient Ricci soliton.*

Proof. Using $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$ in (3.18), we see that

$$\nabla_{V_1}D\lambda=\{2(f_1-f_3)+1\}V_1.$$

The foregoing equation can be written as

$$Hess(\lambda)(V_1,V_2)+S(V_1,V_2)-\{2(f_1-f_3)(\lambda+1)+1\}g(V_1,V_2)=0,$$

which is the gradient Ricci soliton, where the soliton constant is $2(f_1-f_3)(\lambda+1)+1$. □

Theorem 3.6 ([36]). *If $\tilde{\lambda}$ is a solution of the Fischer-Marsden equation (FME, in short) on a three-dimensional trans-Sasakian GSSF, then the curvature tensor R is given by*

$$R(V_1,V_2)D\tilde{\lambda}=V_1(\tilde{\lambda})QV_2-V_2(\tilde{\lambda})QV_1+\tilde{\lambda}\{(\nabla_{V_1}Q)V_2-(\nabla_{V_2}Q)V_1\}+V_1(\tilde{f})V_2-V_2(\tilde{f})V_1, \tag{3.22}$$

for every vector fields V_1, V_2 on M and $\tilde{f}=-\frac{r\tilde{\lambda}}{2}$.
Moreover,

$$\nabla_{V_1}D\tilde{\lambda}=\tilde{\lambda}QV_1+\tilde{f}V_1. \tag{3.23}$$

Theorem 3.7. *In a three-dimensional trans-Sasakian GSSF, if the FME admits a solution then either the solution is trivial or, the scalar curvature is a constant.*

Proof. Using (2.4) in (3.22), one can obtain

$$\begin{aligned}R(V_1,V_2)D\tilde{\lambda}=&(2f_1+3f_2-f_3)V_1(\tilde{\lambda})V_2-(3f_2+f_3)V_1(\tilde{\lambda})\eta(V_2)\theta-(2f_1+3f_2-f_3)V_2(\tilde{\lambda})V_1+(3f_2+f_3)V_2(\tilde{\lambda})\eta(V_1)\theta\\&+\tilde{\lambda}\{(\nabla_{V_1}Q)V_2-(\nabla_{V_2}Q)V_1\}+V_1(\tilde{f})V_2-V_2(\tilde{f})V_1.\end{aligned} \tag{3.24}$$

Contracting (3.24) along V_1, we infer

$$S(V_2,D\tilde{\lambda})=(2f_1+3f_2-f_3)V_2(\tilde{\lambda})-(3f_2+f_3)\theta(\tilde{\lambda})\eta(V_2)+\frac{\tilde{\lambda}}{2}V_2(r), \tag{3.25}$$

where we have used $\tilde{f}=-\frac{r\tilde{\lambda}}{2}$. Comparing (3.25) with (3.12), we find that $\tilde{\lambda}V_2(r)=0$, which gives either $\tilde{\lambda}=0$, i.e., the solution is trivial or, $V_2(r)=0$, i.e., the scalar curvature is a constant.
This establishes the theorem. □

Theorem 3.8. *In a three-dimensional trans-Sasakian GSSF, if the FME admits a solution then either the structure is β-Kenmotsu or, the manifold is Einstein or, the solution is trivial.*

Proof. Taking inner product of (3.22) with θ, we find that

$$\begin{aligned} g(R(V_1,V_2)D\tilde{\lambda},\theta) =& 2(f_1-f_3)\{V_1(\tilde{\lambda})\eta(V_2)-V_2(\tilde{\lambda})\eta(V_1)\} \\ &+\tilde{\lambda}\{2V_1(f_1-f_3)\eta(V_2)-2V_2(f_1-f_3)\eta(V_1)+2(3f_2+f_3)\alpha g(\phi V_1,V_2)\} \\ &+V_1(\tilde{f})\eta(V_2)-V_2(\tilde{f})\eta(V_1). \end{aligned} \tag{3.26}$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 in (3.26), one can obtain

$$g(R(\phi V_1,\phi V_2)D\tilde{\lambda},\theta) = -2\tilde{\lambda}(3f_2+f_3)\alpha g(V_1,\phi V_2). \tag{3.27}$$

Also, from(1.4), we have

$$g(R(\phi V_1,\phi V_2)D\tilde{\lambda},\theta) = 0. \tag{3.28}$$

Comparing (3.27) and (3.28), we obtain

$$\tilde{\lambda}(3f_2+f_3)\alpha g(V_1,\phi V_2) = 0.$$

Thus three possibility arise: (1) $\tilde{\lambda}=0$, (2) $(3f_2+f_3)=0$ and (3) $\alpha=0$.
Let us discuss the case when $(3f_2+f_3)=0$. Then, from (2.6), we find that $r=6(f_1-f_3)$. From (3.22), we get

$$g(R(\theta,V_2)D\tilde{\lambda},V_1) = \theta(\tilde{\lambda})S(V_1,V_2)-V_2(\tilde{\lambda})S(V_1,\theta)+\theta(\tilde{f})g(V_1,V_2)-V_2(\tilde{f})\eta(V_1). \tag{3.29}$$

Also, from (1.4), we infer

$$g(R(\theta,V_2)D\tilde{\lambda},V_1) = -(f_1-f_3)\{\theta(\tilde{f})g(V_1,V_2)-V_2(\tilde{f})\eta(V_1)\}. \tag{3.30}$$

Comparing (3.29) and (3.30) and using $r=6(f_1-f_3)$, $f=-\frac{r\tilde{\lambda}}{2}$ and the equation (2.4), one can obtain

$$\theta(\tilde{\lambda})(S(V_1,V_2)-2(f_1-f_3)g(V_1,V_2)) = 0,$$

which implies either $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$, i.e., the manifold is Einstein or, $\theta(\tilde{\lambda})=0$. Let us discuss the case when $(\theta\tilde{\lambda})=0$. Then we have $g(\theta,D\tilde{\lambda})=0$, which gives

$$g(\nabla_{V_2}\theta,D\tilde{\lambda})+g(\theta,\nabla_{V_2}D\tilde{\lambda}) = 0.$$

Applying (2.3), (2.4), (3.23) and $\tilde{f}=-\frac{r\tilde{\lambda}}{2}$ in the foregoing equation, we see that

$$-\alpha\phi V_2(\tilde{\lambda})+\beta V_2(\tilde{\lambda})-(f_1-f_3)\tilde{\lambda}\eta(V_2) = 0, \tag{3.31}$$

where we have used $\theta(\tilde{\lambda})=0$. Replacing V_2 by θ and taking $f_1\neq f_3$ in (3.31), we find that $\tilde{\lambda}=0$, i.e., the solution is trivial.
This ensures the validity of the theorem.

□

4. Ricci Solitons on Three-Dimensional Generalized Sasakian Space-forms with Trans-Sasakian Structures

In the present section, we study Ricci solitons on three-dimensional generalized Sasakian space-forms with trans-Sasakian structure.

Theorem 4.1. *In a three-dimensional trans-Sasakian GSSF obeying Ricci solitons, the potential vector field is an infinitesimal contact transformation.*

Proof. From (1.2), we have

$$(\pounds_V g)(V_1,V_2)+2S(V_1,V_2)+2\psi g(V_1,V_2) = 0.$$

Applying $V_2=\theta$ in the foregoing equation and using (2.4), we have

$$(\pounds_V g)(V_1,\theta) = -2(2(f_1-f_3)+\psi)\eta(V_1). \tag{4.1}$$

Again, changing V_1 by θ in (4.1), we get

$$\pounds_V\theta = (2(f_1-f_3)+\psi)\theta. \tag{4.2}$$

Applying Lie derivative of $\eta(V_1)=g(V_1,\theta)$ with respect to V and then using (4.1) and (4.2), we find that

$$(\pounds_V\eta)(V_1) = -(2(f_1-f_3)+\psi)\eta(V_1),$$

an infinitesimal contact transformation. □

From the above theorem, we prove the following:

Theorem 4.2. *In a three-dimensional trans-Sasakian GSSF obeying Ricci solitons, the soliton is shrinking, expanding or steady if $f_1 - f_3$ is positive, negative or zero, respectively.*

Proof. We have

$$(\pounds_V d\eta)(V_1,V_2) = (\pounds_V g)(V_1,\phi V_2) + g(V_1,(\pounds_V \phi)V_2).$$

Using (2.4) and (1.2) in the foregoing equation, we infer

$$\begin{aligned}(\pounds_V d\eta)(V_1,V_2) =& -2(2f_1+3f_2-f_3+\psi)g(V_1,\phi V_2)\\ &+g(V_1,(\pounds_V\phi)V_2).\end{aligned} \tag{4.3}$$

According to Theorem 4.1, V is an infinitesimal contact transformation. Also, since $\pounds$ and d commutes, equation (2.10) gives

$$\begin{aligned}(\pounds_V d\eta)(V_1,V_2) =&((d\kappa)\wedge\eta)(V_1,V_2)+\kappa g(V_1,\phi V_2)\\ =&\frac{1}{2}(V_1(\kappa)\eta(V_2)-V_2(\kappa)\eta(V_1))+\kappa g(V_1,\phi V_2).\end{aligned} \tag{4.4}$$

Comparing (4.3) and (4.4), we have

$$g(V_1,(\pounds_V\phi)V_2) =\frac{1}{2}(V_1(\kappa)\eta(V_2)-V_2(\kappa)\eta(V_1))+(2(2f_1+3f_2-f_3+\psi)+\kappa)g(V_1,\phi V_2),$$

which gives

$$(\pounds_V\phi)V_2 = \frac{1}{2}(\eta(V_2)D\kappa - V_2(\kappa)\theta)+(2(2f_1+3f_2-f_3+\psi)+\kappa)\phi V_2.$$

Changing V_2 by θ in the previous equation, we find

$$(\pounds_V\phi)\theta = \frac{1}{2}(D\kappa-\theta(\kappa)\theta). \tag{4.5}$$

But

$$(\pounds_V\phi)\theta = \pounds_V\phi\theta - \phi(\pounds_V\theta) = 0, \tag{4.6}$$

where we used (4.2) and $\phi\theta = 0$. Using (4.6) in (4.5), we obtain

$$D\kappa = \theta(\kappa)\theta,$$

which gives

$$d\kappa = \theta(\kappa)\eta. \tag{4.7}$$

By exterior derivative we find from (4.7) that

$$0 = d^2\kappa = d(\theta(\kappa))\wedge\eta + \theta(\kappa)d\eta.$$

Taking wedge product with η in the foregoing equation, we get

$$\theta(\kappa)\eta\wedge d\eta = 0.$$

As $\eta\wedge d\eta \neq 0$, the previous equation gives $\theta(\kappa)=0$. Thus, from (4.7), we have $d\kappa = 0$, i.e., κ is a constant.
Due to Cartan's formula, for the closed volume form $\Omega(=\eta\wedge d\eta)$, we have

$$\pounds_V\Omega = (divV)\Omega, \tag{4.8}$$

where *div* is the divergence operator. Again, taking Lie derivative of the volume form $\Omega(=\eta\wedge d\eta)$ and using (4.4) and (4.8), we get

$$(divV)\Omega = 2\kappa\Omega,$$

which implies

$$divV = 2\kappa.$$

Integrating the above equation and using divergence theorem, we see that $\kappa = 0$. Thus V is the strict infinitesimal contact transformation and hence, we get $\psi = -2(f_1 - f_3)$.
This establishes the theorem. □

Theorem 4.3. *In a three dimensional trans-Sasakian GSSF obeying gradient Ricci solitons, either the structure is β-Kenmotsu or, the potential function is constant, i.e., the soliton is trivial.*

Proof. Let us suppose that a three dimensional trans-Sasakian generalized Sasakian space-form admit gradient Ricci solitons. Then, from (1.3), we can write

$$\nabla_{V_1} D\zeta = -QV_1 - \psi V_1. \tag{4.9}$$

Applying covariant derivative on (4.9), we get

$$\nabla_{V_2}\nabla_{V_1} D\zeta = -\nabla_{V_2} QV_1 - \psi \nabla_{V_2} V_1. \tag{4.10}$$

Interchanging V_1 and V_2 in the previous equation, we obtain

$$\nabla_{V_1}\nabla_{V_2} D\zeta = -\nabla_{V_1} QV_2 - \psi \nabla_{V_1} V_2. \tag{4.11}$$

Also, equation (4.9) gives

$$\nabla_{[V_1,V_2]} D\zeta = -Q[V_1,V_2] - \psi [V_1,V_2]. \tag{4.12}$$

Using (4.10)-(4.12), we get the curvature tensor as

$$\begin{aligned} R(V_1,V_2)D\zeta = &-\{V_1(2f_1+3f_2-f_3)V_2 - V_1(3f_2+f_3)\eta(V_2)\theta - V_2(2f_1+3f_2-f_3) + V_2(3f_2+f_3)\eta(V_1)\theta \\ &+2(3f_2+f_3)g(\phi V_1,V_2)\theta - (3f_2+f_3)(-\alpha\phi V_1 + \beta(V_1-\eta(V_1)\theta))\eta(V_2) \\ &+(3f_2+f_3)(-\alpha\phi V_2 + \beta(V_2-\eta(V_2)\theta))\eta(V_1)\}. \end{aligned} \tag{4.13}$$

Replacing V_1 by θ in (4.13) and then taking inner product with V_1, we see that

$$\begin{aligned} g(R(\theta,V_2)D\zeta,V_1) = &-\{\theta(2f_1+3f_2-f_3)g(V_1,V_2) - \theta(3f_2+f_3)\eta(V_1)\eta(V_2) - 2V_2(f_1-f_3)\eta(V_1) \\ &+(3f_2+f_3)(-\alpha g(V_1,\phi V_2) + \beta(g(V_1,V_2)-\eta(V_1)\eta(V_2)))\}. \end{aligned} \tag{4.14}$$

Also, the equation (1.4) can be written as

$$g(R(\theta,V_2)D\zeta,V_1) = (f_1-f_3)\{V_2(\zeta)\eta(V_1) - \theta(\zeta)g(V_1,V_2)\}. \tag{4.15}$$

Comparing (4.14) and (4.15), we obtain

$$\begin{aligned} &\theta(2f_1+3f_2-f_3)g(V_1,V_2) - \theta(3f_2+f_3)\eta(V_1)\eta(V_2) - 2V_2(f_1-f_3)\eta(V_1) \\ &+(3f_2+f_3)(-\alpha g(V_1,\phi V_2) + \beta(g(V_1,V_2)-\eta(V_1)\eta(V_2))) + (f_1-f_3)\{V_2(\zeta)\eta(V_1) - \theta(\zeta)g(V_1,V_2)\} = 0. \end{aligned} \tag{4.16}$$

Interchanging V_1 and V_2 in (4.16), we have

$$\begin{aligned} &\theta(2f_1+3f_2-f_3)g(V_1,V_2) - \theta(3f_2+f_3)\eta(V_1)\eta(V_2) - 2V_1(f_1-f_3)\eta(V_2) \\ &+(3f_2+f_3)(\alpha g(V_1,\phi V_2) + \beta(g(V_1,V_2)-\eta(V_1)\eta(V_2))) + (f_1-f_3)\{V_1(\zeta)\eta(V_2) - \theta(\zeta)g(V_1,V_2)\} = 0. \end{aligned} \tag{4.17}$$

Subtracting (4.17) from (4.16), we see that

$$2V_1(f_1-f_3)\eta(V_2) - 2V_2(f_1-f_3)\eta(V_1) - 2(3f_2+f_3)\alpha g(V_1,\phi V_2) + (f_1-f_3)\{V_2(\zeta)\eta(V_1) - V_1(\zeta)\eta(V_2)\} = 0. \tag{4.18}$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 in (4.18), we obtain

$$(3f_2+f_3)\alpha g(\phi V_1,V_2) = 0,$$

which indicates that either $\alpha = 0$, i.e., the structure is β-Kenmotsu or, $3f_2+f_3 = 0$. For the later case, with the help of (2.6) and (3.2), we get

$$V_1(f_1-f_3) = 0, \tag{4.19}$$

for every vector field V_1, i.e., $f_1 - f_3$ is a constant. Thus, from (4.18), we obtain

$$(f_1-f_3)\{V_2(\zeta)\eta(V_1) - V_1(\zeta)\eta(V_2)\} = 0,$$

which gives either $f_1 = f_3$ or

$$V_2(\zeta)\eta(V_1) = V_1(\zeta)\eta(V_2). \tag{4.20}$$

Let us discuss the second possibility. Putting $V_2 = \theta$ in (4.20), we obtain

$$D\zeta = \theta(\zeta)\theta. \tag{4.21}$$

Taking covariant derivative of (4.21) with respect to V_1 and using (2.3), we obtain

$$\nabla_{V_1} D\zeta = V_1(\theta(\zeta))\theta + \theta(\zeta)(-\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta)). \tag{4.22}$$

Comparing (4.22) with (4.9), we find that

$$V_1(\theta(\zeta))\eta(V_2) = -S(V_1,V_2) - \psi g(V_1,V_2) - \theta(\zeta)(-\alpha g(\phi V_1,V_2) + \beta(g(V_1,V_2) - \eta(V_1)\eta(V_2))).$$

Since $3f_2+f_3=0$, using (2.4) in the above equation, we get

$$V_1(\theta(\zeta))\eta(V_2)=-\{2(f_1-f_3)+\psi\}g(V_1,V_2)-\theta(\zeta)(-\alpha g(\phi V_1,V_2)+\beta(g(V_1,V_2)-\eta(V_1)\eta(V_2))). \tag{4.23}$$

Replacing V_2 by ϕV_2 in (4.23), we see that

$$\{2(f_1-f_3)+\psi\}g(V_1,\phi V_2)+\theta(\zeta)(-\alpha(g(V_1,V_2)-\eta(V_1)\eta(V_2))+\beta g(V_1,\phi V_2))=0.$$

Contracting the above equation and using $\mathrm{tr}\phi=0$, we get

$$\alpha\theta(\zeta)=0,$$

which gives $\theta(\zeta)=0$, as we consider $\alpha\neq 0$. Thus, from (4.21), we see that $D\zeta=0$, i.e., ζ is a constant.
Hence the proof is completed. □

From the equation(4.19), we can state the following corollary

Corollary 4.4. *If a three-dimensional trans-Sasakian GSSF admits gradient Ricci solitons, then either the structure is β-Kenmotsu or, f_1-f_3 is a constant.*

5. Example

Let $M=\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\}$ be a three-dimensional manifold, where (x,y,z) are the standard co-ordinates in $\mathbb{R}^3$. We choose the basis vectors on M as

$$u_1=e^{-2z}\frac{\partial}{\partial x},\quad u_2=e^{-2z}\frac{\partial}{\partial y},\quad u_3=\frac{\partial}{\partial z}.$$

Then we find by direct computation that

$$[u_1,u_2]=0,\quad [u_1,u_3]=2u_1,\quad [u_2,u_3]=2u_2.$$

Let g be the metric tensor defined by

$$g(u_1,u_1)=1,\quad g(u_2,u_2)=1,\quad g(u_3,u_3)=1,\quad g(u_1,u_2)=0,\quad g(u_1,u_3)=0,\quad g(u_2,u_3)=0.$$

The 1-form η is given by $\eta(V_1)=g(V_1,u_3)$ for all V_1 on M. Let us define the $(1,1)$-tensor field ϕ as

$$\phi u_1=-u_2,\quad \phi u_2=u_1,\quad \phi u_3=0.$$

Then we see that

$$\eta(u_3)=1,\quad \phi^2V_1=-V_1+\eta(V_1)u_3,\quad g(\phi V_1,\phi V_2)=g(V_1,V_2)-\eta(V_1)\eta(V_2),\quad d\eta(V_1,V_2)=g(V_1,\phi V_2).$$

Thus the given manifold admits a contact metric structure (ϕ,u_3,η,g).Now, using Koszul's formula, we obtain

$$\nabla_{u_1}u_1=-2u_3,\quad \nabla_{u_1}u_2=0,\quad \nabla_{u_1}u_3=2u_1,\quad \nabla_{u_2}u_1=0,\quad \nabla_{u_2}u_2=-2u_3,\quad \nabla_{u_2}u_3=2u_2,\quad \nabla_{u_3}u_1=0,$$
$$\nabla_{u_3}u_2=0,\quad \nabla_{u_3}u_3=0.$$

Thus the given structure is a trans-Sasakian structure with $\alpha=0$, $\beta=2$. The components of the curvature tensor are given by

$$R(u_1,u_2)u_2=-4u_1,\quad R(u_2,u_1)u_1=-4u_2,\quad R(u_1,u_3)u_3=-4u_1,\quad R(u_2,u_3)u_3=-4u_2,\quad R(u_3,u_1)u_1=-4u_3,$$
$$R(u_3,u_2)u_2=-4u_3,\quad R(u_1,u_2)u_3=0,\quad R(u_1,u_3)u_2=0,\quad R(u_2,u_3)u_1=0.$$

From the above expressions, the given manifold is a generalized Sasakian space-form with $f_1=\omega-1$, $f_2=-\frac{\omega+3}{3}$ and $f_3=\omega+3$, where ω is a smooth function on M.
The non-zero components of the Ricci tensor are given by

$$S(u_1,u_1)=-8,\quad S(u_2,u_2)=-8,\quad S(u_3,u_3)=-8.$$

Thus we see that $S(V_1,V_2)=-8g(V_1,V_2)$, for every vector fields V_1, V_2 on M. Hence the space-form is an Einstein manifold. The scalar curvature of the manifold is -24.
Let $\lambda=e^{-\frac{az}{2}}+b$, where a and b are scalars, so that, $e^{-\frac{az}{2}}=\lambda-b$. Now $D\lambda=-\frac{a}{2}e^{-\frac{az}{2}}u_3=-\frac{a}{2}(\lambda-b)u_3$. Then

$$\nabla_{u_1}D\lambda=-a(\lambda-b)u_1,\quad \nabla_{u_2}D\lambda=-a(\lambda-b)u_2,\quad \nabla_{u_3}D\lambda=\frac{a^2}{4}(\lambda-b)u_3.$$

Thus $(\Delta_g\lambda)=(\frac{a^2}{4}-2a)(\lambda-b)$. Now $-(\Delta_g\lambda)g(u_i,u_j)+g(\nabla_{u_i}D\lambda,u_j)-\lambda S(u_i,u_j)=g(u_i,u_j)$, $i,j=1,2,3$, gives the following two equations

$$(a-\frac{a^2}{4})(\lambda-b)+8\lambda=1$$

and

$$2a(\lambda-b)+8\lambda=1.$$

Comparing the above two equations, we see that $a=0$, $b=-\frac{7}{8}$ and $\lambda=\frac{1}{8}$ or $a=-4$, $b=\frac{1}{8}$ and $\lambda=e^{2z}+\frac{1}{8}$. Thus the non-trivial solution of the Miao-Tam equation exists on the given manifold. Since the manifold is Einstein and the structure is β-Kenmotsu (as $\alpha=0$), the Theorem 3.3 holds good.
Again, let $\tilde{\lambda}=e^{-\frac{az}{2}}+b$, where a and b are scalars, so that, $e^{-\frac{az}{2}}=\tilde{\lambda}-b$. Now $D\tilde{\lambda}=-\frac{a}{2}e^{-\frac{az}{2}}u_3=-\frac{a}{2}(\tilde{\lambda}-b)u_3$. Then

$$\nabla_{u_1}D\tilde{\lambda}=-a(\tilde{\lambda}-b)u_1,\quad \nabla_{u_2}D\tilde{\lambda}=-a(\tilde{\lambda}-b)u_2,\quad \nabla_{u_3}D\tilde{\lambda}=\frac{a^2}{4}(\tilde{\lambda}-b)u_3.$$

Thus $(\Delta_g\tilde{\lambda})=(\frac{a^2}{4}-2a)(\tilde{\lambda}-b)$. Now $-(\Delta_g\tilde{\lambda})g(u_i,u_j)+g(\nabla_{u_i}D\tilde{\lambda},u_j)-\tilde{\lambda}S(u_i,u_j)=0,\quad i,j=1,2,3$, gives the following two equations

$$(a-\frac{a^2}{4})(\lambda-b)+8\lambda=0$$

and

$$2a(\lambda-b)+8\lambda=0.$$

Solving the last two equations, we see that $\tilde{\lambda}=0$, i.e., the solution is trivial, which ensures the validity of the Theorem 3.8.
Let us consider the potential vector field $V=xe^{2z}u_1+ye^{2z}u_2+\frac{1}{2}(e^{2z}-1)u_3$. Then equation (1.2) is satisfied for that V with $\psi=8-e^{2z}$, i.e., the soliton is steady at $z=\frac{3}{2}\log 2$ and it is expanding or shrinking if z is less than or greater than $\frac{3}{2}\log 2$, respectively. Also $(\pounds_V\eta)(V_1)=e^{2z}\eta(V_1)$, for any vector field V_1 on M. Hence V is an infinitesimal contact transformation. In this way Theorem 4.1 is satisfied.
Next, we suppose that the potential vector field V is the gradient of a smooth function ζ, i.e., $V=D\zeta$. Then

$$D\zeta=e^{-2z}\frac{\partial\zeta}{\partial x}u_1+e^{-2z}\frac{\partial\zeta}{\partial y}u_2+\frac{\partial\zeta}{\partial z}u_3.$$

Therefore,

$$\nabla_{u_1}D\zeta=e^{-4z}\frac{\partial^2\zeta}{\partial x^2}u_1-2e^{-2z}\frac{\partial\zeta}{\partial x}u_3+e^{-4z}\frac{\partial^2\zeta}{\partial x\partial y}u_2+e^{-2z}\frac{\partial^2\zeta}{\partial x\partial z}u_3+2\frac{\partial\zeta}{\partial z}u_1,$$
$$\nabla_{u_2}D\zeta=e^{-4z}\frac{\partial^2\zeta}{\partial y^2}u_2-2e^{-2z}\frac{\partial\zeta}{\partial y}u_3+e^{-4z}\frac{\partial^2\zeta}{\partial y\partial x}u_1+e^{-2z}\frac{\partial^2\zeta}{\partial y\partial z}u_3+2\frac{\partial\zeta}{\partial z}u_2,$$
$$\nabla_{u_3}D\zeta=-2e^{-2z}\frac{\partial\zeta}{\partial x}u_1+e^{-2z}\frac{\partial^2\zeta}{\partial z\partial x}u_1-2e^{-2z}\frac{\partial\zeta}{\partial y}u_2+e^{-2z}\frac{\partial^2\zeta}{\partial z\partial y}u_2+\frac{\partial^2\zeta}{\partial z^2}u_3.$$

Thus the equation $\nabla_{V_1}D\zeta+QV_1+\psi V_1=0$ gives

$$e^{-4z}\frac{\partial^2\zeta}{\partial x^2}+2\frac{\partial\zeta}{\partial z}-8+\psi=0,$$
$$e^{-4z}\frac{\partial^2\zeta}{\partial y^2}+2\frac{\partial\zeta}{\partial z}-8+\psi=0,$$

and

$$\frac{\partial^2\zeta}{\partial z^2}-8+\psi=0.$$

The last three equations satisfy simultaneously only when ζ is a constant. Thus we see that the soliton is trivial, which verifies the Theorem 4.3.

Article Information

Acknowledgements: The authors are thankful to the referees for their valuable suggestions towards the improvement of the paper.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

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