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Some Curvature Tensor Relations on Nearly Cosymplectic Manifolds with Tanaka-Webster Connection

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Abstract

In this article, some curvature properties with respect to Tanaka-Webster connection on nearly cosymplectic manifolds have been studied.

1. Introduction

In 1959 Liberman [18] and in 1967 Blair [6] described cosymplectic manifolds similar to Kähler manifolds as odd-dimensional, respectively. Later in 1970, nearly Kähler's structure manifolds (M,J,g) were introduced by Gray as almost Hermitian manifolds. According to the Levi-Civita connection, the covariant derivative of the almost complex structure is skew symmetric operator. The covariant derivative operator also satisfies

$$(\nabla_X J)X = 0,$$

for every vector field X on M [17]. Blair defined an almost contact manifold with Killing structure tensors the following year, which is a nearly cosymplectic manifold [5]. Furthermore, $(2m+1)-$dimensional manifold M is a normal almost contact metric structure (J,ξ,η,g) with cosymplectic structure in which both the fundamental $2-$form Φ and $1-$form η are closed ([4,6,11]). Nearly cosymplectic manifolds are defined in the same way as cosymplectic (also known as coKähler) manifolds. By the way, almost contact metric structure (φ,ξ,η,g) that provides the condition

$$(\nabla_X \varphi)X = 0, \tag{1.1}$$

is called a nearly cosymplectic structure. Also a smooth manifold M which endowed with almost contact metric structure (φ,ξ,η,g) (1.1) is said to be nearly cosymplectic manifolds.
On the other hand, Tanno first explored a generalized Tanaka-Webster connection for contact metric manifolds through the canonical connection [22]. If the associated CR structure is integrable this connection coincides with the Tanaka-Webster connection. Many authors have studied the Tanaka-Webster connection later. In relation to the generalized Tanaka-Webster connection Ghosh, Prakasha, Ünal and Montano have done important studies on this connection in various structures [13, 19–21]. We, on the other hand, tried to define some curvature tensors on nearly cosymplectic structures according to the tanaka webster connection, taking into account some previous studies. In this study, after the introduction section, the definition and basic curvature properties of the nearly cosymplectic manifolds are given. In later section, according to the conditions, Riemannian curvature tensor, Ricci tensor and scalar curvature tensor of a nearly cosymplectic manifold satisfying Tanaka-Webster connection have been obtained. In section 3, Weyl projective curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, $M-$ projective curvature tensor, pseudo projective and quasi conformal curvature tensor with respect to the generalized Tanaka-Webster connection on nearly cosymplectic manifolds have been defined and finally new studies that can be done depending on these curvature tensor definitions are mentioned.
Throughout this study R is the Riemannian curvature tensor, S is the Ricci tensor, Q is the Ricci operator, r is the scaler curvature tensor.

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2. Nearly Cosymplectic Manifolds

We present some information and curvature properties of nearly cosymplectic manifolds in this section.
Let (M,φ,ξ,η,g) be an $n=(2m+1)-$ dimensional almost contact Riemannian manifold, where φ is a $(1,1)-$tensor field, ξ is the structure vector field, η is a $1-$form and g is the Riemannian metric. This (φ,ξ,η,g)-structure satisfies the following conditions [4],

$$\varphi\xi=0, \quad \eta(\varphi X)=0, \quad \eta(\xi)=1, \tag{2.1}$$

$$\varphi^2X=-X+\eta(X)\xi, \quad \eta(X)=g(X,\xi), \tag{2.2}$$

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y). \tag{2.3}$$

for any vector fields X and Y on M.
From the above definition, φ is skew-symmetric operator according to g, in order that the bilinear form $\Phi := g(.,\varphi.)$ defines a $2-$form on M called fundamental $2-$form. An almost contact metric manifold satisfying $d\eta=2\Phi$ is called a contact metric manifold. Under the circumstances, η is a contact form, i.e., $\eta\wedge(d\eta)^n\neq 0$ everywhere on M [12].
A nearly cosymplectic manifold with the (M,φ,ξ,η,g) form is an almost contact metric manifold such that

$$(\nabla_X\varphi)Y+(\nabla_Y\varphi)X=0, \tag{2.4}$$

for all vector fields X,Y. It is clear that this condition is equivalent to $(\nabla_X\varphi)X=0$.
It is known that in a nearly cosymplectic manifold with the Reeb vector field ξ is Killing and satisfies the $\nabla_\xi\xi=0$ and $\nabla_\xi\eta=0$ conditions. Also the tensor field H of type $(1,1)$ defined by

$$\nabla_X\xi=HX, \tag{2.5}$$

is skew symmetric and anti-commutative with φ. Also H providing $H\xi=0$, $\eta\circ H=0$ features and the following formulas hold ([3, 12, 14, 15, 25]):

$$(\nabla_\xi\varphi)X=\varphi HX=\frac{1}{3}(\nabla_\xi\varphi)X,$$

$$g((\nabla_X\varphi)Y,HZ)=\eta(Y)g(H^2X,\varphi Z)-\eta(X)g(H^2Y,\varphi Z), \tag{2.6}$$

$$(\nabla_XH)Y=g(H^2X,Y)\xi-\eta(Y)H^2X, \tag{2.7}$$

$$tr(H^2)=constant, \tag{2.8}$$

$$R(Y,Z)\xi=\eta(Y)H^2Z-\eta(Z)H^2Y, \tag{2.9}$$

$$S(\xi,Z)=-\eta(Z)tr(H^2), \tag{2.10}$$

$$S(\varphi Y,Z)=S(Y,\varphi Z), \quad \varphi Q=Q\varphi, \tag{2.11}$$

$$S(\varphi Y,\varphi Z)=S(Y,Z)+\eta(Y)\eta(Z)tr(H^2). \tag{2.12}$$

3. Properties of Nearly Cosymplectic Manifolds Satisfying Tanaka Webster Connection

We associate $\widetilde{\nabla}$ with the quantities with respect to the generalized Tanaka-Webster connection throughout this paper.
The generalized Tanaka-Webster connection $\widetilde{\nabla}$ associated to the Levi-Civita connection ∇ is given by [24]

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi - \eta(X)\phi Y \tag{3.1}$$

for any vector fields X, Y on M.
Using (2.3) and (2.4), the above equation yields,

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)HX + g(\nabla_X \xi, Y)\xi - \eta(X)\phi Y \tag{3.2}$$

By taking $Y = \xi$ in (3.2) and using (1.1) and (2.3) we obtain

$$\widetilde{\nabla}_X \xi = 0 \tag{3.3}$$

We now calculate the Riemannian curvature tensor $\widetilde{R}$ using (3.2) as follows:

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z\eta(X)\eta(Z)\phi HY + g(H^2X,Z)\eta(Y)\xi \\ &\quad - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX \\ &\quad + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi\end{aligned} \tag{3.4}$$

Using (2.5) and taking $Z = \xi$ in (3.4) we get

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi + \eta(X)H^2Y - \eta(Y)H^2X. \tag{3.5}$$

On contracting (3.4), we obtain the Ricci tensor $\widetilde{S}$ of a nearly cosymplectic manifold with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ as

$$\widetilde{S}(Y,Z) = S(Y,Z) + 2g(HY,\phi Z) - \eta(Y)(div\phi)(Z) + g(Z,HY)tr(H) - \eta(Z)\eta(Y)tr(H^2) - \eta(Y)\eta(Z)tr(\phi H) + 2g(HZ,HY) \tag{3.6}$$

This gives

$$\widetilde{Q}Y = QY - tr(H^2)Y. \tag{3.7}$$

Contracting with respect to Y and Z in (3.6) , we get

$$\widetilde{r} = r - tr(H^2)(2m+1), \tag{3.8}$$

where $\widetilde{r}$ and r are the scalar curvatures with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ respectively.

4. Some Curvature Tensors On Nearly Cosymplectic Manifolds with Respect to the Generalized Tanaka Webster Connection

The exploration of curvature tensors and curvature properties has an important place in the literature within the scope of the study of structures on Riemannian manifolds. The properties provided by a curvature tensor give us important information about the structure of the manifold.
Until this time, curvature tensors have been defined by many mathematicians and their properties have been studied. Some of these curvature tensors are; weyl protective curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor $M-$projective curvature tensor ... etc.
In this study, we tried to define Weyl protective curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor $M-$ projective curvature tensor, pseudo projective curvature tensor and quasi conformal curvature tensor with respect to the generalized Tanaka-Webster connection on nearly cosymplectic manifolds, based on the curvature tensors previously defined.
In the $n-$dimensional space Vn, the Weyl projective curvature tensor is given by [12].

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2m}\{S(Y,Z)X - S(X,Z)Y\}.$$

where g is the associated Riemannian metric, R, S are Riemannian curvature tensor, Ricci tensor respectively.
Based on this definition, we give the Weyl curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka-Webster connection is as follows.

Definition 4.1. *On a nearly cosymlectic manifold M of dimension $n>2$, the Weyl projective curvature tensor $\widetilde{W}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{W}(X,Y)Z=\widetilde{R}(X,Y)Z-\frac{1}{2m}[\widetilde{S}(Y,Z)X-\widetilde{S}(X,Z)Y] \tag{4.1}$$

for all vector fields X,Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$are the Riemannian curvature tensor and Ricci tensor respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.2. *On a nearly cosymlectic manifold M of dimension $n>2$, the Weyl projective curvature tensor $\widetilde{W}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}\widetilde{W}(X,Y)Z&=R(X,Y)Z-g(Z,HX)HY-g(H^2Y,Z)\eta(X)\xi-2g(Y,HX)\phi Z+\eta(X)\eta(Z)\phi HY+g(H^2X,Z)\eta(Y)\xi-\eta(Y)(\nabla_X\phi)Z\\&-\eta(Y)g(HX,\phi Z)\xi+g(Z,HY)HX+\eta(Z)\eta(X)H^2Y-\eta(Z)\eta(Y)H^2X-\eta(Y)\eta(Z)\phi HX\\&+\eta(X)(\nabla_Y\phi)Z+\eta(X)g(HY,\phi Z)\xi-\frac{1}{2m}[S(Y,Z)X+2g(HY,\phi Z)X-\eta(Y)(div\phi)(Z)X+g(Z,HY)tr(H)X\\&-\eta(Z)\eta(Y)tr(H^2)X-\eta(Y)\eta(Z)tr(\phi H)X+2g(HZ,HY)XS(X,Z)Y-2g(HX,\phi Z)Y+\eta(X)div(\phi)ZY-g(Z,HX)tr(H)Y\\&+\eta(Z)\eta(X)tr(H^2)Y+\eta(X)\eta(Z)tr(\phi H)Y-2g(HZ,HX)Y]\end{aligned}$$

Proof. Using (3.4) and (3.6) in (4.1), we have the equation above. □

In the $n-$dimensional space Vn, the concircular curvature tensor is given by [12].

$$C(X,Y)Z=R(X,Y)Z-\frac{r}{2m(2m+1)}\{g(Y,Z)X-g(X,Z)Y\}$$

where g is the associated Riemannian metric, R, S and r are Riemannian curvature tensor, Ricci tensor and scalar curvature tensor respectively. Based on this definition, we give the concircular curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka-Webster connection is as follows.

Definition 4.3. *The concircular curvature tensor [26] $\widetilde{C}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{C}(X,Y)Z=\widetilde{R}(X,Y)Z-\frac{\widetilde{r}}{2m(2m+1)}\{g(Y,Z)X-g(X,Z)Y\} \tag{4.2}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$ and $\widetilde{r}$are the Riemannian curvature tensor, scalar curvature tensor respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.4. *In a nearly cosymlectic manifold M, the concircular curvature $\widetilde{W}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}\widetilde{C}(X,Y)Z&=R(X,Y)Z-g(Z,HX)HY-g(H^2Y,Z)\eta(X)\xi-2g(Y,HX)\phi Z\\&+\eta(X)\eta(Z)\phi HY+g(H^2X,Z)\eta(Y)\xi-\eta(Y)(\nabla_X\phi)Z-\eta(Y)g(HX,\phi Z)\xi\\&+g(Z,HY)HX+\eta(Z)\eta(X)H^2Y-\eta(Z)\eta(Y)H^2X-\eta(Y)\eta(Z)\phi HX\\&+\eta(X)(\nabla_Y\phi)Z+\eta(X)g(HY,\phi Z)\xi-\frac{r-tr(H^2)(2m+1)}{2m(2m+1)}\{g(Y,Z)X-g(X,Z)Y\}\end{aligned}$$

Proof. Using (3.4) and (3.8) in (4.2), we have the equation above. □

In the $n-$dimensional space Vn, the conharmonic curvature tensor is given by [12].

$$K(X,Y)Z=R(X,Y)Z-\frac{1}{(2m-1)}\{S(Y,Z)X-gSX,Z)Y+g(Y,Z)QX-g(X,Z)QY\}$$

where g is the associated Riemannian metric, R, S and Q are Riemannian curvature tensor, Ricci tensor and the Ricci operator respectively. Based on this definition, we give the conharmonic curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka Webster connection is as follows.

Definition 4.5. *On a nearly cosymlectic manifold M of dimension $n>2$, the conharmonic curvature tensor $\widetilde{K}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{K}(X,Y)Z=\widetilde{R}(X,Y)Z-\frac{1}{(2m-1)}[\widetilde{S}(Y,Z)X-\widetilde{S}(X,Z)Y+g(Y,Z)\widetilde{Q}X-g(X,Z)\widetilde{Q}Y] \tag{4.3}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$and $\widetilde{Q}$are the Riemannian curvature tensor, Ricci tensor and Ricci operator, respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.6. *On a nearly cosymlectic manifold M of dimension $n>2$, the conharmonic curvature tensor $\widetilde{K}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by [1]*

$$\begin{aligned}\widetilde{K}(X,Y)Z=&R(X,Y)Z-g(Z,HX)HY-g(H^2Y,Z)\eta(X)\xi-2g(Y,HX)\phi Z+\eta(X)\eta(Z)\phi HY+g(H^2X,Z)\eta(Y)\xi-\eta(Y)(\nabla_X\phi)Z\\&-\eta(Y)g(HX,\phi Z)\xi+g(Z,HY)HX+\eta(Z)\eta(X)H^2Y-\eta(Z)\eta(Y)H^2X-\eta(Y)\eta(Z)\phi HX+\eta(X)(\nabla_Y\phi)Z\\&+\eta(X)g(HY,\phi Z)\xi-\frac{1}{(2m-1)}[S(Y,Z)X+2g(HY,\phi Z)X-\eta(Y)div(\phi)ZX+g(Z,HY)tr(H)X-\eta(Z)\eta(Y)tr(H^2)X\\&-\eta(Y)\eta(Z)tr(\phi H)X+2g(HZ,HY)X-S(X,Z)Y-2g(HX,\phi Z)Y+\eta(X)div(\phi)ZY-g(Z,HX)tr(H)Y+\eta(Z)\eta(X)tr(H^2)Y\\&+\eta(X)\eta(Z)tr(\phi H)Y-2g(HZ,HX)Y+g(Y,Z)QX-g(Y,Z)tr(H^2)X-g(X,Z)QY+g(X,Z)tr(H^2)Y].\end{aligned}$$

Proof. Using (3.4) ,(3.6) and (3.7) in (4.3), we have the equation above. □

In the $n-$dimensional space Vn, the conformal curvature tensor is given by [12].

$$V(X,Y)Z=R(X,Y)Z-\frac{1}{2m-1}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]+\frac{r}{(2m)(2m-1)}[g(Y,Z)X-g(X,Z)Y]$$

where g is the associated Riemannian metric, R, S, r and Q are Riemannian curvature tensor, Ricci tensor, scalar curvature tensor and the Ricci operator respectively.
Based on this definition, we give the conformal curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka Webster connection is as follows.

Definition 4.7. *On a nearly cosymlectic manifold M of dimension $n>2$, the conformal curvature tensor $\widetilde{V}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{V}(X,Y)Z=\widetilde{R}(X,Y)Z-\frac{1}{2m-1}[\widetilde{S}(Y,Z)X-\widetilde{S}(X,Z)Y+g(Y,Z)\widetilde{Q}X-g(X,Z)\widetilde{Q}Y]+\frac{\widetilde{r}}{(2m)(2m-1)}[g(Y,Z)X-g(X,Z)Y] \tag{4.4}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$, $\widetilde{Q}$ and $\widetilde{r}$ are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.8. *On a nearly cosymlectic manifold M of dimension $n>2$, the conformal curvature tensor $\widetilde{V}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}\widetilde{V}(X,Y)Z=&R(X,Y)Z-g(Z,HX)HY-g(H^2Y,Z)\eta(X)\xi-2g(Y,HX)\phi Z+\eta(X)\eta(Z)\phi HY+g(H^2X,Z)\eta(Y)\xi-\eta(Y)(\nabla_X\phi)Z\\&-\eta(Y)g(HX,\phi Z)\xi+g(Z,HY)HX+\eta(Z)\eta(X)H^2Y\\&-\eta(Z)\eta(Y)H^2X-\eta(Y)\eta(Z)\phi HX+\eta(X)(\nabla_Y\phi)Z+\eta(X)g(HY,\phi Z)\xi\\&-\frac{1}{2m-1}[S(Y,Z)X+2g(HY,\phi Z)X-\eta(Y)(div\phi)(Z)X+g(Z,HY)tr(H)X\\&-\eta(Z)\eta(Y)tr(H^2)X-\eta(Y)\eta(Z)tr(\phi H)X+2g(HZ,HY)X\\&-S(X,Z)Y-2g(HX,\phi Z)Y+\eta(X)div(\phi)ZY-g(Z,HX)tr(H)Y\\&+\eta(Z)\eta(X)tr(H^2)Y+\eta(X)\eta(Z)tr(\phi H)Y-2g(HZ,HX)Y\\&+g(Y,Z)QX-g(Y,Z)tr(H^2)X-g(X,Z)QY+g(X,Z)tr(H^2)Y]+\frac{r-tr(H^2)(2m+1)}{(2m)(2m-1)}[g(Y,Z)X-g(X,Z)Y]\end{aligned}$$

Proof. Using (3.4), (3.6), (3.7) and (3.8) in (4.4), we have the equation above. □

In 1971 on an $n-$dimensional Riemannian manifold, G. P. Pokhariyal and R.S. Mishra [22] defined $M-$projective curvature tensor field as

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{4m}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$

It has been further studied in [7–9].

Definition 4.9. *On a nearly cosymlectic manifold M of dimension $n > 2$, the $M-$projective curvature tensor $\widetilde{M}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{M}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{4m}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y] \tag{4.5}$$

for all vector fields X,Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$ and $\widetilde{Q}$ are the Riemannian curvature tensor, Ricci tensor and Ricci operator respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.10. *On a nearly cosymlectic manifold M of dimension $n > 2$, the $M-$projective curvature tensor $\widetilde{M}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}
\widetilde{M}(X,Y)Z &= R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z + \eta(X)\eta(Z)\phi HY \\
&\quad + g(H^2X,Z)\eta(Y)\xi - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y \\
&\quad - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi \\
&\quad - \frac{1}{4m}[S(Y,Z)X + 2g(HY,\phi Z)X - \eta(Y)(div\phi)(Z)X + g(Z,HY)tr(H)X \\
&\quad - \eta(Z)\eta(Y)tr(H^2)X - \eta(Y)\eta(Z)tr(\phi H)X + 2g(HZ,HY)X - S(X,Z)Y - 2g(HX,\phi Z)Y + \eta(X)div(\phi)ZY \\
&\quad - g(Z,HX)tr(H)Y + \eta(Z)\eta(X)tr(H^2)Y \\
&\quad + \eta(X)\eta(Z)tr(\phi H)Y - 2g(HZ,HX)Y + g(Y,Z)QX - g(Y,Z)tr(H^2)X - g(X,Z)QY + g(X,Z)tr(H^2)Y]
\end{aligned}$$

Proof. Using (3.4), (3.6) and (3.7) in (4.5), we have the equation above. □

In 2002 on a $n-$dimensional $n > 2$ Riemannian manifold, Prasad [12] defined pseudo-projective curvature tensor P as

$$P(X,Y)Z = \alpha R(X,Y)Z + \beta\,[S(Y,Z)X - S(X,Z)Y] - \frac{r}{2m+1}\left(\frac{\alpha}{2m} + \beta\right)[g(Y,Z)X - g(X,Z)Y]$$

where α, β are non-zero constants, g is the associated Riemannian metric, R, S and r are Riemannian curvature tensor, Ricci tensor and scalar curvature tensor respectively.
Based on this definition, we give the quasi conformal curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka Webster connection is as follows.

Definition 4.11. *On a nenarly cosymlectic manifold M of dimension $n > 2$, the pseuso-projective curvature tensor $\widetilde{P}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{P}(X,Y)Z = \alpha\widetilde{R}(X,Y)Z + \beta\left[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y\right] - \frac{\widetilde{r}}{2m+1}\left(\frac{\alpha}{2m} + \beta\right)[g(Y,Z)X - g(X,Z)Y] \tag{4.6}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$, $\widetilde{Q}$ and $\widetilde{r}$ are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.12. *On a nenarly cosymlectic manifold M of dimension $n > 2$, the pseuso-projective curvature tensor $\widetilde{P}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}
\widetilde{P}(X,Y)Z &= \alpha[R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z + \eta(X)\eta(Z)\phi HY + g(H^2X,Z)\eta(Y)\xi - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi \\
&\quad + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi] - \beta[S(Y,Z)X + 2g(HY,\phi Z)X \\
&\quad - \eta(Y)(div\phi)(Z)X + g(Z,HY)tr(H)X - \eta(Z)\eta(Y)tr(H^2)X - \eta(Y)\eta(Z)tr(\phi H)X \\
&\quad + 2g(HZ,HY)X - S(X,Z)Y - 2g(HX,\phi Z)Y + \eta(X)div(\phi)ZY - g(Z,HX)tr(H)Y \\
&\quad + \eta(Z)\eta(X)tr(H^2)Y + \eta(X)\eta(Z)tr(\phi H)Y - 2g(HZ,HX)Y] - \frac{r - tr(H^2)(2m+1)}{2m+1}\left(\frac{\alpha}{2m} + \beta\right)[g(Y,Z)X - g(X,Z)Y]
\end{aligned}$$

where α and β are constants such that $\alpha = 1$ and $\beta = -\frac{1}{2m}$ then the definition of pseudo projective curvature tensor takes the form

$$\widetilde{P}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{(n-1)}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y] = \widetilde{W}(X,Y)Z$$

where $\widetilde{W}$ is the Weyl projective curvature tensor of a nearly cosymlectic manifold.

Proof. Using (3.4), (3.6), (3.7) and (3.8) in (4.6), we have the equation above. □

Quasi-conformal curvature tensor has introduced by K. Yano and S. Sawaki in 1968 for or a $n-$dimensional Riemannian manifold and the quasi-conformal curvature tensor Q is given by [12].

$$Q(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{2m+1}\left(\frac{a}{2m} + 2b\right)[g(Y,Z)X - g(X,Z)Y]$$

where a and b are two scalars, and r is the scalar curvature of the manifold.
Based on this definition, we give the quasi conformal curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka-Webster connection is as follows.

Definition 4.13. *On a nearly cosymlectic manifold M of dimension $n > 2$, the quasi conformal curvature tensor $\widetilde{Q}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{Q}(X,Y)Z = a\widetilde{R}(X,Y)Z + b\left[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y\right] - \frac{\widetilde{r}}{2m+1}\left(\frac{a}{2m} + 2b\right)[g(Y,Z)X - g(X,Z)Y] \tag{4.7}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$, $\widetilde{Q}$ and $\widetilde{r}$ are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.14. *On a nearly cosymlectic manifold M of dimension $n > 2$, the quasi conformal curvature tensor $\widetilde{Q}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}
\widetilde{Q}(X,Y)Z &= a[R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z + \eta(X)\eta(Z)\phi HY + g(H^2X,Z)\eta(Y)\xi - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi \\
&\quad + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi] + b[S(Y,Z)X + 2g(HY,\phi Z)X \\
&\quad - \eta(Y)(div\phi)(Z)X + g(Z,HY)tr(H)X - \eta(Z)\eta(Y)tr(H^2)X - \eta(Y)\eta(Z)tr(\phi H)X + 2g(HZ,HY)X - S(X,Z)Y - 2g(HX,\phi Z)Y + \eta(X)div(\phi)ZY \\
&\quad - g(Z,HX)tr(H)Y + \eta(Z)\eta(X)tr(H^2)Y + \eta(X)\eta(Z)tr(\phi H)Y - 2g(HZ,HX)Y + g(Y,Z)QX - g(Y,Z)tr(H^2)X - g(X,Z)QY + g(X,Z)tr(H^2)Y] \\
&\quad - \frac{r - tr(H^2)(2m+1)}{2m+1}\left(\frac{a}{2m} + 2b\right)[g(Y,Z)X - g(X,Z)Y].
\end{aligned}$$

From the definition of quasi-conformal curvature tensor, if we take $a = 1$ and $b = -\frac{1}{2m-1}$, then the above equality takes the form as

$$\widetilde{Q}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{2m-1}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y] + \frac{\widetilde{r}}{(2m)(2m-1)}[g(Y,Z)X - g(X,Z)Y] = \widetilde{V}(X,Y)Z \tag{4.8}$$

where $\widetilde{V}$ is the conformal curvature tensor.

Proof. Using (3.4), (3.6), (3.7) and (3.8) in (4.7), we have the equation above. □

5. Conclusion

In this study, various curvature tensors have been defined in nearly cosyplectic structures with respect to the generalized Tanaka-Webster connection. Starting from these new connection curvature tensor definitions, conditions of being flat, conditions of being symmetri such as Ricci symmetric, semi-symmetric, conditions of being recurrent can be examined for each curvature tensor.

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Author's contributions

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