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Cyclic (α,β)-Admissible Mappings in Modular Spaces and Applications to Integral Equations

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Abstract

The main concern of this study is to present a generalization of Banach's fixed point theorem in some classes of modular spaces, where the modular is convex and satisfying the Δ_2-condition. In this work, the existence and uniqueness of fixed point for $(\alpha,\beta)-(\psi,\varphi)-$contractive mapping and $\alpha-\beta-\psi-$weak rational contraction in modular spaces are proved. Some examples are supplied to support the usability of our results. As an application, the existence of a solution for an integral equation of Lipschitz type in a Musielak-Orlicz space is presented.

1. Introduction and Preliminaries

It is well known fixed point theorems play important roles and have applications in mathematics analysis, particularly in differential and integral equations. One of the most popular fixed point theorem is Banach fixed point theorem [6]. By using this theorem, most authors have proved several fixed point theorems for various mappings [13, 21, 28]. Such as, Dutta and Choudhury proved $(\psi,\phi)-$contractive mappings in complete metric space [11]. Samet et al. introduced the concept of $\alpha-\varphi-$ contractive type mappings and established various fixed point theorems [32]. Later, Salimi et al. modified the concept of $\alpha-\varphi-$ contractive type mappings [31]. Alizadeh et al. [4] developed a new fixed point theorem in complete metric spaces. They introduced the concept of cyclic $(\alpha,\beta)-$admissible and $(\alpha,\beta)-(\psi,\phi)-$contractive mappings and established some fixed point results in metric spaces.
On the other hand, some authors introduced a new concept of modular vector spaces which are natural generalizations of many classical function spaces. Firstly, Nakano initiated the concept of modular spaces [26]. Later, some authors proved new fixed point theorems of Banach type in modular spaces [12, 18, 19, 22, 23, 24, 29, 33]. Then, also the concept of the fixed point theory was studied in modular metric, modular function and modular vector spaces. [1, 2, 3, 5, 8, 9, 10, 14, 15, 16, 17, 20, 30, 34].
In this work, some fixed point results as a generalization of Banach's fixed point theorem are presented using some convenient constants in the contraction assumption in modular spaces. Motivated by [4] and [25], some fixed point results for $(\alpha,\beta)-(\psi,\phi)-$contractive mappings in modular spaces are proved. Some examples are supplied in order to support the usability of our results. As an application the existence and uniqueness of solutions for an integral equation of Lipschitz type in a Musielak-Orlicz space are showed.

Definition 1.1. *[25, 27] Let X be an arbitrary vector space. A functional $\rho : X \to [0,\infty)$ is called a modular if, for any x,y in X, the following conditions hold:*

(a) $\rho(x) = 0$ *if and only if* $x = 0$,
(b) $\rho(-x) = \rho(x)$,
(c) $\rho(\alpha x+\beta y) \le \rho(x)+\rho(y)$, *whenever* $\alpha+\beta = 1$ *and* $\alpha,\beta \ge 0$.

If (c) is replaced with $\rho(\alpha x+\beta y) \le \alpha^s\rho(x)+\beta^s\rho(y)$ where $\alpha^s+\beta^s = 1$, $\alpha,\beta \ge 0$, and $s \in (0,1]$, then ρ is called s-convex modular. If $s = 1$, then we say that ρ is convex modular. The following are some consequences of condition (c).

Remark 1.2. *[7]*

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(a) For $a,b \in \mathbb{R}$ with $|a| < |b|$ we have
$\rho(ax) < \rho(bx)$ for all $x \in X$.

(b) For $a_1,...,a_n \in R^+$ with $\sum_{i=1}^n a_i = 1$, we have

$$\rho(\sum_{i=1}^n a_i x_i) = \rho(\sum_{i=1}^n x_i) \text{ for any } x_1,...,x_n \in X.$$

Remark 1.3. *[26] A modular ρ defines a corresponding modular space, i.e. the space is given by*

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Definition 1.4. *A sequence $\{x_n\}$ in modular space X_ρ is said to be:*

(a) ρ-convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(b) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n,m \to \infty$.

(c) X_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(d) ρ satisfies Δ_2-condition if $\rho(2x_n) \to 0$ as $n \to \infty$, whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Definition 1.5. *[4] Let $T : X \to X$ be a mapping and $\alpha,\beta : X \to \mathbb{R}^+$ be two functions. We say that T is a cyclic (α,β)-admissible mapping if*

(i) $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(Tx) \geq 1$,
(ii) $\beta(x) \geq 1$ for some $x \in X$ implies $\alpha(Tx) \geq 1$.

Definition 1.6. *[4] Let Ψ be the set of continuous and increasing functions $\psi : [0,\infty) \to [0,\infty)$ and Φ be the set of lower semicontinuous functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) = 0$ iff $t = 0$. Let X be a metric space and $T : X \to X$ be a cyclic (α,β)-admissible mapping. We say that T is a $(\alpha,\beta)-(\psi,\phi)-$contractive mapping if*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$$

for $x,y \in X$, where $\psi \in \Psi$ and $\phi \in \Phi$.

2. Main Results

Let Ψ and Φ be defined as in Definition 1.6. Let X_ρ be a nonempty set and $T : X_\rho \to X_\rho$ be an arbitrary mapping. We say that $x \in X_\rho$ is a fixed point of T, if $x = Tx$. We denote by $Fix(T)$ the set of all fixed points of T. In the sequel, suppose the modular ρ is convex and satisfies the Δ_2-condition.

Definition 2.1. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping. We say that T is a $(\alpha,\beta)-(\psi,\phi)-$contractive mapping if*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(\rho(Tx - Ty)) \leq \psi(\rho(x-y)) - \phi(\rho(x-y)) \tag{2.1}$$

for $x,y \in X_\rho$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Theorem 2.2. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a $(\alpha,\beta)-(\psi,\phi)-$contractive mapping. Suppose that the following conditions hold:*

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a cyclic $(\alpha,\beta)-$admissible mapping and $\alpha(x_0) \geq 1$ then $\beta(x_1) = \beta(Tx_0) \geq 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geq 1$. By continuing this process, we get $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. Again, since T is a cyclic (α,β)-admissible mapping and $\beta(x_0) \geq 1$, by the similar method, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$. Therefore by (2.1), we have

$$\begin{aligned} \psi(\rho(x_n - x_{n+1})) &\leq \psi(\rho(x_{n-1} - x_n)) - \phi(\rho(x_{n-1} - x_n)) \\ &\leq \psi(\rho(x_{n-1} - x_n)) \end{aligned} \tag{2.2}$$

and since ψ is increasing, we get

$$\rho(x_n - x_{n+1}) \leq \rho(x_{n-1} - x_n)$$

for all $n \in \mathbb{N}$. So, $\{\rho_n := \rho(x_n - x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Then, there exists $r \geq 0$ such that $\lim_{n\to\infty} \rho_n = r$. We shall show that $r = 0$. By taking the limsup on both sides of (2.2), we have

$$\lim_{n\to\infty} \rho(x_n - x_{n+1}) = 0. \tag{2.3}$$

Now, we want to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose to the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, and for $n(k) > m(k) > k$, we have

$$\rho(x_{2n(k)} - x_{2m(k)}) \geq \varepsilon \text{ and } \rho(2(x_{2n(k)-1} - x_{2m(k)})) < \varepsilon. \tag{2.4}$$

Now for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leq \rho(x_{2n(k)} - x_{2m(k)}) \\ &\leq \rho(2(x_{2n(k)} - x_{2n(k)-1})) + \rho(2(x_{2n(k)-1} - x_{2m(k)})) \\ &< \rho(2(x_{2n(k)} - x_{2n(k)-1})) + \varepsilon. \end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (2.3), we get

$$\lim_{k\to\infty} \rho(x_{2n(k)} - x_{2m(k)}) = \varepsilon. \tag{2.5}$$

Since

$$\begin{aligned} \rho(x_{2n(k)+1} - x_{2m(k)+1}) &= \rho(x_{2n(k)+1} - x_{2n(k)} + x_{2n(k)} - x_{2m(k)+1}) \\ &\leq \rho(2(x_{2n(k)+1} - x_{2n(k)})) + \rho(2(x_{2n(k)} - x_{2m(k)+1})) \end{aligned}$$

and

$$\rho(2(x_{2n(k)} - x_{2m(k)+1})) = \rho(2(x_{2n(k)} - x_{2m(k)} + x_{2m(k)} - x_{2m(k)+1})) \leq \rho(4(x_{2n(k)} - x_{2m(k)}) + \rho(4(x_{2m(k)} - x_{2m(k)+1}))$$

then by taking the limit as $k \to +\infty$ in above inequality and using (2.3) and (2.5), we deduce that

$$\lim_{k\to\infty} \rho(x_{2n(k)+1} - x_{2m(k)+1}) = \varepsilon. \tag{2.6}$$

Now, by (2.1) and $\alpha(x_{2n(k)})\beta(x_{2m(k)}) \geq 1$ for all $k \in \mathbb{N}$, we get

$$\psi(\rho(x_{2n(k)+1} - x_{2m(k)+1})) \leq \psi(\rho(x_{2n(k)} - x_{2m(k)})) - \phi(\rho(x_{2n(k)} - x_{2m(k)})). \tag{2.7}$$

By taking the limsup on both sides of (2.7), applying (2.4) and (2.6), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon).$$

That is, $\varepsilon = 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X_ρ is a complete modular space, then there is a $z \in X_\rho$ such that $x_n \to z$ as $n \to \infty$. First, we assume that T is continuous. Hence, we deduce

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So z is a fixed point of T . Now, assume that (c) holds. That is, $\alpha(x_n)\beta(z) \geq 1$. From (2.1) we have

$$\psi(\rho(x_{n+1} - Tz)) \leq \psi(\rho(x_n - z)) - \phi(\rho(x_n - z)). \tag{2.8}$$

By taking the limsup on both sides of (2.8), we get $\psi(\rho(z - Tz)) = 0$. Then $\rho(z - Tz) = 0$. i.e., $z = Tz$. To prove the uniqueness of fixed point, suppose that z and z^* are two fixed points of T . From condition (c) we have, $\alpha(z)\beta(z^*) \geq 1$, it follows from (2.1) that

$$\psi(\rho(z - Tz^*)) \leq \psi(\rho(z - z^*)) - \phi(\rho(z - z^*)).$$

So $\phi(\rho(z - z^*)) = 0$ and hence $\rho(z - z^*) = 0$ i.e., $z = z^*$. □

Corollary 2.3. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic $(\alpha, \beta)-$admissible mapping such that*

$$\alpha(x)\beta(y)\psi(\rho(Tx - Ty)) \leq \psi(\rho(x - y)) - \phi(\rho(x - y))$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Let $\alpha(x)\beta(y) \geq 1$ for $x, y \in X_\rho$. Then by (2.8), we have

$$\psi(\rho(Tx-Ty)) \leq \psi(\rho(x-y)) - \phi(\rho(x-y)).$$

This implies that the inequality (2.1) holds. Therefore, the proof follows from Theorem 2.2. □

Corollary 2.4. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α,β)-admissible mapping such that*

$$(\alpha(x)\beta(y)+1)^{\psi(\rho(fx-fy))} \leq 2^{\psi(\rho(x-y))-\phi(\rho(x-y))}$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$ and $\phi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Example 2.5. *Let $X_\rho = [-2,\infty] \to \mathbb{R}$, $\rho(x) = |x|$ for all $x \in X_\rho$, and $T : X_\rho \to X_\rho$ by*

$$Tx = \begin{cases} \frac{x^2}{3} & , \quad x \in [-2,2] \\ \sqrt{x} & , \quad otherwise. \end{cases}$$

Define $\psi, \phi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = 3t, \phi(t) = t$ and $\alpha, \beta : X_\rho \to [0,+\infty)$ by

$$\alpha(x) = \begin{cases} 1 & , \quad x \in [-2, \frac{4}{3}] \\ 0 & , \quad otherwise. \end{cases}$$

and

$$\beta(x) = \begin{cases} 1 & , \quad x \in [\frac{4}{3}, 2] \\ 0 & , \quad otherwise. \end{cases}$$

Now, we prove that the hypotheses (a) and (c) of Corollary 2.4 are satisfied by T and hence T has a fixed point. Let $\alpha(x) \geq 1$ for some $x \in X_\rho$. Then $x \in [-2, \frac{4}{3}]$ and so $Tx \in [\frac{4}{3}, 2]$. Therefore, $\beta(Tx) \geq 1$. Similarly, if $\beta(x) \geq 1$ then $\alpha(x) \geq 1$. Then T is a cyclic (α,β)-admissible mapping and that the hypotheses (a) and (c) of Corollary 2.4 hold.
Now, for all $x \in [-2, \frac{4}{3}]$ and $y \in [\frac{4}{3}, 2]$, we get

$$\begin{aligned}(\alpha(x)\beta(y)+1)^{\psi(\rho(fx-fy))} &= 2^{3\rho(fx-fy)} \\ &= 2^{3\left|\frac{x^2}{3}-\frac{y^2}{3}\right|} \\ &= 2^{|x-y||x+y|} \\ &\leq 2^{2|x-y|} = 2^{3|x-y|-|x-y|} \\ &= 2^{\psi(\rho(x-y))-\phi(\rho(x-y))}\end{aligned}$$

Otherwise, if $\alpha(x)\beta(y) = 0$, we have

$$(\alpha(x)\beta(y)+1)^{\psi(\rho(fx-fy))} = 1 \leq 2^{\psi(\rho(x-y))-\phi(\rho(x-y))}$$

Therefore, Corollary 2.4 implies that T has a fixed point.

Corollary 2.6. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α,β)-admissible mapping. Assume that there exists $\ell > 1$ such that*

$$(\psi(\rho(Tx-Ty))+\ell)^{\alpha(x)\beta(y)} \leq \psi(\rho(x-y)) - \phi(\rho(x-y)) + \ell$$

for all $x, y \in X_\rho$ where $\psi \in \Psi$ and $\varphi \in \Phi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Example 2.7. *Let $X_\rho = \mathbb{R}^+$, $\rho(x) = |x|$ for all $x \in X_\rho$, and $T : X_\rho \to X_\rho$ by*

$$Tx = \begin{cases} \frac{x^2+x}{4} & , \quad x \in [0,1] \\ 2x & , \quad otherwise. \end{cases}$$

Define $\psi, \phi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = t, \phi(t) = \frac{t}{4}$ and $\alpha, \beta : X_\rho \to [0,+\infty)$ by

$$\alpha(x) = \beta(x) = \begin{cases} 1 & , \quad x \in [0,1] \\ 0 & , \quad otherwise. \end{cases}$$

Now, we prove that the hypotheses (a) and (c) of Corollary 2.6 are satisfied by T and hence T has a fixed point. Proceeding as in the Example 2.5, we deduce that T is a cyclic (α,β)-admissible mapping and that the hypotheses (a) and (c) of Corollary 2.6 hold.
Now, for all $x \in [0,1]$ and all $y \in [0,1]$, we get

$$\begin{aligned}(\psi(\rho(Tx-Ty))+\ell)^{\alpha(x)\beta(y)} &= |Tx-Ty|+\ell \\ &\le \frac{1}{4}|x-y|\,|x+y+1|+\ell \\ &\le \frac{3}{4}|x-y|+\ell \\ &= |x-y|-\frac{1}{4}|x-y|+\ell \\ &= \psi(\rho(x-y))-\phi(\rho(x-y))+\ell.\end{aligned}$$

Otherwise, if $\alpha(x)\beta(y) = 0$, we have

$$(\psi(\rho(Tx-Ty))+\ell)^{\alpha(x)\beta(y)} = 1 \le \psi(\rho(x-y))-\phi(\rho(x-y))+\ell.$$

Therefore, Corollary 2.6 implies that T has a fixed point.

Definition 2.8. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping. We say that T is $\alpha-\beta-\psi-$ weak rational contraction if $\alpha(x)\beta(y) \ge 1$ for some $x,y \in X_\rho$ such that*

$$\rho(Tx-Ty) \le M(x,y) - \psi(M(x,y))$$

where $\psi \in \Psi$ and

$$M(x,y) = \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{[1+\rho(x-Tx)]\rho(y-Ty)}{\rho(x-y)+1}\}.$$

Theorem 2.9. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be $\alpha-\beta-\psi-$weak rational contraction. Assume that the following assertions hold:*

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \ge 1$ and $\beta(x_0) \ge 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \ge 1$ for all $n \in \mathbb{N}$, then $\beta(x) \ge 1$,

then T has a fixed point. Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a cyclic $(\alpha,\beta)-$admissible mapping and $\alpha(x_0) \ge 1$ then $\beta(x_1) = \beta(Tx_0) \ge 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \ge 1$. By continuing this process, we get $\alpha(x_{2n}) \ge 1$ and $\beta(x_{2n-1}) \ge 1$ for all $n \in \mathbb{N}$. Again, since T is a cyclic $(\alpha,\beta)-$admissible mapping and $\beta(x_0) \ge 1$, by the similar method, we have $\beta(x_{2n}) \ge 1$ and $\alpha(x_{2n-1}) \ge 1$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \ge 1$ and $\beta(x_n) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \ge 1$ for all $n \in \mathbb{N}$. Since T is $\alpha-\beta-\psi-$weak rational contraction, we get

$$\rho(x_n - x_{n+1}) \le M(x_{n-1},x_n) - \psi(M(x_{n-1},x_n)) \tag{2.9}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\{\rho(x_{n-1}-x_n), \rho(x_{n-1}-Tx_{n-1}), \rho(x_n-Tx_n), \frac{[1+\rho(x_{n-1}-Tx_{n-1})]\rho(x_n-Tx_n)}{\rho(x_{n-1}-x_n)+1}\} \\ &= \max\{\rho(x_{n-1}-x_n), \rho(x_n-x_{n+1})\}.\end{aligned}$$

Now, suppose that there exists $n_0 \in \mathbb{N}$ such that $\rho(x_{n_0}-x_{n_{0+1}}) > \rho(x_{n_{0-1}}-x_{n_0})$. Therefore $M(x_{n_{0-1}},x_{n_0}) = \rho(x_{n_0}-x_{n_{0+1}})$ and so from (2.9), we get

$$\rho(x_{n_0}-x_{n_0+1}) \le \rho(x_{n_0}-x_{n_{0+1}}) - \psi(\rho(x_{n_0}-x_{n_{0+1}})).$$

This implies that $\psi(\rho(x_{n_0}-x_{n_0+1})) = 0$, i.e., $\rho(x_{n_0}-x_{n_0+1}) = 0$, which is a contradiction. Hence, $\rho(x_n-x_{n+1}) \le \rho(x_{n-1}-x_n)$ for all $n \in \mathbb{N}$. That is the sequence $\{\rho_n : \rho(x_n-x_{n+1})\}$ is decreasing and so there exists $r \ge 0$ such that $\rho_n \to r$ as $n \to \infty$. Taking the limit as $n \to \infty$ in (2.9), we have

$$r \le r - \psi(r).$$

This implies that $\psi(r)=0$. Therefore, the property of ψ implies that $r=0$. That is

$$\lim_{n\to\infty}\rho(x_{n+1}-x_n)=0. \tag{2.10}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon>0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, for $n(k)>m(k)>k$, we have

$$\rho(x_{2n(k)}-x_{2m(k)})\geq\varepsilon \text{ and } \rho(2(x_{2n(k)-1}-x_{2m(k)}))<\varepsilon. \tag{2.11}$$

For all $k\in\mathbb{N}$, we have

$$\varepsilon\leq\rho(x_{2n(k)}-x_{2m(k)})\leq\rho(2(x_{2n(k)}-x_{2n(k)-1}))+\rho(2(x_{2n(k)-1}-x_{2m(k)}))<\rho(2(x_{2n(k)}-x_{2n(k)-1}))+\varepsilon.$$

Taking the limit as $k\to\infty$ in above inequality and from (2.10), we have

$$\lim_{k\to\infty}\rho(x_{2n(k)}-x_{2m(k)})=\varepsilon. \tag{2.12}$$

Then, we get

$$\rho(x_{2n(k)+1}-x_{2m(k)+1})=\rho(x_{2n(k)+1}-x_{2n(k)}+x_{2n(k)}-x_{2m(k)+1})\leq\rho(2(x_{2n(k)+1}-x_{2n(k)}))+\rho(2(x_{2n(k)}-x_{2m(k)+1}))$$

and

$$\rho(2(x_{2n(k)}-x_{2m(k)+1}))=\rho(2(x_{2n(k)}-x_{2m(k)}+x_{2m(k)}-x_{2m(k)+1}))\leq\rho(4(x_{2n(k)}-x_{2m(k)})+\rho(4(x_{2m(k)}-x_{2m(k)+1})).$$

Taking the limit as $k\to+\infty$ in above inequality and using (2.12) and (2.11), we deduce that

$$\lim_{k\to\infty}\rho(x_{2n(k)}-x_{2m(k)+1})=\varepsilon. \tag{2.13}$$

Now, by (2.1), we get

$$\rho(x_{2n(k)+1}-x_{2m(k)+1})\leq M(x_{2n(k)}-x_{2m(k)})-\varphi(M(x_{2n(k)}-x_{2m(k)})) \tag{2.14}$$

where

$$M(x_{2n(k)}-x_{2m(k)})=\max\{\rho(x_{2n(k)}-x_{2m(k)}),\rho(x_{2n(k)}-x_{2n(k)+1}),\rho(x_{2m(k)}-x_{2m(k)+1}),\frac{[1+\rho(x_{2n(k)}-x_{2n(k)+1})]\rho(x_{2m(k)}-x_{2m(k)+1})}{\rho(x_{2n(k)}-x_{2m(k)})+1}\}.$$

Letting $k\to\infty$ in (2.14) and using (2.10), (2.12) and (2.13), we get

$$\varepsilon\leq\varepsilon-\psi(\varepsilon).$$

That is $\varepsilon=0$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since X_ρ is complete, then there exists a $z\in X_\rho$ such that $x_n\to z$. Suppose that (c) holds. That is, $\alpha(x_{2n})\beta(z)\geq 1$. Since T is $\alpha-\beta-\psi-$weak rational contradiction, then we have

$$\rho(x_{2n+1}-Tz)\leq M(x_{2n},z)-\psi((x_{2n},z)) \tag{2.15}$$

where

$$M(x_{2n},z)=\max\{\rho(x_{2n}-z),\rho(x_{2n}-Tz),\rho(z-Tz),\frac{[1+\rho(x_{2n}-x_{n+1})]\rho(z-Tz)}{\rho(x_{2n}-z)+1}\}.$$

Taking the limit as $n\to\infty$ in (2.15), we have $z=Tz$. Now, let show that T has at most one fixed point. Indeed, if $x,y\in X_\rho$ be two fixed points of T, that is, $Tx=x\neq y=Ty$. From condition (c) we have, $\alpha(x)\beta(y)\geq 1$, it follows that

$$\psi(\rho(x-y))\leq\psi(M(x,y))-\phi(M(x,y))$$

where

$$M(x,y)=\max\{\rho(x-y),\rho(x-Tx),\rho(y-Ty),\frac{[1+\rho(x-Tx)]\rho(y-Ty)}{\rho(x-y)+1}\}.$$

Then, we obtain

$$\psi(\rho(x-y))\leq\psi(\rho(x-y))-\phi(\rho(x-y)).$$

So $\phi(\rho(x-y))=0$ and hence, $\rho(x-y)=0$, that is, $x=y$.

□

We obtain the following corollaries from Theorem 2.9.

Corollary 2.10. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping such that*

$$\alpha(x)\beta(y)\rho(Tx-Ty) \leq M(x,y) - \psi(M(x,y)),$$

where $\psi \in \Psi$ and

$$M(x,y) = \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{[1+\rho(x-Tx)]\rho(y-Ty)}{\rho(x-y)+1}\}.$$

Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Corollary 2.11. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping such that*

$$(\alpha(x)\beta(y)+1)^{\rho(Tx-Ty)} \leq 2^{M(x,y)-\psi(M(x,y))}$$

for all $x,y \in X_\rho$ where $\psi \in \Psi$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

Corollary 2.12. *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic $(\alpha,\beta)-$admissible mapping such that*

$$(\alpha x)(\beta x)+\ell)^{(\alpha x)(\beta x)} \leq M(x,y) - \psi(M(x,y)) + \ell$$

for all $x,y \in X_\rho$ where $\psi \in \Psi$ and $l > 1$. Suppose that the following assertions hold:

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

(b) T is continuous, or

(c) if $\{x_n\}$ is a sequence in X_ρ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$,

then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x,y \in Fix(T)$, then T has a unique fixed point.

3. Application

In this section, firstly we shall apply Corollary 2.3 to show the existence of solution of integral equation. Let φ be a Musielak-Orlicz function on a measurable space $C = ([0,1],\Lambda,\mu)$, where ρ_φ is a modular defined by

$$\rho_\varphi(u) = \int_0^1 \varphi(s,|u(s)|)ds$$

for $\in u\ \mathscr{L}^\varphi$ and $\alpha_0 > e$ and $c_0 \in [\frac{e}{\alpha_0},1)$. Assume that ρ_φ is convex satisfying the Δ_2-condition. Now, we investigate the existence and uniqueness of solution of integral equation:

$$u(t) = e^{-t}f + \int_0^t e^{s-t}(\int_0^1 K(\xi,u(s))d\xi)ds,$$

where $K : [0,1] \times \mathscr{L}^\varphi \to \mathscr{L}^\varphi$ is a measurable function satisfying:

(1) $\lim\limits_{\lambda \to 0^+} \int_0^1 \varphi(\xi,\lambda \left|(\int_0^1 K(s,u)ds)\xi\right| d\xi = 0$ for any $u \in \mathscr{L}^\varphi$.

(2) $\left|\int_0^1 (K(\xi,u(s)) - K(\xi,v(s)))d\xi\right| \leq k|(u-v)(s)|$ for any $u,v \in \mathscr{L}^\varphi$ with $k \in (0,1)$.

(3) We denote by $B = C([0,1],A)$ the space of all ρ-continuous function from $[0,1]$ into A which is a convex, closed, bounded subset of $\mathscr{L}^\varphi$. So, B is a closed, bounded, convex subset of $C([0,1],\mathscr{L}^\varphi)$ satisfying the Δ_2-condition.

Let $T : B \to B$ defined by

$$T(u) = \int_0^1 \frac{c_0}{e} K(s,u)ds.$$

(4) $f \in B$.

(5) There exists $u_0 \in B$ such that $\theta(u_0) \geq 0$, $\eta(u_0) \geq 0$ and

$$\theta(u) \geq 0 \text{ for some } u \in B \text{ implies } \eta(Tu) \geq 0,$$
$$\eta(u) \geq 0 \text{ for some } u \in B \text{ implies } \theta(Tu) \geq 0.$$

(6) if $\{u_n\}$ is a sequence in B such that $\theta(u_n) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to u$ as $n \to \infty$, then $\theta(u) \geq 0$.
(7) Let $\alpha, \beta : B \to [0, \infty)$ by

$$\alpha(u) = \begin{cases} 1, & \theta(u) \geq 0 \\ 0, & otherwise \end{cases} \text{ and } \beta(v) = \begin{cases} 1, & \eta(u) \geq 0 \\ 0, & otherwise. \end{cases}$$

Theorem 3.1. *Under the above assumptions (1)-(7), the integral equation has a solution in* $C([0,1], \mathscr{L}^{\varphi})$.

Proof. Firstly, we show that T is ρ-Lipschitz. By assumption (1), we have $\int_0^1 \varphi(\xi, \lambda |Tu(\xi)| d\xi \to 0$ as $\lambda \to 0^+$. Hence the definition of $\mathscr{L}^{\varphi}$, we get $Tu \in \mathscr{L}^{\varphi}$ for any $u \in \mathscr{L}^{\varphi}$.
Let $x, y \in B$, then we have

$$\begin{aligned} \rho_f(Tu - Tv) &= \rho_f\left(\frac{c_0}{e}\left(\frac{e}{c_0}(Tu - Tv)\right)\right) \\ &\leq \frac{c_0}{e}\rho_f\left(\frac{e}{c_0}(Tu - Tv)\right) \\ &= \frac{c_0}{e}\int_0^1 \varphi\left(s, \frac{e}{c_0}|(Tu - Tv)(s)|\right) ds \\ &= \frac{c_0}{e}\int_0^1 \varphi\left(s, \frac{e}{c_0}\left|\int_0^1 (K(\xi, u(s)) - K(\xi, v(s)))d\xi\right|\right) ds. \end{aligned}$$

Therefore by assumption (2)

$$\begin{aligned} \rho_f(Tu - Tv) &\leq \frac{c_0}{e}\int_0^1 \varphi(s, k|(u - v)(s)|) ds \\ &= \frac{c_0}{e}\rho_{\varphi}(k(u - v)) \\ &= \frac{c_0}{e}k\rho_{\varphi}(u - v). \end{aligned}$$

Then, we get T is ρ-Lipschitz (see Theorem 1.3 in [33]). Also define $\psi, \phi : C([0,1], \mathscr{L}^{\varphi}) \to C([0,1], \mathscr{L}^{\varphi})$ by

$$\psi(u) = u, \text{ and } \phi(u) = \left(1 - \frac{c_0}{e}k\right)u \text{ for } \frac{c_0}{e}k \in (0,1).$$

Consequently, for all $u, v \in B$ we have

$$\alpha(u)\beta(v)\psi(\rho_{\varphi}(Tu - Tv)) \leq \psi(\rho_{\varphi}(u - v)) - \phi(\rho_{\varphi}(u - v)).$$

It shows that all the hypotheses of Corollary 2.3 are satisfied, hence T has a solution $u \in C([0,1], \mathscr{L}^{\varphi})$. □

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