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Free Vibration of Cracked Cantilever Beams: Analytical and Experimental Modelling

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Abstract

This study presents free vibration of cantilever beams with multiple cracks. The problem is solved analytically by the transfer matrix method, and is validated experimentally by the operational modal analysis. Six damage scenarios are considered to study crack effect on the natural frequencies and corresponding mode shapes. Graphs and tables for numerical results are given and discussed. Results show that crack occurrence in a beam significantly changes its dynamic behavior.

Keywords: Crack, Beam, Free vibration, Transfer matrix method, Experimental model

Çatlaklı Konsol Kirişlerin Serbest Titreşimi: Analitik ve Deneysel Modelleme

Özet

Bu çalışmada birden fazla çatlak içere konsol kirişlerin serbest titreşimleri incelenmiştir. Problem analitik olarak transfer matrisi metoduyla çözülmüş ve operasyonel modal analiz ile deneysel doğrulama yapılmıştır. Çatlağın doğal frekanslar ve mod şekillerini üzerindeki etkilerini incelemk üzere altı farklı hasar durumu göz önüne alınmıştır. Sayısal sonuçlar grafik ve tablolarla sunulmuştur. Sonuçlar, çatlak varlığının kirişin dinamik davranışını önemli ölçüde değiştirdiğini göstermiştir.

Anahtar Kelimeler: Çatlak, Serbest titreşim, Transfer matrisi metodu, Deneysel model

1. Introduction

Engineering structures are exposed to different types of environmental loads such as earthquakes, wind and traffic loads etc. Over time, stresses and strains due to these loadings lead to reduce in lifetime of the structure, and may cause damages (cracks), which is a serious threat to performance of structure. Early detection of any structural damage is important to prevent structural failures that causes human casualties and financial costs. Thus, an accurate and comprehensive study on structures including cracks are necessary.

Beams are structural elements in which cracks are commonly observed. Therefore, they have been frequently studied by researchers with through different analytical, numerical and experimental techniques. Dimarogonas [1] presented a comprehensive review of various methods in studying structural members with cracks.

Dimarogonas and Paipetis [2] proposed the local flexibility concept to model an open edge crack in a beam, which can be derived from the stress intensity factors in the theory of fracture mechanics. The cracked section in a beam can be replaced by massless rotational springs representing the local flexibility of the crack. Studies on vibrations of cracked beams using the local flexibility concept have generally focused on two main aspects: The first is to estimate the effects of cracks on the eigen-parameters of beams as a direct problem, and the second is to detect the location and size of the crack from the measured information as an inverse problem. Direct analysis of beam vibrations in the presence of cracks is, however, required for solution of the inverse problem.

Methods in studying free vibration of beams with cracks are, in general, divided into two main groups: continuous and discrete methods. In continuous methods, the beam is divided into several sub-beams connected by massless springs. Differential equations are, then, solved for each sub-beam individually with considering the boundary and continuity conditions. As a continuous method, the transfer matrix method is an efficient tool for free vibration of cracked beams, and have been widely preferred [3-8]. Viola et al. [9] derived the explicit dynamic stiffness matrix of a cracked axially loaded beam under coupled bending–torsion with considering the effects of the rotatory inertia and the shear deformation. Among discrete methods, the finite element method [10-12] and the discrete element method [13] can be mentioned.

Experimental measurements including ambient and forced vibration tests have also been used to extract the dynamic characteristics of cracked beams during operational conditions as well as to verify their analytical and/or numerical models [14-16]. Experimental measurements can also be used to identify cracks in a beam in inverse problems.

As can be seen in the literature summarized above, there are many studies on cracked beam vibrations using different analytical/numerical and experimental methods. However, the studies on extracting dynamic characteristics of cracked beams by operational modal analysis (OMA), and validating the experimental results with analytical solution are limited. This study presents free vibration analysis of cantilever beams with multiple cracks. The problem is solved analytically by the transfer matrix method (TMM), and is validated experimentally by the operational modal analysis (OMA). The cantilever beam is assumed to obey Bernoulli-Euler theory. Six damage scenarios are considered to study crack effect on the natural frequencies and corresponding mode shapes. Comparative graphs and tables for numerical results are given and discussed.

2. Theoretical Formulation

2.1. Analytical model

Consider a cantilever beam with N cracks along its length, and has a rectangular cross-section of width b and height h shown in Fig. 1. The beam is assumed to be connected by massless rotational springs at cracked section as shown in Fig. 2. Equation of motion for each segment of the beam is given by

$$EI\frac{\partial^4 Y_i(x,t)}{\partial x^4}+\rho A\frac{\partial^2 Y_i(x,t)}{\partial t^2}=0 \qquad (i=1,2,...,N+1) \tag{1}$$

where $Y_i(x,t)$ is the deflection function, EI is the flexural rigidity, ρ is the mass density, $A=bh$ is the cross-sectional area of the beam. Introducing

$$\bar{x}=x/L \tag{2}$$

into (1) yields

$$\frac{EI}{L^4}\frac{\partial^4 y_i(\bar{x},t)}{\partial \bar{x}^4}+\rho A\frac{\partial^2 y_i(\bar{x},t)}{\partial t^2}=0 \qquad (i=1,2,...,N+1) \tag{3}$$

where $Y(x,t)\equiv y(\bar{x},t)$.

Assuming the solution of (3) as

$$y_i(\bar{x},t)=X_i(\bar{x})e^{i\omega t} \qquad (i=1,2,...N+1) \tag{4}$$

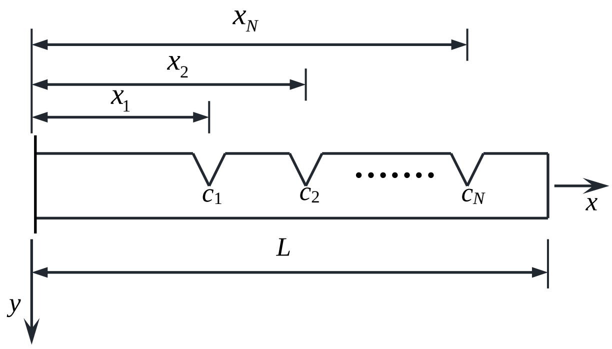

Figure 1. A cantilever beam with multiple cracks

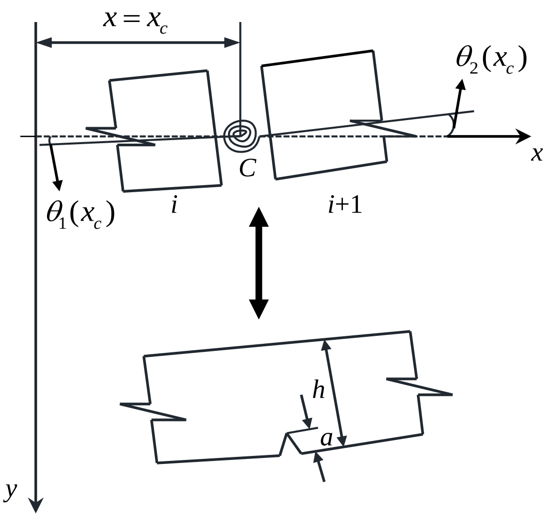

Figure 2. Cracked section represented by massless rotational spring.

where $X_i(\bar{x})$ denotes the modal shape function, and ω is the natural frequency of the beam, and substituting it into (3) gives the following:

$$\frac{d^4 X_i}{d\bar{x}^4} - \beta^4 X_i = 0 \quad (i = 1, 2, ..., N+1) \tag{5}$$

where

$$\beta^4 = \frac{\rho A L^4}{EI}\omega^2 \tag{6}$$

Solution of (5) is

$$\begin{aligned} X_i(\bar{x}) &= A_i \sin\beta\bar{x} + B_i \cos\beta\bar{x} + C_i \sinh\beta\bar{x} \\ &+ D_i \cosh\beta\bar{x} \quad (i = 1, 2, ... N+1) \end{aligned} \tag{7}$$

where A_i, B_i, C_i and D_i are constants to be determined from the boundary and continuity conditions given by

$$y(0,t) = y'(0,t) = -EIy''(1,t) = -EIy'''(1,t) = 0 \tag{8}$$

$$\begin{aligned} & y_i(\bar{x}_i, t) = y_{i+1}(\bar{x}_i, t) \\ & y_i'(\bar{x}_i) = y_{i+1}'(\bar{x}_i) - (h/L) f(a_i) y_{i+1}''(\bar{x}_i) \\ & y_i''(\bar{x}_i, t) = y_{i+1}''(\bar{x}_i, t) \\ & y_i'''(\bar{x}_i) = y_{i+1}'''(\bar{x}_i) \end{aligned} \tag{9}$$

where $f(d_i)$ is a dimensionless function, which is given by single-sided open cracks as

$$\begin{aligned} f(d_i) &= 2[d_i/(1-d_i)]^2 (5.93 - 19.69 d_i \\ &+ 37.14 d_i^2 - 35.64 d_i^3 + 13.12 d_i^4) \end{aligned} \tag{10}$$

where $d_i = a_i / h$ is dimensionless crack depth.

Substituting (4) into (9), we have the following:

$$\mathbf{P}_i \mathbf{a}_i = \mathbf{Q}_i \mathbf{a}_{i+1} \quad (i = 1, 2, ..., N) \tag{11}$$

where $\mathbf{a}_i = \{A_i \quad B_i \quad C_i \quad D_i\}^T$ and

$$\mathbf{P}_i = \begin{bmatrix} \sin\beta\bar{x}_i & \cos\beta\bar{x}_i & \sinh\beta\bar{x}_i & \cosh\beta\bar{x}_i \\ \cos\beta\bar{x}_i & -\sin\beta\bar{x}_i & \cosh\beta\bar{x}_i & \sinh\beta\bar{x}_i \\ -\sin\beta\bar{x}_i & -\cos\beta\bar{x}_i & \sinh\beta\bar{x}_i & \cosh\beta\bar{x}_i \\ -\cos\beta\bar{x}_i & \sin\beta\bar{x}_i & \cosh\beta\bar{x}_i & \sinh\beta\bar{x}_i \end{bmatrix} \tag{12}$$

$$\mathbf{Q}_i = \mathbf{P}_i + \mathbf{S}_i \tag{13}$$

where

$$\mathbf{S}_i = \chi \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\sin\beta\bar{x}_i & -\cos\beta\bar{x}_i & \sinh\beta\bar{x}_i & \cosh\beta\bar{x}_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

where $\chi = -\beta(h/L) f(a_i)$. From (11), we have

$$\mathbf{a}_{i+1} = \mathbf{Q}_i^{-1} \mathbf{P}_i \mathbf{a}_i \quad (i = 1, 2, ..., N) \tag{15}$$

Considering (15), the following relation between the constants of $(N+1)$ th and those of first segment can be written:

$$\mathbf{a}_{N+1} = \mathbf{Q}_N^{-1} \mathbf{P}_N \mathbf{Q}_{N-1}^{-1} \mathbf{P}_{N-1} \cdots \mathbf{Q}_1^{-1} \mathbf{P}_1 \mathbf{a}_1 = \mathbf{T}\mathbf{a}_1 \tag{16}$$

where $\mathbf{T}$ is the transfer matrix.

Using the first two conditions of (8) into (7) gives $D_1 = -B_1$ and $C_1 = -A_1$. Substituting the latter two of (8) into (7) gives

$$\mathbf{W}\mathbf{a}_{N+1} = \mathbf{0} \tag{17}$$

where

$$\mathbf{W} = \begin{bmatrix} -\sin\beta & -\cos\beta & \sinh\beta & \cosh\beta \\ -\cos\beta & \sin\beta & \cosh\beta & \sinh\beta \end{bmatrix} \tag{18}$$

Substituting (16) into (17) gives

$$\mathbf{Z}\mathbf{a}_1 = \mathbf{0} \tag{19}$$

where $\mathbf{Z} = \mathbf{WT}$. Re-calling $D_1 = -B_1$ and $C_1 = -A_1$ and re-arranging equation (19), we have

$$\begin{bmatrix} Z_{11} - Z_{13} & Z_{12} - Z_{14} \\ Z_{21} - Z_{23} & Z_{22} - Z_{24} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

Equation (20) has real roots different from zero when the determinant of its coefficient matrix is zero. Thus,

$$\det \begin{bmatrix} Z_{11} - Z_{13} & Z_{12} - Z_{14} \\ Z_{21} - Z_{23} & Z_{22} - Z_{24} \end{bmatrix} = 0 \tag{21}$$

which gives a characteristic equation as $f(m, \bar{x}_i, d_i) = 0$ depending on natural frequencies, crack size and location. Using the roots β_n $(n = 1, 2, \ldots)$ of the characteristic equation into (6), the natural frequencies of beam can be obtained from

$$\omega_n = \frac{\beta_n^{\ 2}}{L^2} \sqrt{\frac{EI}{\rho A}} \tag{22}$$

For modal shape functions, constants $\mathbf{a}_i$ for each segment can, then, be obtained from (20) for $i = 1$ and (15) for $i = 2, 3, \ldots$.

2.2. Experimental model

Three steel cantilever beams are constructed for laboratory tests. A typical representation of the model is shown in Fig. 3. The beam has uniform rectangular cross-section along its length. In experimental measurements, B&K3560 data acquisition system with 17 channels, B&K8340-type uni-axial accelerometers and uni-axial signal cables are used as test equipment. Six sensitive accelerometers are located on the laboratory model to extract natural frequencies and corresponding mode shapes of the beam shown in Fig. 4.

Measurements are performed during 10 minutes for all cases (damaged and undamaged) considered. Frequency range, FFT analyzers and Multi-buffer are selected to be 0-800Hz, 800 lines and 100 averages, 50 size and 500m update, respectively. The signals from the accelerometers are recorded on the computer with applying FFT process in PULSE [17] software. This transformed data is, then, filtered by the weight functions in OMA [18] software. Modal parameters are obtained by Enhanced Frequency Domain Decomposition (EFDD) method in frequency domain which gives the spectral density functions of the signals in each channel. Natural frequencies and modal damping ratios are, then, determined using the spectral density functions. Fig. 5 shows the spectral density functions obtained from OMA. In there, the peak points which are selected manually show natural frequencies of the beam.

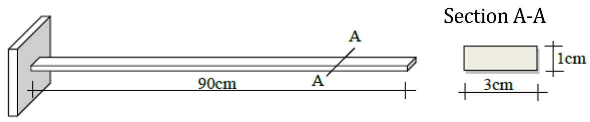

Figure 3. Dimensions of the steel cantilever beam.

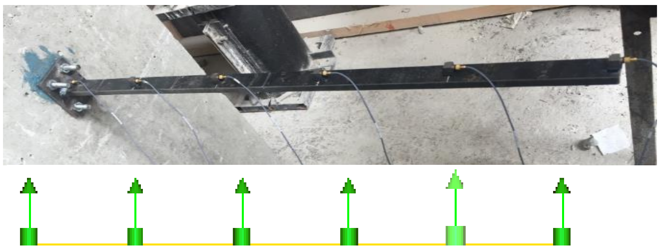

Figure 4. Laboratory model and accelerometer locations.

Table 1. Damage scenarios considered.

Scenario	Crack location (mm)			Crack depth (mm)		
	x_1	x_2	x_2	a_1	a_2	a_3
Damage-1	90	-	-	3	-	-
Damage-2	90	270	-	3	3	-
Damage-3	90	270	450	3	3	3
Damage-4	90	270	450	6	3	3
Damage-5	90	270	450	6	6	3
Damage-6	90	270	450	6	6	6

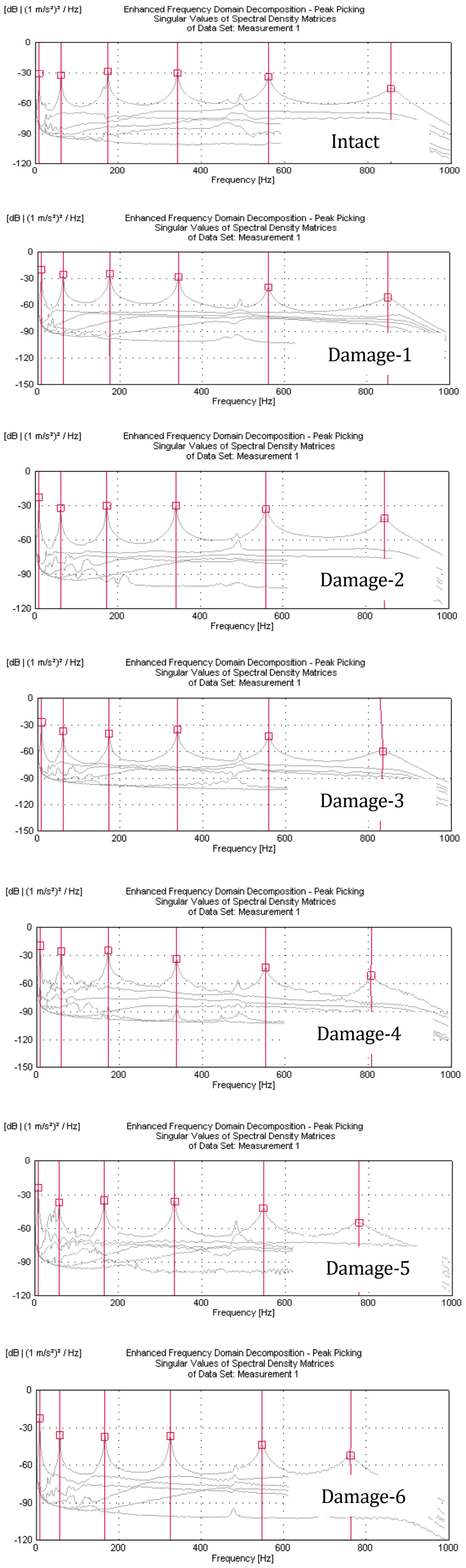

Figure 5. Singular values of spectral density matrices.

Table 2. First six natural frequencies (Hz) calculated from TMM.

Mode	Natural frequencies (Hz)						
	Intact	Damage-1	Damage-2	Damage-3	Damage-4	Damage-5	Damage-6
1	10.25	10.10	10.04	10.03	9.18	8.87	8.77
2	64.24	63.88	63.73	63.16	61.27	60.47	57.07
3	179.83	179.64	178.07	178.02	177.11	168.58	168.58
4	352.42	352.42	351.17	347.91	347.91	342.20	323.51
5	582.60	581.61	580.83	580.75	575.82	571.73	571.73
6	870.24	866.22	857.95	849.86	830.07	796.00	767.36

Table 3. First six natural frequencies (Hz) measured by OMA

Mode	Natural frequencies (Hz)						
	Intact	Damage-1	Damage-2	Damage-3	Damage-4	Damage-5	Damage-6
1	9.92	9.77	9.69	9.75	8.93	8.68	8.59
2	62.53	62.26	61.98	61.56	59.96	59.34	57.54
3	175.30	175.21	173.40	173.70	172.80	167.20	166.90
4	342.10	341.80	340.90	338.60	338.30	334.80	325.30
5	562.00	560.60	559.10	560.00	552.10	548.20	547.50
6	856.70	850.30	844.40	828.60	808.60	776.70	763.50

Table 4. Change in natural frequencies (%) with increasing damage severity.

Case	Change in natural frequencies (%)					
	f_1	f_2	f_3	f_4	f_5	f_6
Undamaged *vs.* Damage-1	1.48	0.55	0.10	0.00	0.17	0.46
Damage-1 *vs.* Damage-2	0.59	0.24	0.87	0.35	0.13	0.95
Damage-2 *vs.* Damage-3	0.11	0.89	0.03	0.93	0.01	0.94
Damage-3 *vs.* Damage-4	8.44	2.98	0.51	0.00	0.85	2.33
Damage-4 *vs.* Damage-5	3.35	1.30	4.82	1.64	0.71	4.11
Damage-5 *vs.* Damage-6	1.14	5.63	0.00	5.46	0.00	3.59

3. Results

Comparisons between the calculated and measured values are given. Material properties are E = 206GPa and ρ = 7800kg/m^3. Free vibration analyses are performed for six damage scenarios in Table 1. Analytical results are obtained through a computer code written in MATLAB environment.

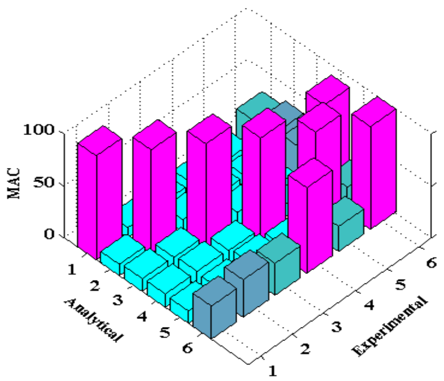

Figure 6. MAC representation between analytical and experimental mode shapes for undamaged case.

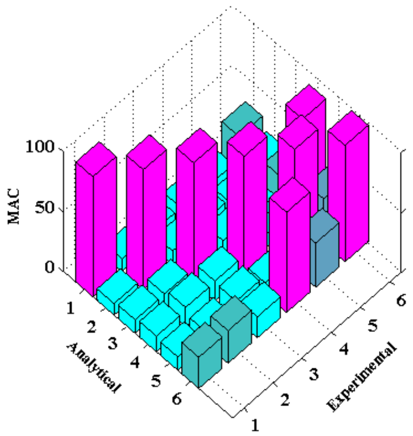

Figure 7. MAC representation between analytical and experimental mode shapes for Damage-6 case.

Tables 2-4 show the first six natural frequencies of the cantilever beam for all cases considered. As can be seen, the natural frequencies decrease with increasing the damage severity. This is more notable when the crack depth increases, i.e., between Damage-3 and Damage-4 cases. Results obtained from TMM and OMA slightly differ. This may be from several reasons such that: (a) fixed support condition at left-end cannot be provided exactly in laboratory environment, and (b) the beam cross-section, thus the flexural rigidity, cannot be uniform along the beam length. The modal updating, which is already out of the scope of this study, is therefore required.

Modal Assurance Criterion (MAC) is used to establish the correlation between the measured and the calculated results, which is defined by

$$\mathrm{MAC}\left(X_a, X_e\right)=\frac{\left|X_a^T X_e\right|^2}{\left(X_a^T X_a\right)\left(X_e^T X_e\right)} \qquad (23)$$

where X_a and X_e denote analytical and experimental mode shapes. MAC values are greater than 90% means the mode shapes are good correlated [19]. Figs. 6 and 7 show MAC values between analytical and experimental mode shapes for undamaged and Damage-6 cases, respectively. As seen, the correlation between the calculated and measured mode shapes are very good, which verifies the laboratory measurements performed

4. Conclusions

Free vibration of cracked cantilever beams is considered by analytical and experimental methods. Results are in good agreement. The followings can be drawn from the study:

a) Natural frequencies are strongly affected by crack presence in the beam.
b) Crack depth is more effective on the natural frequencies compared to the number of cracks.
c) Operational modal analysis is very suitable for experimental analyses of cracked beams. For more accurate results, modal updating should be recommended.
d) Transfer matrix method gives the frequency equation to solve the inverse problem for damage detection. However, the solution is required more symbolic computation and thus computing time when the number of cracks increases.

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