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A note on Hopf bifurcation and steady state analysis for a predator-prey model

İrem Çay[a]

[a] *Department of Mathematics, Faculty of Arts and Sciences, Kocaeli University, Kocaeli, Turkey.*

Abstract. This paper is concerned with the Hopf bifurcation and steady state analysis of a predator-prey model. Firstly, by analyzing the characteristic equation, the local stability of the nonnegative equilibriums is discussed. Then the Hopf bifurcation around the positive equilibrium is obtained, and the direction and the stability of the Hopf bifurcation are investigated. Finally, some numerical simulations are given to support the theoretical results.

1. Introduction

Mathematical ecology is a subject field in which dynamic systems are involved in species, populations, and how these groups interact with the environment. This subject field primarily studies how species population size changes over time and space. Since Lotka–Volterra's groundbreaking work in the 1920s, the predator-prey model has become one of the most important research topics in mathematical ecology for nearly a century. Species compete, evolve and disperse for the purpose of finding resources to sustain their struggle for their existence. Depending on their specific settings of applications, they can take the forms of resource-consumer, plant-herbivore, parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions, etc. Mathematicians used the theory of dynamics to analyze the differential equations based on a predator-prey model. There are some scholars who applied bifurcation theory in dynamics based on models and we can find them in [2]-[11] etc.

In this paper, we consider a predator-prey model satisfies the following differential equations in [1]

$$\frac{dH}{d\tau} = rH\left(1-\frac{H}{K}\right)-\alpha\frac{PH}{H+\beta}, \qquad (1)$$

$$\frac{dP}{d\tau} = \gamma P\left(-1+\delta\frac{H}{H+\beta}\right), \qquad (2)$$

where H is the prey density and P is the predator density. The parameters are $r,\ K,\ \alpha,\ \beta,\ \gamma,\ \delta > 0$, $H(0) > 0$ and $P(0) > 0$.

The rest of the paper is organized as follows. Basic properties of the model are given in Section 2. Sufficient conditions for the existence of the Hopf bifurcation are obtained in Section 3. In Section 4, the numerical examples are given to illustrate the validity of our results.

Corresponding author: İÇ, mail address: irem.atac@kocaeli.edu.tr ORCID:0000-0001-9234-2523

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2. Preliminary

In this section, firstly, we make the following change of variables to put the model in dimensionless form:

$$x = \frac{H}{K}, \quad y = \frac{\alpha}{rK}P, \quad t = r\tau.$$

Thus (1)-(2) can be written as

$$\frac{dx}{dt} = x\Big(1 - x - \frac{y}{x+b}\Big), \tag{3}$$

$$\frac{dy}{dt} = cy\Big(-1 + a\frac{x}{x+b}\Big). \tag{4}$$

We introduce the basic properties of the nonnegative constant solutions for the system (3)-(4). It is obvious that $\vec{u_1} = (x_1, y_1) = (0, 0)$ and $\vec{u_2} = (x_2, y_2) = (1, 0)$ are constant steady states of (3)-(4). Furthermore, $\vec{u_3} = (x_3, y_3) = \left(\frac{b}{a-1}, \frac{ab(a-b-1)}{(a-1)^2}\right)$ is a constant steady state of (3)-(4).

It is clear that when $a < b + 1$, (3)-(4) has no positive equilibrium.

In the following, we discuss the local stability of equilibrium $\vec{u_i} = (x_i, y_i)$ $(i = 1, 2, 3)$. By directly calculating, the Jacobian matrix at $\vec{u_i}$ is

$$J_i \triangleq J(\vec{u_i}) = \begin{pmatrix} 1 - 2x_i - \frac{by_i}{(x_i+b)^2} & -\frac{x_i}{x_i+b} \\ abc\frac{y_i}{(x_i+b)^2} & c\left(\frac{ax_i}{x_i+b} - 1\right) \end{pmatrix}.$$

Theorem 2.1. For system (3)-(4), the following statements are hold.
(i) For all $a,\ b,\ c > 0$, the constant equilibrium solution $\vec{u_1}$ is a saddle point which is unstable.
(ii) The constant equilibrium solution $\vec{u_2}$ is stable when $a < b + 1$ and it is unstable for $a > b + 1$.
(iii) In the case $a < b + 1$, there is no limit cycle since there is no positive equilibrium.

3. Existence of Hopf Bifurcation

In this section, we restrict $a > b + 1$ and only study the Hopf bifurcation around $\vec{u_3}$. Taking a as the bifurcation parameter, we study the existence of Hopf bifurcation for (3)-(4) and so the direction and the stability of Hopf bifurcation are investigated.

Now, we investigate the results of Hopf bifurcation for (3)-(4). We primarily get the Jacobian matrix of (3)-(4) at $\vec{u_3}$

$$J_3 = \begin{pmatrix} -\frac{2b}{a-1} + \frac{b+1}{a} & -\frac{1}{a} \\ c(a-b-1) & 0 \end{pmatrix}.$$

The characteristic equation of J_3 is

$$\lambda^2 - traceJ_3\lambda + detJ_3 = 0, \tag{5}$$

where

$$traceJ_3 = -\frac{2b}{a-1} + \frac{b+1}{a}, \qquad detJ_3 = \frac{c}{a}(a - b + 1) > 0.$$

Let $(\tilde{x}, \tilde{y}) = (x, y) - (x_3, y_3)$. For convenience, we denote $(\tilde{x}, \tilde{y})$ as (x, y). Then the model (3)-(4) is changed to

$$\frac{dx}{dt} = (x + x_3)\Big(1 - (x + x_3) - \frac{y + y_3}{x + x_3 + b}\Big), \tag{6}$$

$$\frac{dy}{dt} = c(y + y_3)\Big(-1 + a\frac{x + x_3}{x + x_3 + b}\Big). \tag{7}$$

Theorem 3.1. The model (3)-(4) undergoes a Hopf bifurcation at (x_3, y_3) for $a = a^H = \frac{b+1}{1-b}$.

Proof. Since we assume that $a > b + 1$, it should be $0 < b < 1$. Clearly, if $a = a^H = \frac{b+1}{1-b}$ holds, then $\pm i\sqrt{bc}$ is a pair of imaginary eigenvalues of J_3. Let $\alpha(a) \pm iw(a)$ be the roots of (5) in the neighborhood of a^H. So we obtain

$$\alpha(a) = \frac{traceJ_3}{2} = \frac{b+1}{2a} - \frac{b}{a-1}, \qquad w(a) = \sqrt{4\frac{c}{a}(a-b-1) - \left(\frac{b+1}{a} - \frac{2b}{a-1}\right)^2}$$

and

$$\alpha'(a) = -\frac{b+1}{4a^2} + \frac{b}{(a-1)^2}.$$

It is clear that $traceJ_3(a^H) = 0,\ detJ_3(a^H) > 0$ and $\alpha'(a^H) \neq 0$. It follows from the Hopf bifurcation theorem [1] that the model (3)-(4) undergoes a Hopf bifurcation at (x_3, y_3, a^H).

Now, we use a computational method to test whether the Hopf bifurcation is supercritical or subcritical. To study the system around the point $a = a^H$ we expand the right hand side of the system (6)-(7) using the Maclaurin series and we rewrite the system (6)-(7) as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = J_3 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F(x,y,a) \\ G(x,y,a) \end{pmatrix}, \tag{8}$$

where

$$F = \left(\frac{by_3}{(x_3+b)^3} - 1\right)x^2 - \frac{b}{(x_3+b)^2}xy + \frac{b}{(x_3+b)^3}x^2y - \frac{by_3}{(x_3+b)^5}x^3$$

and

$$G = -\frac{abcy_3}{(x_3+b)^3}x^2 + \frac{abc}{(x_3+b)^2}xy - \frac{abc}{(x_3+b)^3}x^2y + \frac{abcy_3}{(x_3+b)^5}x^3.$$

Next, we make the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \tag{9}$$

where

$$P = \begin{pmatrix} \frac{1-b}{bc(b+1)}w(a) & 0 \\ 0 & \frac{b+1}{1-b)}w(a) \end{pmatrix},$$

and substitute it into (8). To avoid the abuse of mathematical notation, we still denote $(\tilde{x}, \tilde{y})$ by (x, y). Then we obtain the normal form of (8) as follows

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -w(a) \\ w(a) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y,a) \\ g(x,y,a) \end{pmatrix}, \tag{10}$$

where

$$f(x,y,a) = \frac{bc(b+1)}{(1-b)w(a)} F\left(\frac{1-b}{bc(b+1)}w(a)x, \frac{b+1}{1-b}w(a)y\right),$$

$$g(x,y,a) = \frac{1-b}{(b+1)w(a)} G\left(\frac{1-b}{bc(b+1)}w(a)x, \frac{b+1}{1-b}w(a)y\right).$$

To determine the stability of periodic solutions, we need to calculate the sign of the following coefficient

$$\begin{aligned} \gamma &= \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) \\ &+ \frac{1}{16w(a^H)}\left[f_{xy}\left(f_{xx} + f_{yy}\right) - g_{xy}\left(g_{xx} + g_{yy}\right) - f_{xx}g_{xx} + f_{yy}g_{yy}\right], \end{aligned} \tag{11}$$

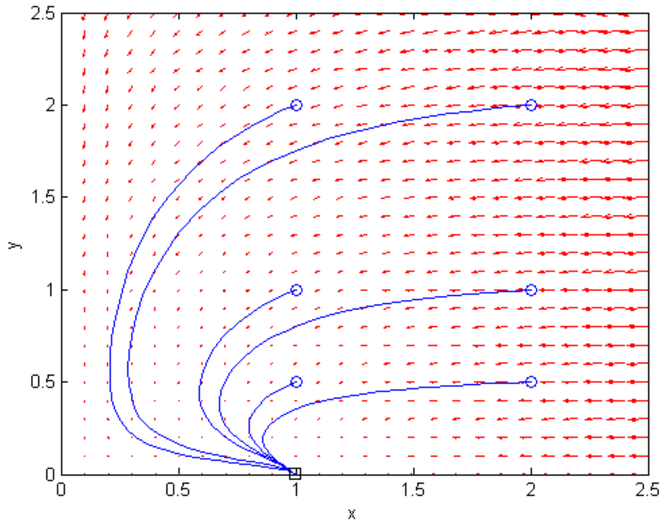

Figure 1: When $a < b + 1$, there is no positive equilibrium. The constant equilibrium $\vec{u_2} = (1, 0)$ is locally stable.

where all the partial derivatives are evaluated at the bifurcation point $(0, 0, a^H)$. Then, by computing we obtain

$$\begin{aligned}\gamma \quad = \quad & -12\frac{b^2}{c^{1/2}(b+1)^5} - \frac{1-b}{(b+1)^4} - \frac{b^{3/2}c^{1/2}}{(1-b)^2}\left(\frac{2b^3(b+1)}{(1-b)^2} - 1\right) \\ + \quad & \frac{(1-b)^2}{b^{1/2}c^{1/2}(b+1)^3} + \frac{1-b}{4b^{1/2}c^{1/2}(b+1)}\left(\frac{2b^3(b+1)}{(1-b)^2} - 1\right). \end{aligned} \tag{12}$$

Therefore, we have the following result.

Theorem 3.2. If $\gamma < 0$, the direction of Hopf bifurcation is supercritical. This means that for $a < a^H$ the positive equilibrium (x_3, y_3) is a stable spiral but for $a > a^H$ there exists a stable periodic solution and (x_3, y_3) is unstable. If $\gamma > 0$, the direction of Hopf bifurcation is subcritical. In this situation, when $a < a^H$ the positive equilibrium (x_3, y_3) is stable and there exists an unstable periodic solution but when $a > a^H$, (x_3, y_3) is unstable.

4. Numerical Simulations

In this section, some numerical simulations are presented, which support and complement the results given in the previous section. There are three parameters a, b, c in our model (3)-(4). We fix $b = 0.5$, $c = 1$ and obtain the following numerical simulations which illustrate the main theoretical results.

Example 4.1. We take $a = 1$, $b = 0.5$, $c = 1$. Then $a < b+1$ and model (3)-(4) has no positive equilibrium. From Fig. 1, we see that $\vec{u_2} = (1, 0)$ is locally stable.

Example 4.2. We take $a = 2.5$, $b = 0.5$, $c = 1$. Then $a > b+1$ and there exists unique positive equilibrium $\vec{u_3} = (x_3, y_3)$. When $a = 2.5$, $b = 0.5$, $a < a^H$. From Fig. 2, we see that (x_3, y_3) is a stable spiral.

Example 4.3. We take $a = 3.5$, $b = 0.5$, $c = 1$, then $a > a^H$. We observe that there exists a stable periodic solution and the positive equilibrium (x_3, y_3) is unstable. This is seem from Fig 3.

In Example 4.2 and Example 4.3, we fix $b = 0.5$, $c = 1$, then we derive $\gamma < 0$. From the numerical simulations (see Fig. 2 and Fig. 3), we can say that there exists a supercritical Hopf bifurcation and this supports our theorical results.

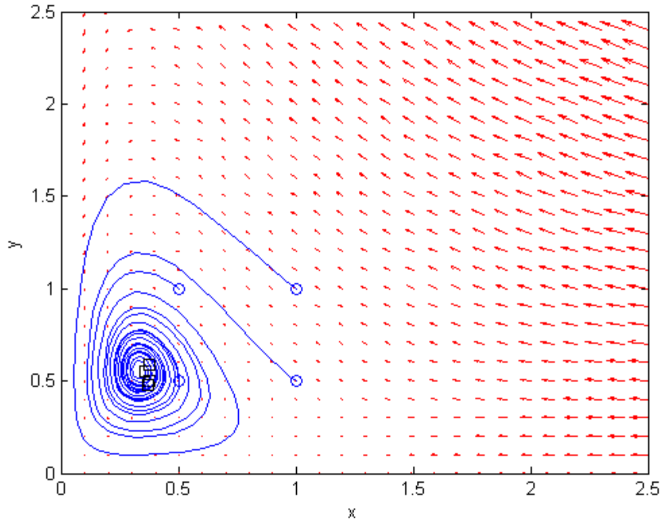

Figure 2: When $a > b + 1$ and $0 < b < 1$, we have a bifurcation parameter and a bifurcation value a and a^H, respectively. If $a < a^H$, (x_3, y_3) is a stable spiral.

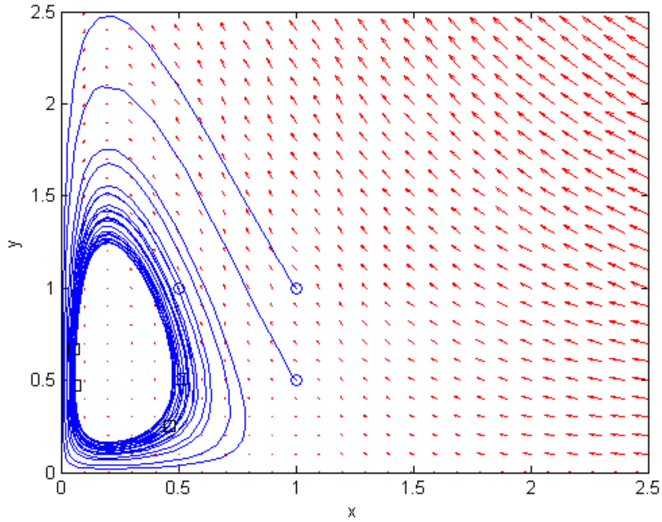

Figure 3: If $a > a^H$, there exist stable periodic orbits and (x_3, y_3) is unstable.

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