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A Note on the Libera Type Meromorphic Close-to-convex Functions

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Abstract. The main purpose of this paper is to introduce a new subclass of meromorphic univalent functions, namely Libera type meromorphic close-to-convex functions, and to obtain general coefficient bounds for functions belonging to this class. For this purpose, we consider a certain convex univalent function in the open unit disk $\mathbb{U}$, that maps $\mathbb{U}$ onto a strip domain.

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1. Introduction

Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} =: \mathbb{U}^* \cup \{0\}.$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ \ (z \in \mathbb{U})\},$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by

$$f(0) = f'(0) - 1 = 0.$$

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Each function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Lemma 1.1. *[1] Let $f \in \mathcal{A}$ and $\alpha < 1 < \beta$. The function $f_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ defined by*

$$f_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}\right) \tag{1.1}$$

is analytic in $\mathbb{U}$ with $f_{\alpha,\beta}(0) = 1$ and maps the unit disk $\mathbb{U}$ onto the vertical strip domain

$$\Omega_{\alpha,\beta} = \{w \in \mathbb{C} : \alpha < \Re(w) < \beta\}$$

conformally.

We note that, the function $f_{\alpha,\beta}$ defined by (1.1) is a convex univalent function in $\mathbb{U}$ and has the form

$$f_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right) \qquad (n \in \mathbb{N}). \tag{1.2}$$

Let σ denote the class of all meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \tag{1.3}$$

defined on the domain

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad 1 < |z| < \infty\}.$$

A function $g \in \sigma$ is said to be meromorphic starlike of order α $(0 \le \alpha < 1)$, if it satisfies the inequality

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

We denote the class which consists of all functions $g \in \sigma$ that are starlike of order α by $\mathcal{S}_\sigma^*(\alpha)$. In particular, we set $\mathcal{S}_\sigma^*(0) = \mathcal{S}_\sigma^*$.

Very recently, Sim and Kwon [3] introduced the subclass of meromorphic functions associated with a vertical strip domain as follows:

Definition 1.2. Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$. The function $g \in \sigma$ belongs to the class $\mathcal{S}_\sigma(\alpha,\beta)$ if it satisfies the inequality

$$\alpha < \Re\left(\frac{zg'(z)}{g(z)}\right) < \beta \qquad (z \in \Delta).$$

Theorem 1.3. *[3] Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$ and let the function $g \in \sigma$ be defined by (1.3). If $g \in \mathcal{S}_\sigma(\alpha,\beta)$, then*

$$|b_0| \le \frac{2(\beta - \alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta - \alpha}$$

and

$$|b_n| \le \frac{2(\beta - \alpha)}{(n+1)\pi} \sin\frac{\pi(1-\alpha)}{\beta - \alpha} \prod_{k=1}^{n}\left(1 + \frac{2(\beta-\alpha)}{k\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right) \qquad (n \in \mathbb{N}).$$

Now, we introduce the class of Libera type meromorphic close-to-convex functions as follows:

Definition 1.4. Let α and β be real such that $0 \leq \alpha < 1 < \beta$. We denote by $C_\sigma(\alpha,\beta)$ the class of functions $g \in \sigma$ satisfying

$$\alpha < \Re\left(\frac{zg'(z)}{h(z)}\right) < \beta \qquad (z \in \Delta),$$

where $h \in \mathcal{S}_\sigma(\delta,\beta)$ with $0 \leq \delta < 1 < \beta$.

Remark 1.5. (i) If we let $\beta \to \infty$ in Definition 1.4, then the class $C_\sigma(\alpha,\beta)$ reduces to the class $C_\sigma(\alpha,\delta)$ of Libera type meromorphic close-to-convex functions of order α and type δ which consists of functions $g \in \sigma$ satisfying

$$\Re\left(\frac{zg'(z)}{h(z)}\right) > \alpha \qquad (z \in \Delta),$$

where $h \in \mathcal{S}^*_\sigma(\delta)$ with $0 \leq \delta < 1$.
(ii) If we let $\delta = 0,\ \beta \to \infty$ in Definition 1.4, then the class $C_\sigma(\alpha,\beta)$ reduces to the class $C_\sigma(\alpha)$ of meromorphic close-to-convex functions of order α which consists of functions $g \in \sigma$ satisfying

$$\Re\left(\frac{zg'(z)}{h(z)}\right) > \alpha \qquad (z \in \Delta),$$

where $h \in \mathcal{S}^*_\sigma$.
(iii) If we let $\alpha = \delta = 0,\ \beta \to \infty$ in Definition 1.4, then the class $C_\sigma(\alpha,\beta)$ reduces to the meromorphic close-to-convex functions class C_σ which consists of functions $g \in \sigma$ satisfying

$$\Re\left(\frac{zg'(z)}{h(z)}\right) > 0 \qquad (z \in \Delta),$$

where $h \in \mathcal{S}^*_\sigma$.

2. Main Results

Lemma 2.1. *[2] Let the function $\mathfrak{g}$ given by*

$$\mathfrak{g}(z) = \sum_{k=1}^{\infty} \mathfrak{b}_k z^k \qquad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. Also let the function $\mathfrak{f}$ given by

$$\mathfrak{f}(z) = \sum_{k=1}^{\infty} \mathfrak{a}_k z^k \qquad (z \in \mathbb{U})$$

be holomorphic in $\mathbb{U}$. If

$$\mathfrak{f}(z) \prec \mathfrak{g}(z) \qquad (z \in \mathbb{U}),$$

then

$$|\mathfrak{a}_k| \leq |\mathfrak{b}_1| \qquad (k \in \mathbb{N}).$$

Theorem 2.2. *Let α,β and δ be real numbers such that $0 \leq \alpha,\delta < 1 < \beta$ and let the function $g \in \sigma$ be defined by (1.3). If $g \in C_\sigma(\alpha,\beta)$, then for all $n \in \mathbb{N}$, we have*

$$\begin{aligned} |b_n| \quad \leq \quad & \frac{2(\beta-\delta)}{n(n+1)\pi} \sin\frac{\pi(1-\delta)}{\beta-\delta} \prod_{k=1}^{n}\left(1+\frac{2(\beta-\delta)}{k\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right) \\ & + \frac{2(\beta-\alpha)}{n\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha} \prod_{k=1}^{n}\left(1+\frac{2(\beta-\alpha)}{k\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right), \end{aligned}$$

where $h \in \mathcal{S}_\sigma(\delta,\beta)$.

Proof. Let the function $g \in C_\sigma(\alpha,\beta)$ be of the form (1.3). Therefore, there exists a function

$$h(z) = z + \sum_{n=0}^{\infty} \frac{h_n}{z^n} \in \mathcal{S}_\sigma(\delta,\beta),$$

so that

$$\alpha < \Re\left(\frac{zg'(z)}{h(z)}\right) < \beta \qquad (z \in \Delta).$$

Note that by Theorem 1.3, we have

$$|h_0| \leq \frac{2(\beta-\delta)}{\pi} \sin \frac{\pi(1-\delta)}{\beta-\delta}$$

and

$$|h_n| \leq \frac{2(\beta-\delta)}{(n+1)\pi} \sin \frac{\pi(1-\delta)}{\beta-\delta} \prod_{k=1}^{n}\left(1+\frac{2(\beta-\delta)}{k\pi} \sin \frac{\pi(1-\delta)}{\beta-\delta}\right) \qquad (n \in \mathbb{N}). \tag{2.1}$$

Define the functions $\varphi, \psi : \mathbb{U}^* \to \mathbb{C}$ by

$$\varphi(z) = g\left(\frac{1}{z}\right) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$$

and

$$\psi(z) = h\left(\frac{1}{z}\right) = \frac{1}{z} + \sum_{n=0}^{\infty} h_n z^n.$$

Since $g \in C_\sigma(\alpha,\beta)$, we get

$$\alpha < \Re\left(-\frac{z\varphi'(z)}{\psi(z)}\right) < \beta \qquad (z \in \mathbb{U}).$$

Let us define the function $p(z)$ by

$$p(z) = -\frac{z\varphi'(z)}{\psi(z)} \qquad (z \in \mathbb{U}). \tag{2.2}$$

It is clear that $p(z)$ is an analytic function and $p(0) = 1$. Then according to the assertion of Lemma 1.1, we get

$$p(z) \prec f_{\alpha,\beta}(z) \qquad (z \in \mathbb{U}),$$

where $f_{\alpha,\beta}(z)$ is defined by (1.1). Hence, using Lemma 2.1, we obtain

$$\left|\frac{p^{(m)}(0)}{m!}\right| = |c_m| \leq |B_1| \qquad (m \in \mathbb{N}), \tag{2.3}$$

where

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U})$$

and (by (1.2))

$$|B_1| = \left|\frac{\beta-\alpha}{\pi} i \left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right)\right| = \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}.$$

Also from (2.2), we find

$$-z\varphi'(z) = p(z)\psi(z). \tag{2.4}$$

In view of (2.4), we obtain

$$-h_0 = c_1$$

and

$$-nb_n - h_n = c_{n+1} + c_n h_0 + \cdots + c_1 h_{n-1} = c_{n+1} + \sum_{j=1}^{n} c_j h_{n-j} \qquad (n \in \mathbb{N}). \tag{2.5}$$

Now we get from (2.1), (2.3) and (2.5),

$$n|b_n| \leq |h_n| + |B_1|\left(1 + \sum_{k=0}^{n-1} |h_k|\right) \leq |h_n| + |B_1| \prod_{k=1}^{n}\left(1 + \frac{|B_1|}{k}\right) \qquad (n \in \mathbb{N}).$$

This evidently completes the proof of Theorem 2.2. □

Letting $\beta \to \infty$ in Theorem 2.2, we have the coefficient bounds for Libera type meromorphic close-to-convex functions of order α and type δ.

Corollary 2.3. *Let α and δ be real numbers such that $0 \leq \alpha, \delta < 1$ and let the function $g \in \sigma$ be defined by* (1.3)*. If $g \in C_\sigma(\alpha, \delta)$, then*

$$|b_n| \leq \frac{2(1-\delta)}{n(n+1)} \prod_{k=1}^{n} \left(1 + \frac{2(1-\delta)}{k}\right) + \frac{2(1-\alpha)}{n} \prod_{k=1}^{n} \left(1 + \frac{2(1-\alpha)}{k}\right) \qquad (n \in \mathbb{N}).$$

Letting $\delta = 0,\ \beta \to \infty$ in Theorem 2.2, we have the following coefficient bounds for meromorphic close-to-convex functions of order α.

Corollary 2.4. *Let α be a real number such that $0 \leq \alpha < 1$ and let the function $g \in \sigma$ be defined by* (1.3)*. If $f \in C_\sigma(\alpha)$, then*

$$|b_n| \leq \frac{n+2}{n} + \frac{2(1-\alpha)}{n} \prod_{k=1}^{n} \left(1 + \frac{2(1-\alpha)}{k}\right) \qquad (n \in \mathbb{N}).$$

Letting $\alpha = \delta = 0,\ \beta \to \infty$ in Theorem 2.2, we have the following coefficient bounds for meromorphic close-to-convex functions.

Corollary 2.5. *Let the function $g \in \sigma$ be defined by* (1.3)*. If $f \in C_\sigma$, then*

$$|b_n| \leq \frac{(n+2)^2}{n} \qquad (n \in \mathbb{N}).$$

Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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