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On The Solutions of The Singular Differential Equations

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Abstract. In this study, new asymptotic expressions for solving the singular coefficient Sturm-Liouville equation were obtained. New asymptotic formulas have been obtained for eigenvalues of Newmann and Drichlet problems for the Sturm-Liouville equation using these asymptotic expressions. Some features of scattering function have also been studied.

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1. Introduction

The spectral theory of differential operators has an important place in applied sciences. Especially in quantum theory there are many applications of singular Shrödinger operator. For example, the problem of finding the hydrogen atom and corresponding wave functions is reduced to the problem of learning the behaviour of the eigenvalues of the Shrödinger operator with Columb potential and the corresponding eigenfunctions. Therefore, some of the characteristics of the Shrödinger operator with a special type of potential have been examined in our study. Differential operators with singularity and discontinuity conditions at the interval have been studied by Amirov and Yurko (2001). In this study, for Sturm-Liouville operator with non-self adjoint Bessel potential with singularities at x=0, we have investigated the case where the solution of the end point of the finite interval has discontinuity. The spectral properties of the given operator and location of inverse problem with respect to these spectral properties and the uniqueness theorem for the solution have been proven.

We will investigate equation of Sturm-Liouville

$$-y'' + q(x)\,y(x) = \lambda^2 y(x)$$

on the finite interval $(0,\pi)$ with real potential $q(x)$, which have nonintegrable singularity at the x=0, satisfying to the following condition

$$\int_0^\pi t\,|q(t)|\,dt < +\infty$$

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and to separate boundary conditions of the form

$$y(0) = 0 = y(\pi).$$

By virtue of singularity of potential, generally, there doesn't exist any finite values for the derivative of solution of Sturm-Liouville equation at the point x=0 of interval. All considered problem self-adjoint, it eigenvalues are real. In this study, properties of solutions, properties of spectral characteristics for Sturm-Liouville differential operators with nonintegrable singularity.

2. Asymptotics of Solutions and Eigenvalues

Consider boundary value problem generated on the interval $(0,\pi)$ of one dimensional Shrödinger equation:

$$-y'' + q(x)y(x) = \lambda^2 y(x) \tag{2.1}$$

with singular potential

$$q(x) = \frac{A}{x^\alpha} + q_0(x) \tag{2.2}$$

where $A, \alpha \in [1,2)$ are orbitrary real numbers and $q_0(x) \in L_2(0,\pi)$ and with separable boundary conditions of the form

$$y(0) = 0 = y(\pi), \tag{2.3}$$

$$y(0) = 0 = y'(\pi). \tag{2.4}$$

By virtue of singularity of potential, generally, there doesn't exist finite values for the derivative of solution of equation (2.1) at the point $x = 0$.

Let us assume that the function $q(x)$ provides the condition

$$\int_0^\pi x|q(x)|dx < +\infty. \tag{2.5}$$

Then, in this case there is a $s(x,\lambda): s(0,\lambda) = 0, s'(0,\lambda) = 1$ solution for all values of λ of the (2.1) equation and it provides

$$|s(x,\lambda)| \le xe^{|Im\lambda|x}\exp\left(\int_0^x t|q(t)|dt\right)$$

$$\left|s(x,\lambda) - \frac{\sin\lambda x}{\lambda}\right| \le x\int_0^x t|q(t)|dt\exp\left(|Im\lambda|x + \int_0^x t|q(t)|dt\right)$$

$$|\lambda s(x,\lambda) - \sin\lambda x| \le \left(\sigma_1(0) - \sigma_1\left(\frac{1}{|\lambda|}\right)\right)\exp\left(|Im\lambda|x + \int_0^x t|q(t)|dt\right)$$

inequalities. Where

$$\sigma_1(x) = \int_x^\pi \sigma(t)dt, \sigma(x) = \int_x^\pi |q(t)|dt.$$

Let's take the

$$R(\lambda) = \int_0^{\frac{1}{|\lambda|}} t|q(t)|dt + \frac{1}{|\lambda|}\int_{\frac{1}{|\lambda|}}^\pi |q(t)|dt$$

function. Clearly,

$$\lim_{|\lambda|\to\infty} R(\lambda) = 0 \tag{2.6}$$

is provided

It is obvious that (2.5) is actually $\lim\limits_{\epsilon\to 0}\int\limits_0^{\epsilon} t\,|q(t)|\,dt = 0$.

$$\epsilon\int\limits_{\epsilon}^{\pi}|q(t)|\,dt = \epsilon\left(\int\limits_{\epsilon}^{\sqrt{\epsilon}}|q(t)|\,dt + \int\limits_{\sqrt{\epsilon}}^{\pi}|q(t)|\,dt\right) \le \int\limits_0^{\sqrt{\epsilon}} t\,|q(t)|\,dt + \sqrt{\epsilon}\int\limits_{\sqrt{\epsilon}}^{\pi} t\,|q(t)|\,dt$$

$$\lim_{\epsilon\to 0}\epsilon\int\limits_{\epsilon}^{\pi}|q(t)|\,dt \le \lim_{\epsilon\to 0}\int\limits_0^{\sqrt{\epsilon}} t\,|q(t)|\,dt + \lim_{\epsilon\to 0}\sqrt{\epsilon}\int\limits_{\sqrt{\epsilon}}^{\pi} t\,|q(t)|\,dt = 0$$

or $\lim\limits_{\epsilon\to 0}\epsilon\int\limits_{\epsilon}^{\pi}|q(t)|\,dt = 0$ is obtained from the other side inequality. So, we get the correctness of (2.6) equality.

Theorem 2.1. *For each $\lambda \epsilon C$, there exists fundamental system of solutions $s(x,\lambda)$ and $c(x,\lambda)$ of equation (2.1) satisfying asymptotic formulas for $|\lambda| \to \infty$:*

$$s(x,\lambda) = \frac{\sin\lambda x}{\lambda} + e^{|Im\lambda|x} O\left(\frac{R(\lambda)}{|\lambda|}\right) \tag{2.7}$$

$$s(x,\lambda) = \frac{\sin\lambda x}{\lambda} + \frac{1}{\lambda^2}\int\limits_0^x \sin\lambda(x-t)\sin\lambda t q(t)\,dt + e^{|Im\lambda|x} O\left(\frac{R^2(\lambda)}{|\lambda|}\right) \tag{2.8}$$

$$s'(x,\lambda) = \cos\lambda x + e^{|Im\lambda|x} O(R(\lambda)) \tag{2.9}$$

$$s'(x,\lambda) = \cos\lambda x + \frac{1}{\lambda}\int\limits_0^x \cos\lambda(x-t)\sin\lambda t q(t)\,dt + e^{|Im\lambda|x} O\left(R^2(\lambda)\right) \tag{2.10}$$

$$c(x,\lambda) = \cos\lambda x + e^{|Im\lambda|x} O(R(\lambda)) \tag{2.11}$$

Proof. For the deduction of asymptotic formulas (2.7) and (2.8) we use well known integral equation

$$s(x,\lambda) = \frac{\sin\lambda x}{\lambda} + \int\limits_0^x \frac{\sin\lambda(x-t)}{\lambda} q(t)\, s(t,\lambda)\,dt \tag{2.12}$$

which is equivalent to differential equation (2.1) with initial conditions:$s(0,\lambda) = 0$, $s'(0,\lambda) = 1$.
For the deduction of formulas (2.9), (2.10) for $s'(x,\lambda)$ we differentiate (2.12):

$$s'(x,\lambda) = \cos\lambda x + \int\limits_0^x \cos\lambda(x-t)\, q(t)\, s(t,\lambda)\,dt \tag{2.13}$$

By evaluating the integral forms in equations (2.12) and (2.13),

$$\left|\int\limits_0^x \sin\lambda(x-t)\, q(t)\, r(t,\lambda)\,dt\right| \le \left|\int\limits_0^{\frac{1}{|\lambda|}} e^{|Im\lambda|(x-t)} q(t)\, r(t,\lambda)\,dt\right| + \left|\int\limits_{\frac{1}{|\lambda|}}^x e^{|Im\lambda|(x-t)} q(t)\, r(t,\lambda)\,dt\right|$$

$$\le c\left[\left|\int\limits_0^{\frac{1}{|\lambda|}} e^{|Im\lambda|(x-t)} q(t)\, e^{|Im\lambda|t}\left(\int\limits_0^t \xi\,|q(\xi)|\,d\xi\right)dt\right| + \left|\int\limits_{\frac{1}{|\lambda|}}^x e^{|Im\lambda|(x-t)} q(t)\,\frac{e^{|Im\lambda|t}}{|\lambda|}\left(\sigma_1(0) - \sigma_1\tfrac{1}{|\lambda|}\right)dt\right|\right]$$

$$= \quad ce^{|Im\lambda|x}\left(\frac{1}{2}\left(\int\limits_0^{\frac{1}{|\lambda|}} t\,|q(t)|\,dt\right)^2 + \frac{1}{|\lambda|}\left(\int\limits_0^{\frac{1}{|\lambda|}} t\,|q(t)|\,dt + \frac{1}{|\lambda|}\int\limits_{\frac{1}{|\lambda|}}^{\pi}|q(t)|\,dt\right)\int\limits_{\frac{1}{|\lambda|}}^{\pi}|q(t)|\,dt\right)$$

$$\le ce^{|Im\lambda|x}\left(\int\limits_0^{\frac{1}{|\lambda|}} t\,|q(t)|\,dt + \frac{1}{|\lambda|}\int\limits_{\frac{1}{|\lambda|}}^{\pi}|q(t)|\,dt\right)^2$$

or

$s(x,\lambda) = \frac{\sin\lambda x}{\lambda} + \frac{1}{\lambda^2}\int\limits_0^x \sin\lambda(x-t)\sin\lambda t q(t)\,dt + e^{|Im\lambda|x}O\left(\frac{R^2(\lambda)}{|\lambda|}\right)$

is obtained.

Similarly,

$$\left|\int\limits_0^x \cos\lambda(x-t)\,q(t)\,r(t,\lambda)\,dt\right| \le ce^{|Im\lambda|x}\left(\int\limits_0^{\frac{1}{|\lambda|}} t\,|q(t)|\,dt + \frac{1}{|\lambda|}\int\limits_{\frac{1}{|\lambda|}}^{\pi} |q(t)|\,dt\right)^2$$

is shown. We substitute these inequalities in equations (2.12) and (2.13) to obtain equations (2.7) and (2.11). □

$$\begin{aligned} s(x,\lambda) &= \frac{\sin\lambda x}{\lambda} + A\frac{\sin\lambda x}{2\lambda^2}\int\limits_0^x \frac{\sin 2\lambda t}{t^\alpha}dt - A\frac{\cos\lambda x}{2\lambda^2}\int\limits_0^x q_0(t)\,dt \\ &+ \frac{1}{2\lambda^2}\int\limits_0^x \cos\lambda(x-2t)\,q_0(t)\,dt + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^{5-2\alpha}}\right) \end{aligned} \tag{2.14}$$

If $q(x) = \frac{A}{x^\alpha} + q_0(x)$ is taken into account and

$$\begin{aligned} s'(x,\lambda) &= \cos\lambda x + \frac{A\cos\lambda x}{2\lambda}\int\limits_0^x \frac{\sin 2\lambda t}{t^\alpha}dt + \frac{A\sin\lambda x}{\lambda}\int\limits_0^x \frac{\sin^2\lambda t}{t^\alpha}dt \\ &+ \frac{\sin\lambda x}{2\lambda}\int\limits_0^x q_0(t)\,dt - \frac{1}{2\lambda}\int\limits_0^x \sin\lambda(x-2t)\,q_0(t)\,dt + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^{4-2\alpha}}\right) \end{aligned} \tag{2.15}$$

In the case of $\alpha = 1$, the term $O\left(e^{|Im\lambda|x}\frac{\ln^2|\lambda|}{|\lambda|^3}\right)$ remains in the equation (2.14) and become $O\left(e^{|Im\lambda|x}\frac{\ln^2|\lambda|}{|\lambda|^2}\right)$ in the case of the equation (2.15). In the case of $\alpha\epsilon\left[\frac{3}{2},2\right)$, the terms containing the function $q_0(x)$ in equations (2.14) and (2.15) are in the remaining forms.

Theorem 2.2. *The boundary value problems (2.1), (2.3) and (2.1), (2.4) have countable number of eigenvalues and largest by modulus are equal to correspondingly:* $\sqrt{\lambda_k} = k + O(R(k))$
and
$\sqrt{\mu_k} = k - \frac{1}{2} + O\left(R(k-\frac{1}{2})\right)$
at the first approximation and

$$\sqrt{\lambda_k} = k + \frac{1}{\pi k}\int\limits_0^x q(t)\sin^2 ktdt + O(R^2(k)) \tag{2.16}$$

and

$$\sqrt{\mu_k} = k - \frac{1}{2} + \frac{1}{\pi\left(k-\frac{1}{2}\right)}\int\limits_0^\pi q(t)\sin^2(k-\frac{1}{2})tdt + O\left(R^2(k-\frac{1}{2})\right) \tag{2.17}$$

the second approximation.

It is easy to verify, that eigenvalue of bounded value problems (2.1), (2.3) and (2.1), (2.4) coincides correspondingly with squares of roots of its characteristic equations
$\psi_{1,3}(\lambda) = s(\pi,\lambda) = 0, \psi_{1,4}(\lambda) = s'(\pi,\lambda) = 0.$

Asymptotic formulas of Theorem 2.2 could be proved by well known methods (see [1, 2]) on the base of asymptotic formulas for the solution $s(x,\lambda)$ mentioned in Theorem 2.1.

Substituting potential (2.2), we obtain

$$\sqrt{\lambda_k}=k+\tfrac{1}{\pi k}\int\limits_0^{\pi}\left[\tfrac{A}{t^{\alpha}}+q_0(t)\right]\sin^2 ktdt+O\left(\left[\int\limits_0^{\frac{1}{k-1}}t(\tfrac{A}{t^{\alpha}}+q_0(t))dt+\tfrac{1}{k}\int\limits_{\frac{1}{k-1}}^{\pi}(\tfrac{A}{t^{\alpha}}+q_0(t))dt\right]^2\right)$$

$$\sqrt{\mu_k}=k-\tfrac{1}{2}+\tfrac{1}{\pi\left(k-\frac{1}{2}\right)}\int\limits_0^{\pi}\left[\tfrac{A}{t^{\alpha}}+q_0(t)\right]\sin^2 ktdt+O\left(\left[\int\limits_0^{\frac{1}{k-1}}t(\tfrac{A}{t^{\alpha}}+q_0(t))dt+\tfrac{1}{k}\int\limits_{\frac{1}{k-1}}^{\pi}(\tfrac{A}{t^{\alpha}}+q_0(t))dt\right]^2\right)$$

Since,

$$O\left(\left[\int\limits_0^{\frac{1}{k-1}}\frac{1}{t^{\alpha-1}}dt+\frac{1}{k}\int\limits_{\frac{1}{k-1}}^{\pi}\frac{1}{t^{\alpha}}dt+\int\limits_0^{\frac{1}{k-1}}t\left|q_0(t)\right|dt+\frac{1}{k}\int\limits_{\frac{1}{k-1}}^{\pi}\left|q_0(t)\right|dt\right]^2\right) \tag{2.18}$$

$$=\quad O\left(\left[\frac{1}{k^{2-p}}+\sqrt{\int\limits_0^{\frac{1}{k-1}}t^2dt}\sqrt{\int\limits_0^{\frac{1}{k-1}}q_0^2(t)dt}\right]^2\right)=O\left(\frac{1}{k^{4-2\alpha}}\right)$$

$$\begin{aligned}\int\limits_0^{\pi}\frac{\sin^2 kt}{t^{\alpha}}dt &= n^{\alpha-1}\left[\int\limits_0^{\infty}\frac{\sin^2\xi}{\xi^{\alpha}}d\xi-\int\limits_{n\pi}^{\infty}\frac{1-\cos 2\xi}{2\xi^{\alpha}}d\xi\right] \\ &= n^{\alpha-1}C_{\alpha}-\frac{1}{2\left(\alpha-1\right)\pi^{\alpha-1}}+O\left(\frac{1}{n}\right)\end{aligned} \tag{2.19}$$

Substituting (2.18) and (2.19) into the formula (2.16) and (2.17), we obtained asymptotics formulas:

$$\sqrt{\lambda_k}=k+\tfrac{AC_{\alpha}}{\pi k^{2-\alpha}}-\tfrac{1}{2\pi k}\left[\tfrac{A}{(\alpha-1)\pi^{\alpha-1}}+\int\limits_0^{\pi}q_0(t)\left(1-\cos 2kt\right)dt\right]+O\left(\tfrac{1}{k^{4-2\alpha}}\right)$$

$$\sqrt{\mu_k}=k-\tfrac{1}{2}+\tfrac{AC_{\alpha}}{\pi\left(k-\frac{1}{2}\right)^{2-\alpha}}-\tfrac{1}{2\pi(k-\frac{1}{2})}\left[\tfrac{A}{(\alpha-1)\pi^{\alpha-1}}+\int\limits_0^{\pi}q_0(t)\left(1-\cos 2kt\right)dt\right]+O\left(\tfrac{1}{k^{4-2\alpha}}\right)$$

where

$$C_{\alpha}=\int\limits_0^{\infty}\tfrac{\sin^2\xi}{\xi^{\alpha}}d\xi=\tfrac{2^{\alpha-3}\pi}{(\alpha-1)\Gamma(\alpha-1)\sin\left[\frac{\pi(\alpha-1)}{2}\right]}$$

It must be noted that from the classical theorems on oscillation follows alternative property of this eigenvalues, more exactly: $-\infty<\lambda_1<\mu_1<\lambda_2<\mu_2<....$

Let us show the necessary for the solution of the problem by two spectrums statements. For its solution, we use method from [1], and we reduce it to the considered inverse problem of quantum theory of scattering. The following boundary value problems are considered:

$$-y''+q\left(x\right)y(x)=\lambda^2 y\left(x\right)\qquad\left(0\le x<+\infty\right) \tag{2.20}$$

$$y\left(0\right)=0 \tag{2.21}$$

which have properties:

$$q\left(x\right)=\begin{cases} q_1\left(x\right)+q_0\left(x\right), & 0\le x\le\pi \\ 0, & x>\pi\end{cases} \tag{2.22}$$

discret spectrum is absent.

Theorem 2.3. *For arbitrary function $S\left(\lambda\right)$ $(-\infty<\lambda<\infty)$ to be the scattering function of same boundary value problem (2.20), (2.21) satisfying to the conditions 1), 2) it is necessary and sufficient the fulfilement of the following conditions:*

1.Function $S(\lambda)$ $(-\infty < \lambda < \infty)$ is continuous on the whole axis,

$S(\lambda) = \overline{S(-\lambda)} = [S(-\lambda)]^{-1}$

and

$\lim\limits_{|\lambda|\to\infty} (1 - S(\lambda)) = 0$

and is the Fourier transformation of function

$F_s(x) = \frac{1}{2} \int\limits_{-\infty}^{\infty} [1 - S(\lambda)] e^{i\lambda x} dx$

that can be represented in the form of the sum of two functions , one of which belong to space $L_1(-\infty, \infty)$ and other one is bounded and belong to $L_2(-\infty, \infty)$. On the positive half axis function, $F_s(x)$ have a derivative, satisfying the condition $\int\limits_0^{\infty} x \left|F_s^{'}(x)\right| dx < \infty$

for the argument of function $s(\lambda)$, the following equality is true

$\frac{\ln S(+0) - \ln S(+\infty)}{\pi i} = \frac{1 - S(0)}{2}$

2. Function $F_s(x) = 0$ for $x > 2\pi$ and is differentable on interval $(0, 2\pi)$ and $\left[F_s^{'}(x) - q_1\left(\frac{x}{2}\right)\right] \in L_2[0, 2\pi]$
The proof of the Theorem 2.2 was considered in the [1].

Theorem 2.4. *If we use the potential of the type* (2.22) *in the equation* (2.20)*, then*

$$e(0, \lambda) = e^{i\lambda x}\left[s^{'}(\pi, \lambda) - i\lambda s(\pi, \lambda)\right] \tag{2.23}$$

where $s(x, \lambda)$ *is a solution of equation* (2.20)*:* $s(0, \lambda) = 0, s^{'}(0, \lambda) = 1$.

Proof. So, the functions $s(x, \lambda)$ and $c(x, \lambda)$ makes fundamental system of solutious of equation (2.20) with potential (2.22), then solution $e(x, \lambda)$ of this equation is equal to the linear combination $e(x, \lambda) = c_1 s(x, \lambda) + c_2 c(x, \lambda)$
By virtue of representation

$$e(x, \lambda) = e^{i\lambda x} + \int\limits_x^{\infty} K(x, t) e^{i\lambda t} dt$$

potential is equal to zero for $x > \pi$, therefore $e(x, \lambda) = e^{i\lambda x}$ for $x > \pi$ and at the point $x = \pi$ the following equalities are fulfilled;

$e^{i\lambda\pi} = c_1 s(\pi, \lambda) + c_2 c(\pi, \lambda)$, $i\lambda e^{i\lambda\pi} = c_1 s^{'}(\pi, \lambda) + c_2 c^{'}(\pi, \lambda)$

From these equalities we determine c_1 and c_2, and obtain:

$e(x, \lambda) = e^{i\lambda\pi}\left[i\lambda c(x, \lambda) - c^{'}(x, \lambda)\right] s(x, \lambda) + \left[s^{'}(\pi, \lambda) - i\lambda s(\pi, \lambda)\right] e(x, \lambda)$

and therefore formula (2.23) is obvious. □

Theorem 2.5. *For the functions* $u(z), v(z)$ *to allow the represatation*

$$u(z) = \sin\pi z + \int\limits_0^{\pi} \frac{\sin z(\pi - t)}{z} q(t) \sin zt dt + B\pi \frac{4z\cos\pi z}{4z^2 - 1} + \frac{f(z)}{z}$$

$$v(z) = \cos\pi z + \int\limits_0^{\pi} \frac{\sin z(\pi - t)}{z} q(t) \sin zt dt - C\pi \frac{\sin\pi z}{z} + \frac{g(z)}{z}$$

where $q(t) = \frac{A}{t^{\alpha}} + q_0(t)$, *B,C,A are constant numbers,* $\alpha \in [1, 2)$, $q_0(t) \in L_2[0, \pi]$, $f(z) = \int\limits_0^{\pi} \widetilde{f(t)} \cos zt dt$, $\widetilde{f(t)} \in L_2[0, \pi]$,

$\int\limits_0^{\pi} \widetilde{f(t)} dt = 0$,

$g(z) = \int\limits_0^\pi \widetilde{g(t)} \sin ztdt, \widetilde{g(t)} \in L_2[0,\pi]$.

It is necessary and sufficient that,

$$u(z) = \pi z \prod_{k=1}^{\infty} \frac{u_k^2 - z^2}{k^2}, v(z) = \prod_{k=1}^{\infty} \frac{v_k^2 - z^2}{(k - \frac{1}{2})^2}$$

where

$u_k = k + \frac{AC_\alpha}{\pi k^{2-\alpha}} - \frac{1}{k}\left[B + \frac{A}{2(\alpha-1)\pi^\alpha}\right] + \frac{a_k}{k}$,

$v_k = k - \frac{1}{2} + \frac{AC_\alpha}{\pi(k-\frac{1}{2})^{2-\alpha}} - \frac{1}{k-\frac{1}{2}}\left[C + \frac{A}{2(\alpha-1)\pi^\alpha}\right] + \frac{b_k}{k-\frac{1}{2}}$

and moreover, $\sum\limits_{k=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right) < \infty$.

The proof of this theorem is similar to analogous at [1]. In Theorem 2.5 functions $u(z)$ and $v(z)$ are taken as analogies of functions $s(\pi, z)$ and $s^{'}(\pi, z)$ correspondingly.

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