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Research Article

BLOW UP OF SOLUTIONS FOR A NONLINEAR VISCOELASTIC WAVE EQUATIONS WITH VARIABLE EXPONENTS

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Abstract: *The main purpose of this work is to study the blow up of solutions for the viscoelastic wave equation with variable exponents in a bounded domain. Our result extends the one in [11] to problems with variable exponent nonlinearities.*

Keywords: *Viscoelastic wave equation; Blow up; Variable exponent.*

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1. Introduction

In this work, we investigate the nonlinear viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u, \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

with the inital-boundary conditions

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{2}$$

and

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{3}$$

here Ω is a regular and bounded domain in R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$.

The variable exponents $p(.)$ and $q(.)$ are given as measurable functions on Ω satisfying

$$2 \leq p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ \leq q^* \tag{4}$$

where

$$p^{-} = ess\inf_{x\in\Omega} p(x), \quad p^{+} = ess\sup_{x\in\Omega} p(x),$$
$$q^{-} = ess\inf_{x\in\Omega} q(x), \quad q^{+} = ess\sup_{x\in\Omega} q(x),$$

and

$$q^{*} = \begin{cases} \infty, & \text{if } n = 1, 2, \\ \frac{2n}{n-2}, & \text{if } n \geq 3. \end{cases}$$

We make some assumptions on g :

(A1) Let $g \in C^1$ and

$$1 - \int_0^\infty g(\tau)\, d\tau = l > 0.$$

(A2) $g(\tau) \geq 0$, $g'(\tau) \leq 0$ and

$$\int_0^\infty g(\tau)\, d\tau < \frac{q^{-}(1-\xi) - 2}{q^{-}(1-\xi) - 2 + \frac{1}{4q^{-}(1-\xi)}}, \quad 0 < \xi < 1.$$

Remark 1 *There are some functions $g(\tau)$ satisfying (A1) and (A2). An example is*

$$g(\tau) = e^{-(\tau+1)}.$$

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When $p(x)$ and $q(x)$ are constants, (1) become the following the viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\, \Delta u(\tau)\, d\tau + |u_t|^{p-2} u_t = |u|^{q-2} u. \tag{5}$$

In [11], the author proved nonexistence of global solutions, for the equation (5). In [12], the same author extended this result in the case of positive initial energy. Later, some authors studied nonexistence of solutions of the equation (5) (see [17, 18]).

Without the viscoelastic term ($g = 0$) the problem (1) reduces to the following form

$$u_{tt} - \triangle u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \tag{6}$$

Messaoudi et al. [13] studied the local existence and nonexistence of the solutions of the equation (6). For more results concerning nonexistence of global solutions, see [2, 7, 8, 10]. For more results about the variable exponent spaces we refer the readers to [1, 14].

Motivated by the above results, in this work, we prove the blow up of solutions (1) under some conditions.

The outline of this work is as follows: In section 2, we state some results about the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ and Sobolev spaces $W^{1,p(x)}(\Omega)$. In section 3, the blow up results will be proved.

2. Preliminaries

In this part, we state some results about the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ and Sobolev spaces $W^{1,p(x)}(\Omega)$ (see [5, 6, 9, 15]).

Let $p : \Omega \to [1, \infty]$ be a measurable function, here Ω is a domain of R^n. We define the Lebesgue space with a variable exponent $p(.)$ by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to R, \text{ measurable and } \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

$L^{p(x)}(\Omega)$ is a Banach space.

The Sobolev space with a variable exponent is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Variable exponent Sobolev space is a Banach space with the following the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}.$$

Let the variable exponent $p(.)$ satisfy the log-Hölder continuity condition:

$$|p(x) - p(y)| \leq \frac{A}{\log \frac{1}{|x-y|}}, \text{ for all } x, y \in \Omega \text{ with } |x - y| < \delta, \tag{7}$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2 *[5] (Poincare inequality) Let Ω be a bounded domain of R^n and $p(.)$ satisfies log-Hölder condition, then*

$$\|u\|_{p(.)} \leq c \|\nabla u\|_{p(.)}, \text{ for all } u \in W_0^{1,p(.)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 3 *[5] Let $p(.) \in C(\overline{\Omega})$ and $q : \Omega \to [1, \infty)$ be a measurable function and satisfy*

$$\underset{x \in \overline{\Omega}}{essinf}(p^*(x) - q(x)) > 0.$$

Then the Sobolev embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact. Where

$$p^*(x) = \begin{cases} \frac{np(x)}{\underset{x \in \overline{\Omega}}{esssup}(n-p(x))}, & if\ p^- < n \\ \infty, & if\ p^- \geq n. \end{cases}$$

The local existence of solutions for the problem (1) that can be established by combining arguments of [3, 7, 13].

Theorem 4 *(Local existence and uniqueness). Assume that (A1), (A2), (4) and (7) holds, and that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then problem (1) has a unuque local solution*

$$u \in C\left([0,T); H_0^1(\Omega)\right), \quad u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{p(.)}(\Omega \times (0,T)).$$

3. Blow up

In this part, we prove that the blow up of the solution for problem (1). Firstly, we give following lemmas:

Lemma 5 *[13] If* $q : \Omega \to [1, \infty)$ *is a measurable function and*

$$2 \le q^- \le q(x) \le q^+ < \frac{2n}{n-2}; \quad n \ge 3 \tag{8}$$

holds. Then, we have following inequalities:

i)

$$\rho_{q(.)}^{\frac{s}{q^-}}(u) \le c\left(\|\nabla u\|^2 + \rho_{q(.)}(u)\right), \tag{9}$$

ii)

$$\|u\|_{q^-}^s \le c\left(\|\nabla u\|^2 + \|u\|_{q^-}^{q^-}\right), \tag{10}$$

iii)

$$\rho_{q(.)}^{\frac{s}{q^-}}(u) \le c\left(|H(t)| + \|u_t\|^2 + \rho_{q(.)}(u)\right), \tag{11}$$

iv)

$$\|u\|_{q^-}^s \le c\left(|H(t)| + \|u_t\|^2 + \|u\|_{q^-}^{q^-}\right), \tag{12}$$

v)

$$c\|u\|_{q^-}^{q^-} \le \rho_{q(.)}(u) \tag{13}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \le s \le q^-$. *Where* $\rho_{q(.)}(u) = \int_\Omega |u|^{q(.)}\,dx$, *and* $c > 1$ *a positive constant and*

$$H(t) = -\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\nabla u\|^2 - \frac{1}{2}(g \circ \nabla u)(t) + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx.$$

Lemma 6 *Suppose that (A1), (A2), (4) and (7) hold. Then*

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\nabla u\|^2 + \frac{1}{2}(g \circ \nabla u)(t) - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx \tag{14}$$

is a nonincreasing function and

$$\begin{aligned} E'(t) &= -\int_\Omega \frac{1}{p(x)}|u_t|^{p(x)}\,dx - \frac{1}{2}g(\tau)\int_\Omega |\nabla u(t)|^2\,dxd\tau \\ &\quad + \frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega [\nabla u(\tau) - \nabla u(t)]^2\,dxd\tau. \end{aligned}$$

Proof. Multiplying u_t on two sides of the problem (1), and integrating by part, we get

$$
\begin{aligned}
& \frac{d}{dt}\left[\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{2}\|\nabla u\|^2-\int_{\Omega} \frac{1}{q(x)}|u|^{q(x)} dx\right] \\
& -\int_0^t \int_{\Omega} g(t-\tau) \nabla u(\tau) \nabla u_t(t)\, dx d\tau \\
= & -\int_{\Omega}\left|u_t\right|^{p(x)} dx. \qquad (15)
\end{aligned}
$$

Now, we estimate last term in the left hand side of (15), we obtain

$$
\begin{aligned}
& \int_0^t \int_{\Omega} g(t-\tau) \nabla u(\tau) \nabla u_t(t)\, dx d\tau \\
= & \int_0^t g(t-\tau) \int_{\Omega} \nabla u_t(t)[\nabla u(\tau)-\nabla u(t)+\nabla u(t)]\, dx d\tau \\
= & \int_0^t g(t-\tau) \int_{\Omega} \nabla u_t(t)[\nabla u(\tau)-\nabla u(t)]\, dx d\tau+\int_0^t g(t-\tau) \int_{\Omega} \nabla u_t(t) \nabla u(t)\, dx d\tau \\
= & -\frac{1}{2} \int_0^t g(t-\tau) \frac{d}{dt}\left[\int_{\Omega}[\nabla u(\tau)-\nabla u(t)]^2 dx\right] d\tau \\
& +\frac{1}{2} \int_0^t g(\tau)\left[\frac{d}{dt} \int_{\Omega}|\nabla u(t)|^2 dx\right] d\tau \\
= & -\frac{1}{2} \frac{d}{dt}\left[\int_0^t g(t-\tau) \int_{\Omega}[\nabla u(\tau)-\nabla u(t)]^2 dx d\tau\right]+\frac{1}{2} \int_0^t g'(t-\tau) \int_{\Omega}[\nabla u(\tau)-\nabla u(t)]^2 dx d\tau \\
& +\frac{1}{2} \frac{d}{dt}\left[\int_0^t g(\tau) \int_{\Omega}|\nabla u(t)|^2 dx d\tau\right]-\frac{1}{2} g(\tau) \int_{\Omega}|\nabla u(t)|^2 dx d\tau. \qquad (16)
\end{aligned}
$$

Finally, inserting (16) into (15), we get

$$
\begin{aligned}
& \frac{d}{dt}\left[\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{2}\left(1-\int_0^t g(\tau)\, d\tau\right)\|\nabla u\|^2+\frac{1}{2}(g \circ \nabla u)(t)-\int_{\Omega} \frac{1}{q(x)}|u|^{q(x)} dx\right] \\
= & -\int_{\Omega} \frac{1}{p(x)}\left|u_t\right|^{p(x)} dx-\frac{1}{2} g(\tau) \int_{\Omega}|\nabla u(t)|^2 dx d\tau+\frac{1}{2} \int_0^t g'(t-\tau) \int_{\Omega}[\nabla u(\tau)-\nabla u(t)]^2 dx d\tau \\
\leq & \ 0, \qquad (17)
\end{aligned}
$$

where

$$
(g \circ \nabla u)(t)=\int_0^t g(t-\tau) \int_{\Omega}[\nabla u(\tau)-\nabla u(t)]^2 dx d\tau .
$$

■

Now, we state and prove our blow up result.

Theorem 7 *Under the assumptions of Theorem 4, and*

$$E(0) < 0.$$

Then the solution of the problem (1) blow up in finite time.

Proof. Let

$$H(t) = -E(t).$$

We see from (17) that $H(t) \geq 0$. Also, by the definition $H(t)$, we have

$$\begin{aligned} H(t) &= -\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\nabla u\|^2 - \frac{1}{2}(g \circ \nabla u)(t) + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx \\ &\leq \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx \\ &\leq \frac{1}{q^-}\rho_{q(.)}(u). \end{aligned} \tag{18}$$

Now, we define $\Psi(t)$ as follows

$$\Psi(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t\,dx, \tag{19}$$

for ε small to be chosen later and

$$0 < \sigma \leq \min\left\{\frac{q^- - p^+}{(p^+ - 1)\,q^-}, \frac{q^- - 2}{2q^-}\right\}. \tag{20}$$

The time derivative of (19) and using Eq. (1), we have

$$\begin{aligned} \Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega \left(u_t^2 + u u_{tt}\right)dx \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 \\ &\quad + \varepsilon\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\nabla u(\tau)\,dx\,d\tau \\ &\quad + \varepsilon\int_\Omega |u|^{q(.)}\,dx - \varepsilon\int_\Omega u u_t|u_t|^{p(.)-2}\,dx \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 \\ &\quad + \varepsilon\int_\Omega |u|^{q(.)}\,dx - \varepsilon\int_\Omega u u_t|u_t|^{p(.)-2}\,dx \\ &\quad + \varepsilon\int_0^t g(\tau)\,d\tau\,\|\nabla u\|^2 + \varepsilon\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\left[\nabla u(\tau) - \nabla u(t)\right]dx\,d\tau. \end{aligned} \tag{21}$$

By using Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned} & \int_0^t g(t-\tau)\int_\Omega \nabla u(t)\left[\nabla u(\tau)-\nabla u(t)\right]dxd\tau \\ \leq & \int_0^t g(t-\tau)\left\|\nabla u(t)\right\|\left\|\nabla u(\tau)-\nabla u(t)\right\|d\tau \\ \leq & \lambda\left(g\circ\nabla u\right)(t)+\frac{1}{4\lambda}\int_0^t g(\tau)\,d\tau\left\|\nabla u\right\|^2, \quad \lambda>0. \end{aligned} \tag{22}$$

Substituting (22) into (21), we have

$$\begin{aligned} \Psi'(t) \geq & (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left\|u_t\right\|^2-\varepsilon\left\|\nabla u\right\|^2 \\ & +\varepsilon\int_\Omega |u|^{q(.)}dx-\varepsilon\int_\Omega uu_t\left|u_t\right|^{p(.)-2}dx+\varepsilon\int_0^t g(\tau)\,d\tau\left\|\nabla u\right\|^2 \\ & -\varepsilon\lambda\left(g\circ\nabla u\right)(t)-\frac{\varepsilon}{4\lambda}\int_0^t g(\tau)\,d\tau\left\|\nabla u\right\|^2 \end{aligned}$$

By using the definition of the $H(t)$, we have

$$\begin{aligned} -\varepsilon q^-(1-\xi)H(t) = & \frac{\varepsilon q^-(1-\xi)}{2}\left\|u_t\right\|^2+\frac{\varepsilon q^-(1-\xi)}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)\left\|\nabla u\right\|^2 \\ & +\frac{\varepsilon q^-(1-\xi)}{2}\left(g\circ\nabla u\right)(t)-\varepsilon q^-(1-\xi)\int_\Omega\frac{1}{q(x)}|u|^{q(.)}dx, \end{aligned} \tag{23}$$

where $0<\xi<1$.

Subtracting and adding (23) on the right hand side of (21), we get

$$\begin{aligned} \Psi'(t) \geq & (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon q^-(1-\xi)H(t)+\varepsilon\left(\frac{q^-(1-\xi)}{2}+1\right)\left\|u_t\right\|^2 \\ & +\varepsilon\left[\frac{q^-(1-\xi)}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)-1+\left(1-\frac{1}{4\lambda}\right)\int_0^t g(\tau)\,d\tau\right]\left\|\nabla u\right\|^2 \\ & \varepsilon\left(\frac{q^-(1-\xi)}{2}-\lambda\right)\left(g\circ\nabla u\right)(t)+\varepsilon\xi\int_\Omega|u|^{q(.)}dx-\varepsilon\int_\Omega uu_t\left|u_t\right|^{p(.)-2}dx. \end{aligned} \tag{24}$$

Then, for ξ small enough, we get

$$\begin{aligned} \Psi'(t) \geq & \varepsilon\beta\left[H(t)+\left\|u_t\right\|^2+\left\|\nabla u\right\|^2+\left(g\circ\nabla u\right)(t)+\rho_{q(.)}(u)\right] \\ & +(1-\sigma)H^{-\sigma}(t)H'(t)-\varepsilon\int_\Omega uu_t\left|u_t\right|^{p(.)-2}dx \end{aligned} \tag{25}$$

where

$$\begin{aligned} \beta = & \min\left\{q^-(1-\xi),\ \frac{q^-(1-\xi)}{2}-\lambda,\ \frac{q^-(1-\xi)}{2}+1,\ \varepsilon\xi,\right. \\ & \left.\frac{q^-(1-\xi)}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)-1+\left(1-\frac{1}{4\lambda}\right)\int_0^t g(\tau)\,d\tau\right\} \\ > & 0 \end{aligned}$$

and

$$\rho_{q(.)}(u) = \int_\Omega |u|^{q(.)}\, dx.$$

By using the following Young's inequality

$$XY \le \frac{\delta^k X^k}{k} + \frac{\delta^{-l} Y^l}{l},$$

where $X, Y \ge 0,\ \delta > 0,\ k, l \in R^+$ such that $\frac{1}{k} + \frac{1}{l} = 1$. As a result, applying the previous we obtain

$$\begin{aligned} \int_\Omega u\, |u_t|^{p(.)-1}\, dx &\le \int_\Omega \frac{1}{p(x)} \delta^{p(x)} |u|^{p(x)}\, dx + \int_\Omega \frac{p(x)-1}{p(x)} \delta^{-\frac{p(x)}{p(x)-1}} |u_t|^{p(x)}\, dx \\ &\le \frac{1}{p^-} \int_\Omega \delta^{p(x)} |u|^{p(x)}\, dx + \frac{p^+-1}{p^+} \int_\Omega \delta^{-\frac{p(x)}{p(x)-1}} |u_t|^{p(x)}\, dx, \end{aligned} \tag{26}$$

where δ is constant depending on the time t and specified later. Inserting estimate (26) into (25), we get

$$\begin{aligned} \Psi'(t) \ \ge\ & \varepsilon\beta \left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + (g \circ \nabla u)(t) + \rho_{q(.)}(u) \right] \\ & + (1-\sigma) H^{-\sigma}(t) H'(t) \\ & - \varepsilon \frac{1}{p^-} \int_\Omega \delta^{p(x)} |u|^{p(x)}\, dx - \varepsilon \frac{p^+-1}{p^+} \int_\Omega \delta^{-\frac{p(x)}{p(x)-1}} |u_t|^{p(x)}\, dx. \end{aligned} \tag{27}$$

Let us choose δ, so that $\delta^{-\frac{p(x)}{p(x)-1}} = k_1 H^{-\sigma}(t)$, where $k_1 > 0$ is specified later, we obtain

$$\begin{aligned} \Psi'(t) \ \ge\ & \varepsilon\beta \left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + (g \circ \nabla u)(t) + \rho_{q(.)}(u) \right] \\ & + (1-\sigma) H^{-\sigma}(t) H'(t) \\ & - \varepsilon \frac{1}{p^-} \int_\Omega k^{1-p(x)} H^{\sigma(p(x)-1)}(t) |u|^{p(x)}\, dx - \varepsilon \frac{p^+-1}{p^+} \int_\Omega k H^{-\sigma}(t) |u_t|^{p(x)}\, dx \\ \ge\ & \varepsilon\beta \left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + (g \circ \nabla u)(t) + \rho_{q(.)}(u) \right] \\ & + (1-\sigma) H^{-\sigma}(t) H'(t) \\ & - \varepsilon \frac{k^{1-p^-}}{p^-} H^{\sigma(p^+-1)}(t) \int_\Omega |u|^{p(x)}\, dx - \varepsilon \left(\frac{p^+-1}{p^+} \right) k H^{-\sigma}(t) \int_\Omega |u_t|^{p(x)}\, dx \\ \ge\ & \varepsilon\beta \left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + (g \circ \nabla u)(t) + \rho_{q(.)}(u) \right] \\ & + \left[(1-\sigma) - \varepsilon \left(\frac{p^+-1}{p^+} \right) k \right] H^{-\sigma}(t) H'(t) - \varepsilon \frac{k^{1-p^-}}{p^-} H^{\sigma(p^+-1)}(t) \int_\Omega |u|^{p(x)}\, dx. \end{aligned}$$

By using (13) and (18), we get

$$\begin{aligned}
H^{\sigma(p^+-1)}(t)\int_\Omega |u|^{p(x)}\,dx &\leq H^{\sigma(p^+-1)}(t)\left[\int_{\Omega_-}|u|^{p^-}\,dx+\int_{\Omega_+}|u|^{p^+}\,dx\right]\\
&\leq H^{\sigma(p^+-1)}(t)\,c\left[\left(\int_{\Omega_-}|u|^{q^-}\,dx\right)^{\frac{p^-}{q^-}}+\left(\int_{\Omega_+}|u|^{q^-}\,dx\right)^{\frac{p^+}{q^-}}\right]\\
&= H^{\sigma(p^+-1)}(t)\,c\left[\|u\|_{q^-}^{p^-}+\|u\|_{q^-}^{p^+}\right]\\
&\leq c\left(\frac{1}{q^-}\rho_{q(.)}(u)\right)^{\sigma(p^+-1)}\left[\left(\rho_{q(.)}(u)\right)^{\frac{p^-}{q^-}}+\left(\rho_{q(.)}(u)\right)^{\frac{p^+}{q^-}}\right]\\
&= c_1\left[\left(\rho_{q(.)}(u)\right)^{\frac{p^-}{q^-}+\sigma(p^+-1)}+\left(\rho_{q(.)}(u)\right)^{\frac{p^+}{q^-}+\sigma(p^+-1)}\right] \qquad (28)
\end{aligned}$$

where $\Omega_-=\{x\in\Omega:|u|<1\}$ and $\Omega_+=\{x\in\Omega:|u|\geq 1\}$.

We then use Lemma 5 and (20), for

$$s=p^-+\sigma q^-\left(p^+-1\right)\leq q^-$$

and

$$s=p^++\sigma q^-\left(p^+-1\right)\leq q^-,$$

to deduce, from (28),

$$H^{\sigma(p^+-1)}(t)\int_\Omega |u|^{p(x)}\,dx\leq c_1\left[\|\nabla u\|^2+\rho_{q(.)}(u)\right]. \qquad (29)$$

Thus, inserting estimate (29) into (25), we have

$$\begin{aligned}
\Psi'(t) &\geq \varepsilon\left(\beta-\frac{k^{1-p^-}}{p^-}c_1\right)\left[H(t)+\|u_t\|^2+\|\nabla u\|^2+(g\circ\nabla u)(t)+\rho_{q(.)}(u)\right]\\
&\quad+\left[(1-\sigma)-\varepsilon\left(\frac{p^+-1}{p^+}\right)k\right]H^{-\sigma}(t)\,H'(t). \qquad (30)
\end{aligned}$$

Let us choose k large enough so that $\gamma=\beta-\frac{k^{1-p^-}}{p^-}c_1>0$, and picking ε small enough such that $(1-\sigma)-\varepsilon\left(\frac{p^+-1}{p^+}\right)k\geq 0$ and

$$\Psi(t)\geq\Psi(0)=H^{1-\sigma}(0)+\varepsilon\int_\Omega u_0u_1dx>0,\ \forall t\geq 0. \qquad (31)$$

Consequently, (30) yields

$$\begin{aligned}
\Psi'(t) &\geq \varepsilon\gamma\left[H(t)+\|u_t\|^2+\|\nabla u\|^2+(g\circ\nabla u)(t)+\rho_{q(.)}(u)\right]\\
&\geq \varepsilon\gamma\left[H(t)+\|u_t\|^2+\|\nabla u\|^2+(g\circ\nabla u)(t)+\|u\|_{q^-}^{q^-}\right], \qquad (32)
\end{aligned}$$

due to (13). Therefore we get

$$\Psi(t)\geq\Psi(0)>0,\ \text{for all } t\geq 0.$$

On the other hand, exploiting Hölder's inequality, we get

$$\begin{aligned}\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}} &\leq \|u\|^{\frac{1}{1-\sigma}} \|u_t\|^{\frac{1}{1-\sigma}} \\ &\leq C\left(\|u\|_{q^-}^{\frac{1}{1-\sigma}} \|u_t\|^{\frac{1}{1-\sigma}}\right).\end{aligned}$$

Young inequality gives

$$\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u\|_{q^-}^{\frac{\mu}{1-\sigma}} + \|u_t\|^{\frac{\theta}{1-\sigma}}\right), \tag{33}$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\sigma)$, to obtain $\frac{\mu}{1-\sigma} = \frac{2}{1-2\sigma} \leq q^-$ by (20). Therefore, (33) becomes

$$\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{s}\right),$$

where $\frac{2}{1-2\sigma} \leq q^-$. By using (12), we get

$$\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{q^-} + H(t)\right).$$

Thus,

$$\begin{aligned}\Psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t) + \varepsilon \int_{\Omega} uu_t dx\right]^{\frac{1}{1-\sigma}} \\ &\leq 2^{\frac{\sigma}{1-\sigma}}\left(H(t) + \varepsilon^{\frac{1}{1-\sigma}}\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}}\right) \\ &\leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{q^-} + H(t)\right) \\ &\leq C\left(H(t) + \|u_t\|^2 + \|\nabla u\|^2 + (g \circ \nabla u)(t) + \|u\|_{q^-}^{q^-}\right)\end{aligned} \tag{34}$$

where

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

is used. Consequently a combining of (32) and (34), for some $\xi > 0$, we have

$$\Psi'(t) \geq \xi \Psi^{\frac{1}{1-\sigma}}(t). \tag{35}$$

Integration of (35) over $(0,t)$ yield

$$\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}}.$$

Therefore $\Psi(t)$ blow up in a finite time

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

Then, the proof is complete. ■

4. Conclusion

In this work, we obtained the blow up for a nonlinear viscoelastic wave equations with variable exponents in a bounded domain. This improves and extends many results in the literature.

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