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Higher Order Real Derivatives Using Parabolic Analytic Functions

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Abstract

Amid the bidimensional hypercomplex numbers, parabolic numbers are defined as $\{z = x + \imath y : \ x, y \in \mathbb{R}, \imath^2 = 0, \imath \neq 0\}$. The analytic functions of a parabolic variable have been introduced as an analytic continuation of the real function of a real variable. Also, their algebraic property has already been discussed. This paper will show the n-th derivative of the real functions using parabolic numbers to further generalize the automatic differentiation. Also, we shall show some of the applications of it.

Keywords: *Automatic differentiation; Dual number; Higher order derivative; Hypercomplex numbers; Parabolic analytic functions*
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1. Introduction

There are three types of hypercomplex numbers: dual numbers, double numbers and complex numbers defined as (see [16])

$$\{x + \imath y : \ \imath^2 = -1, 0, 1, \ (x, y) \in \mathbb{R}^2\}. \tag{1.1}$$

In the same line of complex analysis, an analytic function may be obtained as an analytic continuation of the corresponding real variable functions. The works of Casanova [5] and Catoni et al. [7] can be followed in this regard. In this paper, we only consider parabolic numbers corresponding to $\imath^2 = 0$ (cf. equation (1.1)). This number system is called the set of *dual numbers*, [16]. The underlying geometry behind this two-dimensional hypercomplex number can be found in [8]. Bidimensional hypercomplex numbers are rings. In particular, parabolic numbers are isomorphic to $\mathbb{R}[X]/\langle X^2 \rangle$.
At the end of the article, we shall show some of the applications of our deduced results. This paper can extend the results derived for the first order derivative of real function of a real variable using parabolic numbers (see [7]). However, the approach used in it to obtain the equation

$$f(x + \imath y) = f(x) + iyf'(x)$$

is different, the utmost intention was to arrive at the equation of automatic differentiation and use it further to express the first order real derivative along with its algebra. The novelty of our work lies in the fact that we have tried to compute the second and higher order derivatives in the same line as discussed in [7] with the utlimate motive of developing a generalized automatic differentiation formula. In section 2, we shall discuss some of the properties of parabolic analytic functions and related topics from earlier literature [2, 3, 4, 5, 7, 9]. Section 3 is dedicated to the main results concerning higher order derivatives of parabolic analytic functions. In section 4, we have generalized the automatic differentiation. In section 5, we have discussed certain applications of our deduced results.

2. Preliminaries

Throughout the paper, we denote the set of parabolic numbers as $\mathbb{O} = \{z = a + \imath b : \ (a, b) \in \mathbb{R}^2, \ \imath^2 = 0\}$. In [5], the author defined parabolic functions as $f(v)$ using the expansion of the power series.

$$f(v) = \sum_{n=0}^{\infty} a_n x^n + \imath y \sum_{n=0}^{\infty} n a_n x^{n-1} \text{ or,}$$
$$f(v) = f(x) + \imath y f'(x), \text{ where } v = x + \imath y. \tag{2.1}$$

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For details about the power series in parabolic functions and corresponding applications one can see the work of Casanova [5]. With the help of the equation (2.1), one can define Cauchy-Riemann conditions on parabolic functions (see [6, 7]).

2.1. Derivative for a function of a real variable

If $f(z)$ is a parabolic analytic function and it is the analytic continuation of the real functions of a real variable is $f(x)$, then the first derivative of $f(x)$ is

$$f'(x) = \frac{\Im[f(z)]}{\Im(z)}, \tag{2.2}$$

where $\Im(z)$: imaginary part of $z = x + \imath y \in \mathbb{O}$.

2.1.1. Basic algebraic properties of $\mathbf{f'(x)}$

The algebraic properties of $f'(x)$ have been shown in theorem 2.1. The proof of this theorem can be found in [7].

Theorem 2.1. *If $f(x)$ and $g(x)$ are two functions of real variable with $f(z)$ and $g(z)$ are their analytic parabolic continuations (cf. (2.2)) then*

1. $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x),$
2. $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x),$
3. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}.$
4. $\dfrac{d}{dx}f[g(x)] = f'[g(x)]g'(x).$

2.2. Automatic differentiation

Derivative of real functions of a real variable can be studied using the help of parabolic functions, [7]. In this paper, we shall discuss applications of parabolic numbers in differential calculus, especially the higher order derivatives. The primary feature of these numbers, which no other number system has, is the feature of automatic differentiation. This is a technique using which the value of the function along with its first derivative can be calculated together for a variable x. The main reason why only parabolic numbers are used for this purpose is that for every $n \geq 2$, the value of $\imath^n$ becomes equal to 0, which is not so for the other two number systems. To achieve this, we replace a real number x, with a parabolic number $x + \imath y$ where the real number y is called a *seed* and is arbitrarily chosen. Suppose, we represent a polynomial in x with the function f then,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 .... + a_nx^n.$$

We know all polynomials are analytic functions. Hence, by analytic continuation of f, we can write

$$\begin{aligned} f(x+\imath y) &= a_0 + a_1(x+\imath y) + a_2(x+\imath y)^2 + a_3(x+\imath y)^3 .... + a_n(x+\imath y)^n \\ &= (a_0 + a_1x + a_2x^2 + a_3x^3 .... + a_nx^n) + \imath y(a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}) \text{ (as } \imath^n = 0\ \forall n \geq 2) \\ &= f(x) + iyf'(x), \end{aligned}$$

where f' represents first derivative of f with respect to x. So, we see that the value $f(x)$ and its first derivative $f'(x)$ are being obtained together.

3. Higher order derivative of real functions

3.1. Second order derivative

Equation (2.2) can be extended to get higher order derivative.

Lemma 3.1. *If $z = x + \imath y \in \mathbb{O}$ and $f : \mathbb{O} \to \mathbb{O}$ be an analytic function then the second derivative of $f(x)$ is*

$$f''(x) = \frac{\Im\left[\frac{\partial f}{\partial x}\right] - y'f'(x)}{\Im(z)}, \tag{3.1}$$

where $y' = \dfrac{dy}{dx}$.

Proof. From the definition of derivative (see equation (2.1)) $f(z) = f(x) + \imath y f'(x)$, where $z \in \mathbb{O}$ and $x \in \mathbb{R}$. Differentiating partially with respect to x we get

$$\frac{\partial f}{\partial x} = f'(x) + \imath\left[yf''(x) + y'f'(x)\right].$$

From the above equation

$$yf''(x)=\Im\left[\frac{\partial f}{\partial x}\right]-y'f'(x)\ \text{ or, } f''(x)=\frac{\Im\left[\frac{\partial f}{\partial x}\right]-y'f'(x)}{\Im(z)}.$$

The same can be proved by term-by-term differentiation of the series $f(z)=\sum_{n=0}^{\infty}a_n(x+\imath y)^n=\sum_{n=0}^{\infty}a_n x^{n-1}(x+\imath ny)$ with respect to x which will again give $\left[\frac{\partial f}{\partial x}\right]=f'(x)+\imath\left[yf''(x)+y'f'(x)\right]$ and therefore it gives the required result (cf. equation (3.1)). □

3.2. Verification for double derivative

We now justify our previous lemma by verifying Theorem 2.1 for second order derivatives using Lemma 3.1. We assume, $z=x+\imath y\in\mathbb{O}$ and $f,g:\mathbb{O}\to\mathbb{O}$ are analytic functions.

(i) Verification for $\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$.

$$\frac{\partial f}{\partial x}=f'(x)+\imath\left[yf''(x)+y'f'(x)\right].$$

$$\frac{\partial g}{\partial x}=g'(x)+\imath\left[yg''(x)+y'g'(x)\right].$$

Then,

$$\frac{\partial(f\pm g)}{\partial x}=\frac{\partial f}{\partial x}\pm\frac{\partial g}{\partial x}=\left[f'(x)\pm g'(x)\right]+\imath\left[y'\{f'(x)\pm g'(x)\}+y\{f''(x)\pm g''(x)\}\right].$$

Again,

$$\frac{d^2}{dx^2}(f\pm g)=\frac{\Im\left[\frac{\partial(f\pm g)}{\partial x}\right]-y'[(f\pm g)'(x)]}{\Im(z)}=\frac{y'\{f'(x)\pm g'(x)\}+y\{f''(x)\pm g''(x)\}-y'\{f'(x)\pm g'(x)\}}{y}=f''(x)\pm g''(x).$$

Hence, the lemma is justified for addition and subtraction.

(ii) Verification for $\frac{d^2}{dx^2}[f(x)g(x)]=f''(x)g(x)+2f'(x)g'(x)+g''(x)f(x)$. We know,

$$\begin{aligned}\frac{\partial(fg)}{\partial x}&=\frac{\partial f}{\partial x}g+f\frac{\partial g}{\partial x}=\left[f'(x)+\imath\{y'f'(x)+yf''(x)\}\right]\left[g(x)+\imath yg'(x)\right]+\left[f(x)+\imath yf'(x)\right]\left[g'(x)+\imath\{y'g'(x)+yg''(x)\right]\\&=f'(x)g(x)+\imath yf'(x)g'(x)+\imath y'f'(x)g(x)+\imath yf''(x)g(x)+f(x)g'(x)+\imath y'f(x)g'(x)+\imath yf(x)g''(x)+\imath yf'(x)g'(x)\\&=f'(x)g(x)+f(x)g(x)+\imath[\,2yf'(x)g'(x)+y'\{f'(x)g(x)+f(x)g'(x)\}+yf''(x)g(x)+yf(x)g''(x)].\end{aligned}$$

Now,

$$\begin{aligned}\frac{d^2}{dx^2}[f(x)g(x)]&=\frac{\Im\left[\frac{\partial(fg)}{\partial x}\right]-y'[(fg)'(x)]}{\Im(z)}\\&=\frac{2yf'(x)g'(x)+yf''(x)g(x)+yf(x)g''(x)}{y}+\frac{y'\{f'(x)g(x)+f(x)g'(x)\}}{y}-\frac{y'\{f'(x)g(x)+f(x)g'(x)\}}{y}\\&=f''(x)g(x)+2f'(x)g'(x)+f(x)g''(x).\end{aligned}$$

Hence, the lemma is justified for multiplication.

(iii) Verification for

$$\frac{d^2}{dx^2}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\{f''(x)g(x)-g''(x)f(x)\}}{g(x)^3}-\frac{2g'(x)\{f'(x)g(x)-f(x)g'(x)\}}{g(x)^3}.$$

Now,

$$\frac{1}{g(z)}=\frac{1}{g(x)+\imath yg'(x)}=\frac{g(x)-\imath yg'(x)}{g(x)^2}$$

Again,

$$\begin{aligned}&\frac{\partial(f/g)}{\partial x}\\&=\frac{1}{g(z)}\frac{\partial f}{\partial x}+f(z)\frac{\partial\{1/g(z)\}}{\partial x}\\&=\frac{1}{g(z)}\frac{\partial f}{\partial x}+f(z)\frac{\partial\left[\{g(x)-\imath yg'(x)\}/g(x)^2\right]}{\partial x}\\&=\left\{\frac{g(x)-\imath yg'(x)}{g(x)^3}\right\}\left[g(x)\{f'(x)+\imath y'f'(x)+\imath yf''(x)\}\right]+\left[f(x)+\imath yf'(x)\right]\left[\frac{g(x)\{g'(x)-\imath y'g'(x)-\imath yg''(x)\}-2g'(x)\{g(x)-\imath yg'(x)\}}{g(x)^3}\right]\\&=\frac{g(x)^2f'(x)-f(x)g(x)g'(x)}{g(x)^3}+\imath y\left[\frac{2f(x)g'(x)^2+g(x)^2f''(x)-f(x)g(x)g''(x)-2g(x)g'(x)f'(x)}{g(x)^3}\right]+\imath y'\left[\frac{f'(x)g(x)^2-f(x)g(x)g'(x)}{g(x)^3}\right].\end{aligned}$$

Therefore,

$$\Im\frac{\partial(f/g)}{\partial x}=\frac{y\{2f(x)g'(x)^2+g(x)^2f''(x)-f(x)g(x)g''(x)-2g(x)g'(x)f'(x)\}}{g(x)^3}+\frac{y'\{f'(x)g(x)^2-f(x)g(x)g'(x)\}}{g(x)^3}.$$

Now,

$$\begin{aligned}\frac{d^2}{dx^2}\left[\frac{f(x)}{g(x)}\right]&=\frac{\Im\frac{\partial(f/g)}{\partial x}-y'\left[\frac{f}{g}\right]'}{\Im(z)}\\&=\frac{y\{2f(x)g'(x)^2+g(x)^2f''(x)-f(x)g(x)g''(x)-2g(x)g'(x)f'(x)\}}{g(x)^3}\\&+\frac{y'\{f'(x)g(x)^2-f(x)g(x)g'(x)\}}{g(x)^3}-\frac{y'\{f'(x)g(x)^2-f(x)g(x)g'(x)\}}{g(x)^3}\\&=\frac{g(x)\{f''(x)g(x)-g''(x)f(x)\}}{g(x)^3}-\frac{2g'(x)\{f'(x)g(x)-f(x)g'(x)\}}{g(x)^3}\end{aligned}$$

Hence, the lemma is justified for division.

(iv) Verification for $\frac{d^2}{dx^2}[f(g(x))]=f''(g(x))g'(x)^2+f'(g(x))g''(x)$.
Let

$$g(z)=g(x)+\imath yg'(x)=X+\imath Y,$$

where $X=g(x)$, $Y=yg'(x)$. Then,

$$f(g(z))=f(X+\imath Y)=f(X)+\imath Yf'(X)$$

or,

$$f(Z)=f(X)+\imath Yf'(X)\ \ (\text{where } g(z)=X+\imath Y=Z)$$

Then,

$$\frac{\partial f(Z)}{\partial x}=\frac{\partial f(X)}{\partial x}+\imath\left\{\frac{\partial Y}{\partial x}.f'(X)+Y.\frac{\partial f'(X)}{\partial x}\right\}$$

Using Chain rule, the above equation implies

$$\begin{aligned}\frac{\partial f(g(z))}{\partial x}&=\frac{\partial f(X)}{\partial X}.\frac{\partial X}{\partial x}+\imath\left\{\frac{\partial(yg'(x))}{\partial x}.f'(X)+Y.\frac{\partial f'(X)}{\partial X}.\frac{\partial X}{\partial x}\right\}\\&=f'(X)g'(x)+\imath\{y'g'(x)f'(X)+yg''(x)f'(X)+Yf''(X)g'(x)\}\\&=f'(g(x))g'(x)+\imath\{y'g'(x)f'(g(x))+yg''(x)f'(g(x))+yg'(x)^2f''(g(x))\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2}{dx^2}[f(g(x))]=\frac{\Im\frac{\partial f(g(z))}{\partial x}-y'[f\{g(z)\}]'}{\Im(z)}&=\frac{y'g'(x)f'(g(x))+yg''(x)f'(g(x))+yg'(x)^2f''(g(x))-y'g'(x)f'(g(x))}{y}\\&=f''(g(x))g'(x)^2+f'(g(x))g''(x).\end{aligned}$$

Hence, the lemma is justified for composition of functions.

3.3. Higher order derivatives

Using the Lemma 3.1 and equation (2.2), the third and subsequent derivatives can be deduced.

1. $f'(x)=\frac{\Im[f(z)]}{\Im(z)}$.
2. $f''(x)=\frac{\Im\left[\frac{\partial f}{\partial x}\right]-y'f'(x)}{\Im(z)}$.
3. $f'''(x)=\frac{\Im\left[\frac{\partial^2 f}{\partial x^2}\right]-y''f'(x)-2y'f''(x)}{\Im(z)}$.
4. $f^{(iv)}(x)=\frac{\Im\left[\frac{\partial^3 f}{\partial x^3}\right]-y'''f'(x)-3y''f''(x)-3y'f'''(x)}{\Im(z)}$.
5.
$$f^{(v)}(x)=\frac{\Im\left[\frac{\partial^4 f}{\partial x^4}\right]-y^{(iv)}f'(x)-4y'''f''(x)--6y''f'''(x)-4y'f^{(iv)}(x)}{\Im(z)}.$$

6.

$$f^{(vi)}(x)=\frac{\Im\left[\frac{\partial^5 f}{\partial x^5}\right]-y^{(v)}f'(x)-5y^{(iv)}f''(x)-10y'''f'''(x)}{\Im(z)}-\frac{10y''f^{(iv)}-4y'f^{(v)}(x)}{\Im(z)}.$$

7.

$$f^{(vii)}(x)=\frac{\Im\left[\frac{\partial^6 f}{\partial x^6}\right]-y^{(v)}f'(x)-5y^{(iv)}f''(x)-10y'''f'''(x)}{\Im(z)}-\frac{10y''f^{(iv)}(x)-4y'f^{(v)}(x)}{\Im(z)}.$$

8.

$$f^{(viii)}(x)=\frac{\Im\left[\frac{\partial^7 f}{\partial x^7}\right]-y^{(vii)}f'(x)-7y^{(vi)}f''(x)-21y^{v}f'''(x)}{\Im(z)}-\frac{35y^{(iv)}f^{(iv)}(x)-35y'''f^{(v)}(x)-21y''f^{(vi)}(x)-7y'f^{(vii)}(x)}{\Im(z)}.$$

3.4. Main result: n-th order derivative

By studying the patterns of all the higher order real derivatives using the parabolic function, the general term of the n-th derivative can be estimated as

$$f^{(n+1)}(x)=\frac{\Im\left[\frac{\partial^n f}{\partial x^n}\right]-\sum_{m=1}^{n}\binom{n}{m}y^{(m)}f^{(n-m+1)}(x)}{\Im(z)}.$$

This equality is proved in the next theorem (see Theorem 3.2).

Theorem 3.2. *The n-th order real derivative of a real function f is*

$$f^{n+1}(x)=\frac{\Im\left[\frac{\partial^n f}{\partial x^n}\right]-\sum_{m=1}^{n}\binom{n}{m}y^{(m)}f^{(n-m+1)}(x)}{\Im(z)}, \tag{3.2}$$

where $x\in\mathbb{R}$ *and* $z=x+\imath y\in\mathbb{O}$. *Also* $\imath^2=0$.

Proof. The proof follows using mathematical induction. The base case of the induction process is $n=0,1$.
If $n=0$ then equation (3.2) will give $f'(x)=\frac{\Im[f(z)]}{\Im(z)}$. This matches with the definition given by Catoni et al. [7].
The case $n=1$ will confer $f''(x)=\frac{\Im\left[\frac{\partial f}{\partial x}\right]-y'f'(x)}{\Im(z)}$. Compare this result with equation (3.1) of Lemma 3.1.
Now if the result is true for $n=k$ then

$$f^{(k+1)}(x)=\frac{\Im\left[\frac{\partial^k f}{\partial x^k}\right]-\sum_{m=1}^{k}\binom{k}{m}y^{(m)}f^{(k-m+1)}(x)}{\Im(z)}. \tag{3.3}$$

Equation (3.3) can be written as,
$\Im\left[\frac{\partial^k f}{\partial x^k}\right]=yf^{(k+1)}(x)+\sum\limits_{m=1}^{k}\binom{k}{m}y^{(m)}f^{(k-m+1)}(x)$. Taking derivative and using term-by-term differentiation in right hand side part the equation becomes
$\frac{\partial}{\partial x}\left[\Im\left[\frac{\partial^k f}{\partial x^k}\right]\right]=\frac{d}{dx}\left[yf^{(k+1)}(x)\right]+\sum\limits_{m=1}^{k}\binom{k}{m}\frac{d}{dx}\left[y^{(m)}f^{(k-m+1)}(x)\right]$, or,

$$\Im\left[\frac{\partial^{k+1} f}{\partial x^{k+1}}\right]=yf^{(k+2)}(x)+y'f^{(k+1)}(x)+\sum_{m=1}^{k}\binom{k}{m}\left[y^{(m+1)}f^{(k-m+1)}(x)+y^{(m)}f^{(k-m+2)}(x)\right].$$

By expanding $\sum\limits_{m=1}^{k}\binom{k}{m}\left[y^{(m+1)}f^{(k-m+1)}(x)+y^{(m)}f^{(k-m+2)}(x)\right]$ and using the basic properties of combination, the above equation reduces to

$$\Im\left[\frac{\partial^{k+1} f}{\partial x^{k+1}}\right]=yf^{((k+1)+1)}(x)+\sum_{m=1}^{k+1}\binom{k+1}{m}y^{(m)}f^{((k+1)-m+1)}(x). \tag{3.4}$$

Now the equation (3.4) can be written as

$$f^{((k+1)+1)}(x)=\frac{\Im\left[\frac{\partial^{k+1} f}{\partial x^{k+1}}\right]-\sum_{m=1}^{k+1}\binom{k+1}{m}y^{(m)}f^{((k+1)-m+1)}(x)}{\Im(z)}. \tag{3.5}$$

From equation (3.5) it is proved that the statement of the Theorem 3.2 is true for $n=k+1$ if equation (3.3) is assumed. Hence from the principal of mathematical induction it is proved that the statement of the Theorem 3.2 is true. □

4. Generalization of automatic differentiation

We know that the dual numbers have the unique property of the automatic differentiation, [11]. Recently, in [13], the authors used dual numbers in automatic differentiation for calculating velocity and acceleration. In this section, we shall give a way to calculate higher order derivatives for parabolic analytic functions.
Real value and real derivative of parabolic functions can be found together through the equation,

$$f(z) = f(x) + \imath y f'(x), \text{ where } z = x + \imath y \in \mathbb{O}.$$

In other words 0-th derivative can be found with the along with the first derivative.
Now from the equation (3.2) we have

$$f^{(n+1)}(x) = \frac{\Im\left[\frac{\partial^n f}{\partial x^n}\right] - \sum_{m=1}^{n}\binom{n}{m} y^{(m)} f^{(n-m+1)}(x)}{\Im(z)}.$$

One can deduce that,

$$\Im\left[\frac{\partial^n f}{\partial x^n}\right] = y f^{(n+1)}(x) + \sum_{m=1}^{n}\binom{n}{m} y^{(m)} f^{(n-m+1)}(x) = \sum_{m=0}^{n}\binom{n}{m} y^{(m)} f^{(n-m+1)}(x).$$

Also $\Re\left[\frac{\partial^n f}{\partial x^n}\right] = f^{(n)}(x)$, where $\Re[z]$ is defined as the real part of $z \in \mathbb{O}$. Since $\frac{\partial^n f}{\partial x^n} = \Re\left[\frac{\partial^n f}{\partial x^n}\right] + \imath \Im\left[\frac{\partial^n f}{\partial x^n}\right]$ therefore

$$\frac{\partial^n f}{\partial x^n} = f^{(n)}(x) + \imath \sum_{m=0}^{n}\binom{n}{m} y^{(m)} f^{(n-m+1)}(x). \tag{4.1}$$

Equation (4.1) can be used for *AD* (automatic differentiation).

5. Applications of the generalized automatic differentiation

The generalized *AD* can find several applications in computational physics, where the ability to compute derivatives efficiently is crucial for solving complex mathematical models. Here are some specific applications in computational physics:

(i) Numerical simulations: In computational physics, simulations often involve solving partial differential equations that describe physical phenomena. Generalized *AD* can be used to compute derivatives of solutions for parameters or initial/boundary conditions. This is particularly useful in sensitivity analysis, [10], and uncertainty quantification.
(ii) Optimization problems: Many computational physics problems involve optimization, [12] where the goal is to find the parameters that minimize or maximize a specific objective function. Generalized *AD* is essential for efficiently computing gradients, enabling gradient-based optimization algorithms to find optimal solutions.
(iii) Fluid dynamics: Computational fluid dynamics (CFD) simulations involve solving the Navier-Stokes equations and often require sensitivity analysis for understanding the impact of different parameters on the fluid flow. Generalized *AD* can be applied to compute derivatives efficiently, aiding in optimization and uncertainty quantification, [15].
(iv) Astrophysics: Numerical simulations, [14] in astrophysics, such as modeling the behavior of galaxies or star formation, often involve solving large sets of differential equations. Generalized *AD* can be used to compute derivatives, facilitating sensitivity analysis and parameter optimization in these simulations.
(v) Climate modeling: Climate models involve solving intricate systems of equations that describe the interactions of various components of the Earth's climate system. Generalized *AD* can be applied to study the sensitivity of climate models to different parameters and initial conditions, [1].

6. Conclusions

We have obtained higher order derivatives of real functions of real variables using parabolic functions. This can be an extension of the existing literature. Also, a well known application of parabolic numbers is *AD* (automatic differentiation). We have given a method of calculating *AD* using parabolic functions. Lastly, along with generalizing the *AD*, we have discussed certain physical applications of it.

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