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On f-Biminimal Legendre Curves in (α,β)-Trans Sasakian Generalized Sasakian Space Forms

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Abstract

In this paper, f-biminimal Legendre curves are studied in (α,β)-trans Sasakian generalized Sasakian space forms. Necessary and sufficient conditions are obtained for a Legendre curve to be f-biminimal in such space forms . Besides, some special cases are studied and some nonexistence theorems are obtained.

Keywords: *f-bimininal curves, generalized Sasakian space forms, Legendre curves.*
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1. Introduction

Generalized Sasakian space forms were defined by Alegre et. al in 2004, [1]. Then these authors obtained lots of examples of such manifolds by using the Riemannian submersions, conformal transformations and warped product.

After these studies, Fetcu et. al investigated biharmonic Legendre curves and biharmonic submanifolds in Sasakian space forms between 2008 and 2009, [9, 10]. Then, Özgür et. al obtained some classes of biharmonic Legendre curves in such space forms, [15].

In 2016 Roth et. al, introduced f-biharmonic and bi-f-harmonic submanifolds of generalized space forms, [16]. Then f-biharmonic Legendre curves are discussed and their some curvature characterizations are obtained by Güvenç et. al, [12].

In 2008, Loubeau and Montaldo obtained biminimal immersions, [14].

Karaca et. al introduced f-biminimal immersions and studied f-biminimal curves in Riemannian manifolds,[11]. Then in 2019 Karaca introduced f-biminimal submanifolds of generalized space forms, [13].

Between 2021 and 2022, Bozdağ and Erdoğan handled non-null magnetic, non-Frenet Legendre and Frenet Legendre f-biminimal curves in 3D normal almost paracontact metric manifolds, [3, 8, 4].

In this study, we focused on f-biminimal Legendre curve in (α,β)-trans Sasakian generalized Sasakian space forms and we handled f-biminimality conditions of a Legendre curve in this kind of manifolds.

2. Preliminaries

In this section, we give fundamental notions of this study.

Definition 2.1. *Harmonic maps are defined as critical points of energy functional*

$$E(\varpi)=\frac{1}{2}\int_{\mathbb{M}}|d\varpi|^2 dv_g,$$

for the maps $\varpi:(\mathbb{M},g)\to(\bar{\mathbb{M}},\bar{g})$ which are defined between Riemannian manifolds $(\mathbb{M},g)$ and $(\bar{\mathbb{M}},\bar{g})$. Here v_g is the volume element of $(\mathbb{M},g)$. Also a map defined as a harmonic map iff

$$\dot{\tau}(\varpi):=trace\nabla d\varpi=0, \tag{2.1}$$

where $\dot{\tau}(\varpi)$ is the tension field of map ϖ, ∇ is the connection induced from the Levi-Civita connection $\nabla^{\bar{\mathbb{M}}}$ and the pull-back connection ∇^{ϖ}, [7].

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Definition 2.2. *Biharmonic maps are defined as critical point of the bienergy functional*

$$\mathbb{E}_2(\varpi) = \frac{1}{2}\int_{\mathbb{M}} |\dot{\tau}(\varpi)|^2 dv_g,$$

for the maps $\varpi : (\mathbb{M}, g) \to (\bar{\mathbb{M}}, \bar{g})$ *between Riemannian manifolds. And also a map is biharmonic iff the bitension field*

$$\dot{\tau}_2(\varpi) = trace(\nabla^{\varpi}\nabla^{\varpi} - \nabla^{\varpi}_{\nabla})\dot{\tau}(\varpi) - trace(\mathscr{R}^{\bar{\mathbb{M}}}(d\varpi, \dot{\tau}(\varpi))d\varpi) = 0.$$

where $\mathscr{R}^{\bar{\mathbb{M}}}$ *is the curvature tensor field of* $\bar{\mathbb{M}}$*, [7].*

Definition 2.3. *f-Harmonic maps are defined as critical point of f-energy functional,*

$$\mathbb{E}_f(\varpi) = \frac{1}{2}\int_{\mathbb{M}} f|d\varpi|^2 dv_g,$$

for the maps $\varpi : (\mathbb{M}, g) \to (\bar{\mathbb{M}}, \bar{g})$ *between Riemannian manifolds where* $f \in C^{\infty}(\mathbb{M}, \mathbb{R})$*. Besides a map is f-harmonic iff its f-tension field equals to:*

$$\dot{\tau}_f(\varpi) = f\dot{\tau}(\varpi) + d\varpi(grad f) = 0, \tag{2.2}$$

[5, 6].

Definition 2.4. *f-Biharmonic maps are critical points of f-bienergy functional*

$$\mathbb{E}_{2,f}(\varpi) = \frac{1}{2}\int_{\mathbb{M}} f|\dot{\tau}(\varpi)|^2 dv_g,$$

for the maps $\varpi : (\mathbb{M}, g) \to (\bar{\mathbb{M}}, \bar{g})$ *between Riemannian manifolds. On the other hand,* ϖ *is a f-biharmonic map if*

$$\dot{\tau}_{2,f}(\varpi) = f\dot{\tau}_2(\varpi) + \Delta f\dot{\tau}(\varpi) + 2\nabla^{\varpi}_{grad f}\dot{\tau}(\varpi) = 0, \tag{2.3}$$

where $\dot{\tau}_{2,f}(\varpi)$ *is the f-bitension field of the map* ϖ*.*
Note that if f is a constant then f-biharmonic map turns into a biharmonic map, [6].

Definition 2.5. *Biminimal immersions are defined as critical points of the bienergy functional* $\mathbb{E}_2(\varpi)$ *for variations normal to the image* $\varpi(\mathbb{M}) \subset \bar{\mathbb{M}}$*, with fixed energy for the maps* $\varpi : (\mathbb{M}, g) \to (\bar{\mathbb{M}}, \bar{g})$ *between Riemannian manifolds. Equivalently, a biminimal immersion is a critical point of the λ-bienergy functional,*

$$\mathbb{E}_{2,\lambda}(\varpi) = \mathbb{E}_2(\varpi) + \lambda E(\varpi)$$

where $\lambda \in \mathbb{R}$ *is a constant. The tension field for a λ-biminimal immersion is as follows,*

$$[\dot{\tau}_{2,\lambda}(\varpi)]^{\perp} = [\dot{\tau}_2(\varpi)]^{\perp} - \lambda[\dot{\tau}(\varpi)]^{\perp} = 0,$$

here $[.]^{\perp}$ *denotes the normal component of* $[.]$*, [14, 11].*

Definition 2.6. *f-Biminimal immersions are defined as critical points of the bienergy functional* $\mathbb{E}_{2,f}(\varpi)$ *for variations normal to the image* $\varpi(\mathbb{M}) \subset \bar{\mathbb{M}}$*, with fixed energy for the maps* $\varpi : (\mathbb{M}, g) \to (\bar{\mathbb{M}}, \bar{g})$ *between Riemannian manifolds. Equivalently, a f-biminimal immersion is a critical point of the λ-f-bienergy functional,*

$$\mathbb{E}_{2,\lambda,f}(\varpi) = \mathbb{E}_{2,f}(\varpi) + \lambda\mathbb{E}_f(\varpi)$$

where $\lambda \in \mathbb{R}$ *is a constant. The tension field for a f-biminimal immersion is as follows,*

$$[\dot{\tau}_{2,\lambda,f}(\varpi)]^{\perp} = [\dot{\tau}_{2,f}(\varpi)]^{\perp} - \lambda[\dot{\tau}_f(\varpi)]^{\perp} = 0,$$

for some value of $\lambda \in \mathbb{R}$*, [11].*

Now let recall some basic definitions about almost contact metric manifolds and generalized Sasakian space forms (see [1, 15]).

A differentiable manifold $\mathbb{M}^{2n+1}$ is called an almost contact manifold with an almost contact structure $(\vartheta, \varsigma, \eta)$ if it admits ϑ tensor field of type $(1,1)$, ς vector field and η 1−form satisfying;

$$\vartheta^2 = -I + \eta \otimes \varsigma, \tag{2.4}$$

$$\eta(\varsigma) = 1, \tag{2.5}$$

conditions where I denotes the identity transformation. As consequences of the condition (2.4), we have $\vartheta\varsigma = 0$ and $\eta \circ \vartheta = 0$.
If $\mathbb{M}$ admits a Riemannian metric g such that

$$g(\vartheta K, \vartheta L) = g(K, L) - \eta(K)\eta(L), \qquad K, L \in \Gamma(T\mathbb{M}), \tag{2.6}$$

then $\mathbb{M}$ called as an almost contact metric manifold with an almost contact metric structure $(\vartheta, \varsigma, \eta, g)$.
From (2.6) it is easy to see that

$$g(K, \vartheta L) = -g(\vartheta K, L) \tag{2.7}$$

and

$$g(K,\varsigma)=\eta(K), \tag{2.8}$$

for any $K,L\in\Gamma(T\mathbb{M})$. The fundamental 2-form of $\mathbb{M}$ is defined by

$$\Phi(K,L)=g(K,\vartheta L).$$

An almost contact metric structure becomes a contact metric structure if

$$g(K,\vartheta L)=d\eta(K,L),$$

for all vector fields $K,L\in\Gamma(T\mathbb{M})$, where $d\eta(K,L)=\frac{1}{2}\{K\eta(L)-L\eta(K)-\eta([K,L])\}$.
An almost contact metric manifold is said to be normal if

$$N_{\vartheta}(K,L)+2d\eta(K,L)\varsigma=0, \tag{2.9}$$

where N is the Nijenhuis torsion tensor of ϑ given by

$$N_{\vartheta}(K,L)=\vartheta^2[K,L]+[\vartheta K,\vartheta L]-\vartheta[\vartheta K,L]-\vartheta[K,\vartheta L], \tag{2.10}$$

for all $K,L\in\Gamma(T\mathbb{M})$. A normal contact metric manifold is said to be a Sasakian manifold. On the other hand an almost contact metric manifold is a Sasakian manifold iff

$$(\nabla_K\vartheta)L=g(K,L)\varsigma-\eta(L)K, \tag{2.11}$$

for any K,L.
An almost contact metric manifolds is said to be a Kenmotsu manifold iff $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$ or equivalently

$$(\nabla_K\vartheta)L=-\eta(L)\vartheta K-g(K,\vartheta L)\varsigma, \tag{2.12}$$

holds. Hence, we get

$$\nabla_K\varsigma=K-\eta(K)\varsigma. \tag{2.13}$$

At last, an almost contact metric manifold is called a cosymplectic manifold iff $d\eta=0$ and $d\Phi=0$ or equivalently

$$\nabla\vartheta=0, \tag{2.14}$$

and then we obtain

$$\nabla\varsigma=0. \tag{2.15}$$

As a generalization of both Sasakian and Kenmotsu manifolds, (α,β)-trans Sasakian manifolds were defined by Oubina, [17]. If there exist two functions α and β on an almost contact metric manifold $\mathbb{M}$ satisfying

$$(\nabla_K\vartheta)L=\alpha\left(g(K,L)\varsigma-\eta(L)K\right)+\beta\left(g(\vartheta K,L)\varsigma-\eta(L)\vartheta K\right), \tag{2.16}$$

for any $K,L\in\Gamma(T\mathbb{M})$, then $\mathbb{M}$ is called a trans Sasakian manifold.

- If $\beta=0$, then $\mathbb{M}$ is called a α-Sasakian manifold.
- If $\beta=0$ and $\alpha=1$, then $\mathbb{M}$ is called a Sasakian manifold.
- If $\alpha=0$, then $\mathbb{M}$ is called a β-Kenmotsu manifold.
- If $\alpha=0$ and $\beta=1$, then $\mathbb{M}$ is called a Kenmotsu manifold.
- If $\alpha=\beta=0$ then $\mathbb{M}$ is a cosymplectic manifold.

A ϑ-section of an almost contact metric manifold $(\mathbb{M},\vartheta,\varsigma,\eta,g)$ at a point $p\in\mathbb{M}$ is a section $\Pi\subseteq T_p\mathbb{M}$ which is spanned by a unit vector field U_p orthogonal to ς_p and ϑU_p. The ϑ-sectional curvature $\mathscr{K}(K\wedge\vartheta K)$ is defined by

$$\mathscr{K}(K\wedge\vartheta K)=\mathscr{R}(K,\vartheta K,\vartheta K,K). \tag{2.17}$$

If the ϑ-sectional curvature of a manifold is constant, that manifold is called a space form, [1].
An almost contact metric manifold is called as a generalized Sasakian space form [1], if there exist three functions ρ_1,ρ_2 and ρ_3 on $\mathbb{M}$ such that

$$\begin{aligned}\mathscr{R}(K,L)M &= \rho_1\{g(L,M)K-g(K,M)L\}\\ &\quad+\rho_2\{g(K,\vartheta M)\vartheta L-g(L,\vartheta M)\vartheta K+2g(K,\vartheta L)\vartheta M\}\\ &\quad+\rho_3\{\eta(K)\eta(M)L-\eta(L)\eta(M)K+g(K,M)\eta(L)\varsigma-g(L,M)\eta(K)\varsigma\},\end{aligned} \tag{2.18}$$

for any vector fields on $\mathbb{M}$, where $\mathscr{R}$ denotes the curvature tensor of $\mathbb{M}$.
The contact distribution of an almost contact metric manifold $\mathbb{M}$ is defined by

$$\{U\in\Gamma(T\mathbb{M}):\eta(U)=0\}$$

and an integral curve of the contact distribution is defined as a Legendre curve, [15].
Note that throughout this paper, we will use TSGSSF instead of trans Sasakian generalized Sasakian space form for the sake of simplicity.

3. f-biminimal Legendre curves in (α,β)-trans Sasakian generalized Sasakian space forms

Assume that $\delta: I \to \mathbb{M}$ be a curve parametrized by arclenght on a Riemannian manifold $\mathbb{M}$. Let $\{\mathbb{E}_1,\mathbb{E}_2,...,\mathbb{E}_n\}$ be the Frenet frame in $\mathbb{M}$, defined along δ, where $\mathbb{E}_1=\delta'=\mathbb{T}$ is the unit tangent vector field, $\mathbb{E}_2$ is the unit normal vector field and the vectors $\mathbb{E}_3,...,\mathbb{E}_n$ are the unit vectors obtained from the Frenet equations for δ,

$$\begin{aligned}
\nabla_{\mathbb{T}}\mathbb{E}_1 &= \kappa_1\mathbb{E}_2,\\
\nabla_{\mathbb{T}}\mathbb{E}_2 &= -\kappa_1\mathbb{E}_1+\kappa_2\mathbb{E}_3,\\
&\dots\\
\nabla_{\mathbb{T}}\mathbb{E}_r &= -\kappa_{r-1}\mathbb{E}_{r-1}+\kappa_r\mathbb{E}_{r+1}, \quad r=3,...,n-1,\\
&\dots\\
\nabla_{\mathbb{T}}\mathbb{E}_n &= -\kappa_{n-1}\mathbb{E}_{n-1},
\end{aligned} \tag{3.1}$$

where $\kappa_1=\|\nabla_{\mathbb{T}}\mathbb{E}_1\|$ and $\kappa_2,...,\kappa_n$ are real valued positive functions.
From (2.18) and (3.1), we have

$$\nabla^2_{\mathbb{T}}\mathbb{T} = -\kappa_1^2\mathbb{E}_1+\kappa_1'\mathbb{E}_2+\kappa_1\kappa_2\mathbb{E}_3, \tag{3.2}$$

$$\nabla^3_{\mathbb{T}}\mathbb{T} = -3\kappa_1\kappa_1'\mathbb{E}_1+\left(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2\right)\mathbb{E}_2+\left(2\kappa_1'\kappa_2+\kappa_1\kappa_2'\right)\mathbb{E}_3+\kappa_1\kappa_2\kappa_3\mathbb{E}_4, \tag{3.3}$$

$$\mathscr{R}(\mathbb{T},\nabla_{\mathbb{T}}\mathbb{T})\mathbb{T} = \kappa_1\left[-\rho_3\eta(\mathbb{E}_2)\eta(\mathbb{T})\mathbb{T}+(\rho_3\eta(\mathbb{T})^2-\rho_1)\mathbb{E}_2+3\rho_2 g(\mathbb{T},\vartheta\mathbb{E}_2)\vartheta+\rho_3\eta(\mathbb{E}_2)\varsigma\right]. \tag{3.4}$$

With the help of λ-f-bienergy functional, the f-biminimality condition obtained by using normal components of f-tension and f-bitension field (see [11]);

$$[\dot{\tau}_{2,\lambda,f}(\delta)]^{\perp} = [\dot{\tau}_{2,f}(\delta)]^{\perp}-\lambda[\dot{\tau}_f(\delta)]^{\perp},$$

where

$$\begin{aligned}
\dot{\tau}_f(\delta) &= f\nabla_{\mathbb{T}}\mathbb{T}+f'\mathbb{T},\\
\dot{\tau}_{2,f}(\delta) &= f(\nabla^3_{\mathbb{T}}\mathbb{T}-\mathscr{R}(\mathbb{T},\nabla_{\mathbb{T}}\mathbb{T})\mathbb{T})+f''\nabla_{\mathbb{T}}\mathbb{T}+2f'\nabla^2_{\mathbb{T}}\mathbb{T}.
\end{aligned}$$

With the help of these calculations, we obtained the f-biminimality condition as follows;

$$\begin{aligned}
[\dot{\tau}_{2,\lambda,f}(\delta)]^{\perp} &= [\dot{\tau}_{2,f}(\delta)]^{\perp}-\lambda[\dot{\tau}_f(\delta)]^{\perp}\\
&= \left[(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2+\rho_1\kappa_1-\kappa_1\rho_3\eta(\mathbb{T})^2-\lambda\kappa_1)f+2f'\kappa_1'+f''\kappa_1\right]\mathbb{E}_2\\
&\quad+\left[(2\kappa_1'\kappa_2+\kappa_1\kappa_2')f+2\kappa_1\kappa_2 f'\right]\mathbb{E}_3+\left[f\kappa_1\kappa_2\kappa_3\right]\mathbb{E}_4-\left[3f\kappa_1\rho_2 g(\mathbb{T},\vartheta\mathbb{E}_2)\right]\vartheta\mathbb{T}-\left[f\kappa_1\rho_3\eta(\mathbb{E}_2)\right]\varsigma\\
&= 0.
\end{aligned} \tag{3.5}$$

By using (3.5) we obtain;

Theorem 3.1. *Let $\delta: I \longrightarrow \mathbb{M}$ be a Legendre curve parametrized by its arclength on an (α,β)-TSGSSF. Then δ is a f-biminimal curve iff*

$$\begin{aligned}
&\left[(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2)f+\rho_1\kappa_1 f+f''\kappa_1+2f'\kappa_1'-\lambda f\kappa_1\right]\mathbb{E}_2+\left[(2\kappa_1'\kappa_2+\kappa_1\kappa_2')f+2\kappa_1\kappa_2 f'\right]\mathbb{E}_3\\
&+\left[f\kappa_1\kappa_2\kappa_3\right]\mathbb{E}_4+\left[3f\kappa_1\rho_2 g(\vartheta\mathbb{E}_2,\mathbb{T})\right]\vartheta\mathbb{T}+\left[f\rho_3\beta\right]\varsigma=0.
\end{aligned}$$

Now let $m=min\{r,4\}$ and from (3.5), δ is a f-biminimal Legendre curve iff

- $\rho_2=0$ or $\vartheta\mathbb{T}\perp\mathbb{E}_2$ or $\vartheta\mathbb{T}\in span\{\mathbb{E}_2,...,\mathbb{E}_m\}$ and
- $\rho_3=0$ or $\varsigma\perp\mathbb{E}_2$ or $\varsigma\in span\{\mathbb{E}_2,...,\mathbb{E}_m\}$ and
- $g([\dot{\tau}_{2,\lambda,f}(\delta)]^{\perp},\mathbb{E}_i)=0$ for any $i=1,2,...,m$.

Then we can introduce following theorem.

Theorem 3.2. *Let $\delta: I \longrightarrow \mathbb{M}$ be a Legendre curve parametrized by its arclength on a (α,β)-TSGSSF. Then δ is a f-biminimal curve iff*

- $\rho_2=0$ *or* $\vartheta\mathbb{T}\perp\mathbb{E}_2$ *or* $\vartheta\mathbb{T}\in span\{\mathbb{E}_2,...,\mathbb{E}_m\}$ *and*
- $\rho_3=0$ *or* $\varsigma\perp\mathbb{E}_2$ *or* $\varsigma\in span\{\mathbb{E}_2,...,\mathbb{E}_m\}$ *and*
- *the following differential equations are satisfied:*

$$\begin{cases}
(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2)f+\rho_1\kappa_1 f+f''\kappa_1+2f'\kappa_1'-\lambda f\kappa_1+3f\kappa_1\rho_2 g(\vartheta\mathbb{E}_2,\mathbb{T})-f\rho_3\dfrac{\beta^2}{\kappa_1}=0,\\[2ex]
(2\kappa_1'\kappa_2+\kappa_1\kappa_2')f+2\kappa_1\kappa_2 f'+3f\kappa_1\rho_2 g(\vartheta\mathbb{T},\mathbb{E}_2)g(\vartheta\mathbb{T},\mathbb{E}_3)+f\rho_3\beta\eta(\mathbb{E}_3)=0,\\[2ex]
\kappa_1\kappa_2\kappa_3 f+3f\kappa_1\rho_2 g(\vartheta\mathbb{T},\mathbb{E}_2)g(\vartheta\mathbb{T},\mathbb{E}_4)+f\rho_3\beta\eta(\mathbb{E}_4)=0.
\end{cases} \tag{3.6}$$

Now we consider some special cases for f-biminimal Legendre curves in (α,β)-TSGSSF.
Case I: If $\rho_2=\rho_3=0$, then equation (3.6) reduces to;

$$\begin{cases}(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2)f+\rho_1\kappa_1 f+f''\kappa_1+2f'\kappa_1'-\lambda f\kappa_1=0,\\ (2\kappa_1'\kappa_2+\kappa_1\kappa_2')f+2\kappa_1\kappa_2 f'=0,\\ \kappa_1\kappa_2\kappa_3 f=0.\end{cases}$$

Via third differential equation of Case I, we obtained;

Theorem 3.3. *There is no proper f-biminimal Legendre curve of osculating order $r>3$ in a (α,β)-TSGSSF with $\rho_2=\rho_3=0$.*

And via differential equation system of Case I, we get following theorem.

Theorem 3.4. *Let $\delta: I\longrightarrow\mathbb{M}$ be a Legendre curve of osculating order $r\leq 3$ parametrized by its arclength on a (α,β)-TSGSSF with $\rho_2=\rho_3=0$. Then δ is a f-biminimal curve iff following differential equation system is satisfied;*

$$\begin{cases}(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2)f+\rho_1\kappa_1 f+f''\kappa_1+2f'\kappa_1'-\lambda f\kappa_1=0,\\ (2\kappa_1'\kappa_2+\kappa_1\kappa_2')f+2\kappa_1\kappa_2 f'=0.\end{cases}$$

Sub-Case I-1: If $\rho_2=\rho_3=0$ and $\kappa_1=cons.\neq 0$, $\kappa_2=0$, then we get;

Theorem 3.5. *Let $\delta: I\longrightarrow\mathbb{M}$ be a Legendre curve of osculating order $r\leq 3$ parametrized by its arclength on a (α,β)-TSGSSF form with $\rho_2=\rho_3=0$ and $\kappa_1=cons.\neq 0$, $\kappa_2=0$. Then δ is an f-biminimal curve iff κ_1 satisfy the following differantial equation*

$$\kappa_1^3 f-\rho_1\kappa_1 f-f''\kappa_1+\lambda f\kappa_1=0,$$

where

$$f(s)=c_1 e^{\left(\sqrt{\kappa_1^2+\rho_1+\lambda}\right)s}+c_2 e^{\left(-\sqrt{\kappa_1^2+\rho_1+\lambda}\right)s}.$$

Sub-Case I-2: If $\rho_2=\rho_3=0$ and $\kappa_1=cons.\neq 0$, $\kappa_2=cons.\neq 0$, then we give the following corollary.

Corollary 3.6. *There is no proper f-biminimal Legendre curve of osculating order $r\leq 3$ parametrized by its arclength on a (α,β)-TSGSSF with $\rho_2=\rho_3=0$ and $\kappa_1=cons.\neq 0$, $\kappa_2=cons.\neq 0$.*

Sub-Case I-3: If $\rho_2=\rho_3=0$ and $\kappa_1\neq cons.$, $\kappa_2=cons.\neq 0$, then we state;

Theorem 3.7. *Let $\delta: I\longrightarrow\mathbb{M}$ be a Legendre curve of osculating order $r\leq 3$ parametrized by its arclength on a (α,β)-TSGSSF with $\rho_2=\rho_3=0$ and $\kappa_1\neq cons.$, $\kappa_2=cons.\neq 0$. Then δ is a f-biminimal curve iff κ_1 and κ_2 satisfy the following differantial equation*

$$(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2)f+\rho_1\kappa_1 f+f''\kappa_1+2f'\kappa_1'-\lambda f\kappa_1=0,$$

where

$$f(s)=c\kappa_1(s).$$

Case II: If $\rho_2=0$, $\rho_3\neq 0$ and $\mathbb{E}_2\perp\varsigma$. Then from $\mathbb{E}_2\perp\varsigma$ and $\eta(\mathbb{E}_2)=-\dfrac{\beta}{\kappa_1}$, (see [18]) it is obvious that $\mathbb{M}$ is a α-Sasakian generalized Sasakian space form. Then equation (3.6) reduces to;

$$\begin{cases}(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2)f+\rho_1\kappa_1 f+f''\kappa_1+2f'\kappa_1'-\lambda f\kappa_1=0,\\ (2\kappa_1'\kappa_2+\kappa_1\kappa_2')f+2\kappa_1\kappa_2 f'=0,\\ \kappa_1\kappa_2\kappa_3 f=0.\end{cases}\tag{3.7}$$

Via third equation of (3.7), we have the following theorems.

Theorem 3.8. *There is no proper $f-$biminimal Legendre curve of osculating order $r>3$ in a α-Sasakian generalized Sasakian space form with $\rho_2=0,\rho_3\neq 0$.*

Theorem 3.9. *Let $\delta: I\longrightarrow\mathbb{M}$ be a Legendre curve of osculating order $r\leq 3$ parametrized by its arclength on a α-Sasakian generalized Sasakian space form with $\rho_2=0$, $\rho_3\neq 0$. Then δ is a f-biminimal curve iff following differantial equations are satisfy;*

$$\begin{cases}(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2)f+\rho_1\kappa_1 f+f''\kappa_1+2f'\kappa_1'-\lambda f\kappa_1=0,\\ (2\kappa_1'\kappa_2+\kappa_1\kappa_2')f+2\kappa_1\kappa_2 f'=0.\end{cases}$$

Case III: If $\rho_2 = 0$, $\rho_3 \neq 0$, $\varsigma \in span\{\mathbb{E}_2, ..., \mathbb{E}_m\}$ and $\eta(\mathbb{E}_2) \neq 0$, then equation (3.6) reduces to;

$$\begin{cases} (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)f + \rho_1\kappa_1 f + f''\kappa_1 + 2f'\kappa_1' - \lambda f\kappa_1 - f\kappa_1\rho_3\eta(\mathbb{E}_2)^2 = 0, \\ (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')f + 2\kappa_1\kappa_2 f' + f\rho_3\beta\eta(\mathbb{E}_3) = 0, \\ \kappa_1\kappa_2\kappa_3 f + f\rho_3\beta\eta(\mathbb{E}_4) = 0. \end{cases}$$

Let $m = \min\{r, 4\} = 4$, which implies $r \geq 4$. Then we can write

$$\varsigma = \cos\theta_1\mathbb{E}_2 + \sin\theta_1\cos\theta_2\mathbb{E}_3 + \sin\theta_1\sin\theta_2\mathbb{E}_4, \tag{3.8}$$

which implies

$$\begin{aligned} \eta(\mathbb{E}_2) &= \cos\theta_1, \\ \eta(\mathbb{E}_3) &= \sin\theta_1\cos\theta_2, \\ \eta(\mathbb{E}_4) &= \sin\theta_1\sin\theta_2. \end{aligned} \tag{3.9}$$

Here $\theta_1 : I \to \mathbb{R}$ denotes the angle function between $\mathbb{E}_2$ and ς and $\theta_2 : I \to \mathbb{R}$ is the angle function between $\mathbb{E}_3$ and the orthogonal projection of ς onto $span\{\mathbb{E}_3, \mathbb{E}_4\}$, [15].
From here we get;

Theorem 3.10. *Let $\delta : I \longrightarrow \mathbb{M}$ be a Legendre curve of osculating order r parametrized by its arclength on a (α, β)-TSGSSF with $\rho_2 = 0$, $\rho_3 \neq 0$, $\varsigma \in span\{\mathbb{E}_2, ..., \mathbb{E}_m\}$ and $\eta(\mathbb{E}_2) \neq 0$, Then δ is a f-biminimal curve iff following differantial equations are satisfy;*

$$\begin{cases} f''\kappa_1 + 2f'\kappa_1' + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \rho_1\kappa_1 - \lambda\kappa_1 - \kappa_1\rho_3(cos\theta_1)^2)f = 0, \\ (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')f + 2\kappa_1\kappa_2 f' + f\rho_3\beta sin\theta_1 cos\theta_2 = 0, \\ \kappa_1\kappa_2\kappa_3 f + f\rho_3\beta sin\theta_1 sin\theta_2 = 0. \end{cases}$$

Sub-Case III-1: If $\rho_2 = 0$, $\rho_3 \neq 0$, $\varsigma \in span\{\mathbb{E}_2, ..., \mathbb{E}_m\}$, $\eta(\mathbb{E}_2) \neq 0$, and $\kappa_1 = cons. \neq 0$, $\kappa_2 = 0$, then we state;

Theorem 3.11. *Let $\delta : I \longrightarrow \mathbb{M}$ be a Legendre curve of osculating order r parametrized by its arclength on a (α, β)-TSGSSF with $\rho_2 = 0$, $\rho_3 \neq 0$, $\varsigma \in span\{\mathbb{E}_2, ..., \mathbb{E}_m\}$, $\eta(\mathbb{E}_2) \neq 0$ and $\kappa_1 = cons. \neq 0$, $\kappa_2 = 0$. Then δ is a f-biminimal curve iff*

$$f(s) = c_1 e^{\left(\sqrt{\kappa_1^2 + \rho_3 - \rho_1 + \lambda}\right)s} + c_2 e^{\left(-\sqrt{\kappa_1^2 + \rho_3 - \rho_1 + \lambda}\right)s}$$

where $\theta_1 = 2k\pi$, $k \in \mathbb{Z}$.

Sub-Case III-2: If $\rho_2 = 0$, $\rho_3 \neq 0$, $\varsigma \in span\{\mathbb{E}_2, ..., \mathbb{E}_m\}$, $\eta(\mathbb{E}_2) \neq 0$ and $\kappa_1 = cons. \neq 0$, $\kappa_2 = cons. \neq 0$, then we get the following theorem.

Theorem 3.12. *Let $\delta : I \longrightarrow \mathbb{M}$ be a Legendre curve of osculating order r parametrized by its arclength on a (α, β)-TSGSSF with $\rho_2 = 0$, $\rho_3 \neq 0$, $\varsigma \in span\{\mathbb{E}_2, ..., \mathbb{E}_m\}$, $\eta(\mathbb{E}_2) \neq 0$ and $\kappa_1 = cons. \neq 0$, $\kappa_2 = cons. \neq 0$. Then δ is a f-biminimal curve iff κ_1 and κ_2 satisfy the following differantial equation*

$$f''\kappa_1 - (\kappa_1^3 + \kappa_1\kappa_2^2 - \rho_1\kappa_1 + \lambda\kappa_1 + \kappa_1\rho_3(cos\theta_1)^2)f = 0$$

where

$$f(s) = e^{\int \frac{\kappa_3(s)}{2} cot\theta_2(s) ds} + c.$$

Sub-Case III-3: If $\rho_2 = 0$, $\rho_3 \neq 0$, $\varsigma \in span\{\mathbb{E}_2, ..., \mathbb{E}_m\}$, $\eta(\mathbb{E}_2) \neq 0$ and $\kappa_1 \neq cons.$, $\kappa_2 = cons. \neq 0$, then we have;

Theorem 3.13. *Let $\delta : I \longrightarrow \mathbb{M}$ be a Legendre curve of osculating order r parametrized by its arclength on a (α, β)-TSGSSF with $\rho_2 = 0$, $\rho_3 \neq 0$, $\varsigma \in span\{\mathbb{E}_2, ..., \mathbb{E}_m\}$, $\eta(\mathbb{E}_2) \neq 0$ and $\kappa_1 \neq cons.$, $\kappa_2 = cons. \neq 0$. Then δ is a f-biminimal curve iff κ_1 and κ_2 satisfy the following differantial equation*

$$f''\kappa_1 + 2f'\kappa_1' + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \rho_1\kappa_1 - \lambda\kappa_1 - \kappa_1\rho_3(cos\theta_1)^2)f = 0$$

where

$$f(s) = \frac{1}{\kappa_1(s)} e^{\int \frac{\kappa_3(s)}{2} cot\theta_2(s) ds} + c.$$

4. Conclusion

In this study, we obtained f-biminimality conditions of a Legendre curve in (α, β)-trans Sasakian generalized Sasakian space forms. As a future work, researchers interested in this subject can study such curves in other space forms.

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